Simplices of maximal volume in hyperbolic space, Gromov's norm, and Gromov's proof of Mostow's rigidity theorem (following Thurston).

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§0 Introduction

In my lecture at the conference I gave a relatively detailed proof of the following theorem, which represents joint work with U. Haagerup, and which had been conjectured by Milnor, [2].

Theorem 1 In hyperbolic n-space H^n a geodesic n-simplex σ is of maximal volume if and only if σ is ideal and regular.

Here ideal means that all vertices are on "the sphere at infinity" S^{n-1}_∞ . And regular means that all faces of σ are congruent modulo the isometries of H^n .

I also outlined, very briefly, how this result can be used in a proof of Mostow's rigidity theorem for hyperbolic manifolds.

Theorem 2 (Mostow) Any homotopy equivalence $f:M\to N$ between closed, orientable, hyperbolic n-manifolds with $n\geq 3$ is homotopic to an isometry.

The proof that I refer to was given by Thurston (who attributed it to Gromov) in his 1977/78 Princeton University lecture notes, [4]. Thurston considered only the case $n=3$ because the validity of theorem 1 was unknown for $n>3$.

Since the lecture notes are not easily accessible, and since, at the conference, there was considerable interest in some of the details of Gromov's proof (especially what is below called step 3) I have decided to write down a rather detailed exposition of Gromov's argument. The proof of theorem 1 will then appear elsewhere.

It follows that I claim absolutely no originality concerning the material in this note. It is nothing but my interpretation and expansion of one of Thurston's lectures.

§1. Outline of Gromov's proof

In this section we outline Gromov's proof of Mostow's theorem. Details are given in later sections. Thus let a homotopy equivalence $f:M\to N$ be given. It fits into the commutative diagram

$$\begin{array}{ccc} H^n & \xrightarrow{\tilde{f}} & H^n \\ \downarrow p & & \downarrow p \\ M=\Gamma\backslash H^n & \xrightarrow[f]{} & \Theta\backslash H^n=N \end{array}$$

where p denotes universal covering maps. Also $\tilde{f}$ is φ-equivariant where $\varphi:\Gamma\to\Theta$ is the isomorphism of fundamental groups induced by f .

Step 1 $\tilde{f}:H^n\to H^n$ is a pseudo isometry, i.e. there are constants a,b such that

$$a^{-1}d(x,y)-b\leq d(\tilde{f}(x),\tilde{f}(y))\leq ad(x,y)$$

for all $x,y\in H^n$.

Step 2 Any pseudo isometry g of H^n gives rise to a continuous map $g_+:S_\infty^{n-1}\to S_\infty^{n-1}$ on the sphere at infinity. This association is such that $\tilde{f}_+$ is still φ-equivariant.

Step 3 If $v_o,v_1,\ldots,v_n\in S_\infty^{n-1}$ span a geodesic n-simplex of maximal volume then so do $\tilde{f}_+(v_o),\tilde{f}_+(v_1),\ldots,\tilde{f}_+(v_n)$.

Step 4 $\tilde{f}_+=h_+$ for some isometry $h:H^n\to H^n$.

Let us see how this finishes the proof. It is well known that an isometry h of H^n is completely determined by $h_+:S_\infty^{n-1}\to S_\infty^{n-1}$. Therefore, the above $h:H^n\to H^n$ is φ-equivariant. And the map $\bar{h}:\Gamma\backslash H^n\to\Theta\backslash H^n$ that it covers is the desired isometry. $\bar{h}$ is homotopic to f because it induces φ on fundamental group level, at least up to conjugacy.

§2 Proof of step 1

We may assume that f is simplicial w.r.t. triangulations of M and N . Then f satisfies a Lipschitz condition. Hence so does $\tilde{f}$, i.e.

(2.1) $d(\tilde{f}(x),\tilde{f}(y))\leq ad(x,y)$.

We may also choose a homotopy inverse f_1 covered by an $\tilde{f}_1$ which satisfies (increase a, if need be)

(2.2) $d(\tilde{f}_1(x),\tilde{f}_1(y))\leq ad(x,y)$

(2.3) $\tilde{f}_1\tilde{f}$ is Γ-equivariantly homotopic to 1_{H^n} .

On a compact set the homotopy involved in (2.3) moves any x only a bounded distance. By equivariance, and compactness of M , the same holds on all of H^n , i.e. for some b_1

(2.4) $d(\tilde{f}_1\tilde{f}(x),x)\leq b_1$.

Now one has

$$d(x,y)\leq 2b_1+d(\tilde{f}_1\tilde{f}(x),\tilde{f}_1\tilde{f}(y))\leq 2b_1+ad(\tilde{f}(x),\tilde{f}(y))$$

which implies, with $b=2b_1/a$

(2.5) $d(\tilde{f}(x),\tilde{f}(y))\geq a^{-1}d(x,y)-b$

Q.E.D.

§3 Proof of step 2

The main ingredient is the following proposition which states that pseudo isometries "almost preserve" geodesics and "almost preserve" normal geodesic hyperplanes. If γ is a geodesic in H^n we let P_γ denote the orthogonal projection onto γ .

Proposition 3.1 If $g:H^n\to H^n$ is a pseudo isometry then there exists a constant r so that

(i) Any geodesic γ has $g(\gamma)$ contained in a tubular neighbourhood $N_r(\bar{\gamma})$ of radius r around a unique geodesic $\bar{\gamma}$.

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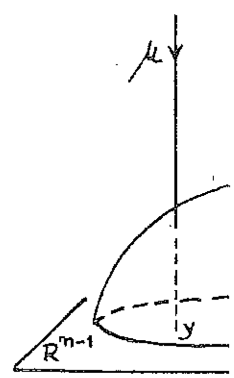
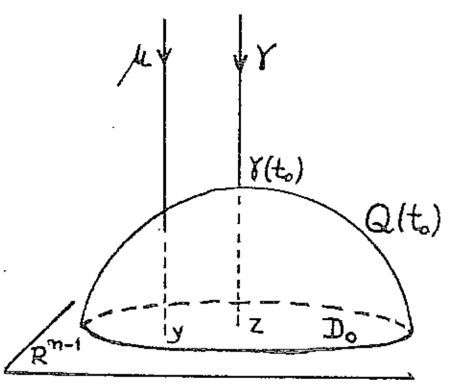
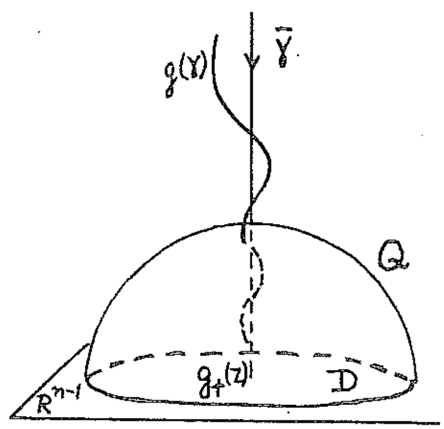


(ii) For any geodesic γ and any geodesic hyperplane Q orthogonal to γ the image $P_{\bar{\gamma}}(g(Q))$ is a segment of length $\leq r$.

Before we outline a proof let us apply the proposition. If we call two geodesics equivalent when $d(\gamma_1(t),\gamma_2(t))$ is bounded for $t\to\infty$ (t a natural parameter) then S_∞^{n-1} is the set of equivalence classes of geodesics (as a set). It is easily seen that $\gamma\to\bar{\gamma}$ respects the equivalence relation. Hence $\gamma\to\bar{\gamma}$ induces a function $g_+ : S_\infty^{n-1}\to S_\infty^{n-1}$. Note that when g is an isometry we may take $r=0$ and we recover the usual extension of isometries over S_∞^{n-1}. Also if g is equivariant w.r.t. to some $\varphi:\Gamma\to\Theta$ (where Γ and Θ are isometry groups) then so is g_+.

To check continuity of g_+ at $z\in S_\infty^{n-1}$ we argue in the upper half space model. Then $S_\infty^{n-1}=R^{n-1}\cup\infty$ and we arrange coordinates so that both z and $g_+(z)$ are $\neq\infty$. Let z be determined by γ which passes through ∞. We may arrange coordinates so that $\infty\in\bar{\gamma}$. Now any neighbourhood of $g_+(z)$ contains a disc D which is "boundary" for a geodesic hyperplane Q orthogonal to $\bar{\gamma}$.





Let $H^+(Q)$ and $H^-(Q)$ be the half spaces determined by Q. One easily checks that

$$d(P_{\overline{\gamma}}(g\gamma(t)),H^-(Q))\to\infty$$

as $t\to\infty$. Hence for suitable t_o

(3.1) $\quad d(P_{\overline{\gamma}}(g\gamma(t)),H^-(Q))>2r\ ,\quad$ for $t\geq t_o$.

If $Q(t)$ is the geodesic hyperplane orthogonal to γ through $\gamma(t)$ then (ii) and (3.1) imply that

(3.2) $\quad d(P_{\overline{\gamma}}(g(Q(t))),H^-(Q))>r\ ,\quad$ for $t\geq t_o$.

Let D_o be the disc in R^{n-1} "bounding" $Q(t_o)$. We finish the proof of continuity by showing that $g_+(D_o)\subseteq D$. In fact let $y\in D_o$ be determined by μ . If $g_+(y)\notin D$ then $\overline{\mu}(t)$, and hence $P_{\overline{\gamma}}(\overline{\mu}(t))$, must be in $H^-(Q)$ for all $t\geq$ some t_1 . Since $P_{\overline{\gamma}}$ decreases distances it follows that

$$d(P_{\overline{\gamma}}(g(\mu(t))),H^-(Q))\leq r$$

for arbitrarily large values of t . But this contradicts (3.2).

The rest of this section contains a proof of part (i) of proposition 3.1. We start by considering geodesics γ and ρ and a fixed $s>0$ with $\cosh(s)>a^2$ (a = the Lipschitz constant for g) . Let ℓ be the length of a bounded, connected component $g(\gamma)_1$ of $g(\gamma)\cap(H^n-N_s(\rho))$. We first want to establish an upper bound for ℓ . Let the endpoints of $g(\gamma)_1$ be $g(p)$ and $g(q)$ and put $p'=P_\rho(g(p))$, $q'=P_\rho(g(q))$. Then $d(g(p),p')=d(g(q),q')=s$. Also, elementary hyperbolic gemoetry shows that $P_\gamma|H^n-N_s(\rho)$ decreases

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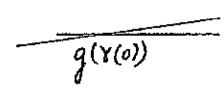
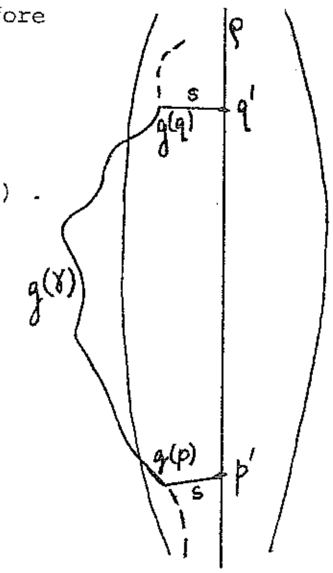


Hence ρ_n has a

show that $g(\gamma)\subseteq$

lengths by a factor $\leq \cosh(s)^{-1}$. Therefore



$$a^{-1}d(p,q)-b \leq d(g(p),g(q))$$
$$\leq 2s+\ell\cosh(s)^{-1}$$
$$\leq 2s+ a\cosh(s)^{-1}d(p,q) .$$

It follows that

$$d(p,q) \leq k = \frac{(2s+b)\, a\cosh(s)}{\cosh(s)-a^2}$$

and, by the Lipschitz condition,

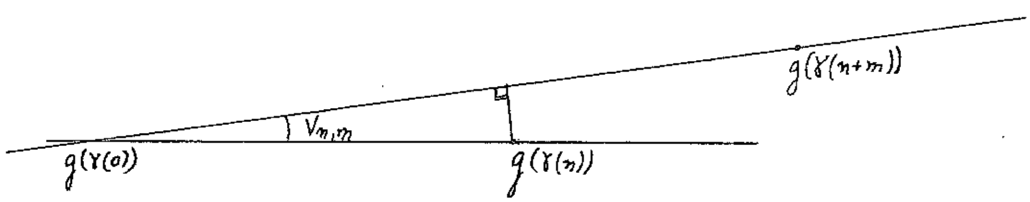
$$\ell \leq ak .$$

Now take $r=s+ak$. We then have

(3.3) If $g\gamma(p)$ and $g\gamma(q)$ lie on ρ then $g\gamma[p,q] \subseteq N_r(\rho)$.

In fact, if $g(\gamma(t))$ leaves $N_s(\rho)$ for some $t \in [p,q]$ then it must return to $N_s(\rho)$ before arc length has grown by ak , so it cannot leave $N_r(\rho)$.

For fixed γ we now let ρ_n be the geodesic through $g(\gamma(0))$ and $g(\gamma(n))$. Since $d(g(\gamma(n)),g(\gamma(0)))\to\infty$ as $n\to\infty$ (3.3) implies that the angle $v_{n,m}$ between ρ_n and ρ_{n+m} at $g(\gamma(0))$ goes to zero as $n\to\infty$ (any $m>0$), see the figure .



Hence ρ_n has a limit geodesic $\overline{\gamma}$ as $n\to\infty$. And one may show that $g(\gamma) \subseteq N_r(\overline{\gamma})$.

Uniqueness of $\overline{Y}$ is clear since $N_r(\overline{Y})$ and $N_r(\overline{\overline{Y}})$ are asymptotically disjoint in at least one end, if $\overline{Y}\neq\overline{\overline{Y}}$.

The proof of (ii) is another relatively simple geometric exercise left to the reader (one may of course have to increase r) .

§4 Gromov's norm

For any smooth manifold M let $C^1(\Delta(k),M)$ be the space (with C^1 topology) of C^1 maps $\sigma:\Delta(k)\to M$ of the standard k-simplex $\Delta(k)$ into M . Let $\mathscr{C}_k(M)$ be the real vector space of compactly supported Borel measures μ of bounded total variation $\|\mu\|$, on the space $C^1(\Delta(k),M)$. The various face inclusions $\eta_i:\Delta(k-1)\to\Delta(k)$ induce maps $\eta_i{}^*:C^1(\Delta(k),M)\to C^1(\Delta(k-1),M)$ and homomorphisms $\partial_i=(\eta_i{}^*)_*:\mathscr{C}_k(M)\to\mathscr{C}_{k-1}(M)$. $\partial=\Sigma(-1)^i\partial_i:\mathscr{C}_k(M)\to\mathscr{C}_{k-1}(M)$ makes $\mathscr{C}_*(M)$ into a chain complex.

If $C_*(M)$ is the real, singular chain complex, based on $C^1(\Delta(k),M)$, then there is an obvious natural transformation $i:C_*(M)\to\mathscr{C}_*(M)$. <u>On homology i induces an isomorphism</u>. Moreover, if $\Lambda^*(M)$ is the deRham cochain complex then the usual pairing

$$<,>:C_*(M)\otimes\Lambda^*(M)\to R$$

extends to a pairing

$$<,>:\mathscr{C}_*(M)\otimes\Lambda^*(M)\to R$$

defined by

$<\mu,\omega>=$

Now let M be a
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<u>Definition 4.1</u>
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$$\langle\mu,\omega\rangle=\int_{\sigma\in C^1(\Delta(k),M)}\left(\int_{\Delta(k)}\sigma^*(\omega)\right)d\mu$$

Now let M be a closed, oriented, hyperbolic n-manifold with volume form Ω_M. If $\mu\in\mathscr{C}_n(M)$ is a cycle then μ represents $\langle\mu,\Omega_M\rangle V(M)^{-1}[M]$, where $V(M)$ is the volume of M and $[M]$ the orientation class.

Definition 4.1 For a closed, oriented n-manifold M one defines Gromov's norm to be

$$||M|| = \inf\{||\mu|| \mid \mu \text{ a cycle representing } [M]\}\ .$$

Theorem 4.2 (Gromov) For any closed, oriented, hyperbolic n-manifold M one has

$$||M|| = V(M)/V_n$$

(V_n = maximal volume of a geodesic n-simplex in H^n) .

Proof We include a proof because it is very nice and because it is used in the next section.

If $\sigma\in C^1(\Delta(k),H^n)$ we have another simplex $s(\sigma)\in C^1(\Delta(k),H^n)$ which is affine and has the same vertices as σ. Obviously this defines a continuous map

$$s:C^1(\Delta(k),H^n)\to C^1(\Delta(k),H^n)$$

which is homotopic to the identity. Represent M as $\Gamma\backslash H^n$ with universal covering projection $p:H^n\to M$. One easily checks that there is a unique map $\bar{s}:C^1(\Delta(k),M)\to C^1(\Delta(k),M)$, homotopic to the identity, which has $p_*s=\bar{s}p_*$. The induced chain map

$$S_M = \overline{s}_* : \mathscr{C}_*(M) \to \mathscr{C}_*(M)$$

is chain homotopic to the identity, and of course

$$p_* S_{H^n} = S_M p_* : \mathscr{C}_*(H^n) \to \mathscr{C}_*(M) \ .$$

Now let μ be a cycle representing $[M]$. Then so does $S_M(\mu)$ so, if $\widetilde{\sigma} \in C^1(\Delta(n), H^n)$ lifts σ,

$$\begin{aligned}
V(M) &= \langle S_M(\mu), \Omega_M \rangle \\
&= \int_{\tau \in C^1(\Delta(n),M)} \left(\int_{\Delta(n)} \tau^*(\Omega_M) \right) d(\overline{s}_* \mu) \\
&= \int_{\sigma \in C^1(\Delta(n),M)} \left(\int_{\Delta(n)} \overline{s}(\sigma)^*(\Omega_M) \right) d\mu \\
&= \int_{\sigma \in C^1(\Delta(n),M)} \left(\int_{\Delta(n)} s(\widetilde{\sigma})^* p^*(\Omega_M) \right) d\mu \\
&= \int_{\sigma \in C^1(\Delta(n),M)} \left(\int_{\Delta(n)} s(\widetilde{\sigma})^*(\Omega_{H^n}) \right) d\mu \ .
\end{aligned}$$

Since $s(\widetilde{\sigma})$ is affine one has

$$\begin{aligned}
\left| \int_{\Delta(n)} s(\widetilde{\sigma})^*(\Omega_{H^n}) \right| &= \int_{s(\widetilde{\sigma})(\Delta(n))} \Omega_{H^n} \\
&= V(s(\widetilde{\sigma})(\Delta(n))) \\
&\le V_n \ .
\end{aligned}$$

Hence

$$V(M) \le \int V_n \, d|\mu| = V_n \|\mu\|$$

and we have proved that

$$\|M\| \ge V(M)/V_n \ .$$

To prove the opposite inequality we need an explicit construction of a cycle representing [M] and of total variation close to $V(M)/V_n$. It proceeds as follows. We have a map of principal K bundles, where K is a maximal compact subgroup of the orientation preserving isometry group $I_+(H^n)$

$$\begin{array}{ccc} K & = & K \\ \downarrow & & \downarrow \\ I_+(H^n) & \longrightarrow & \Gamma\backslash I_+(H^n)=D(M) \\ \downarrow & & \downarrow \\ H^n & \xrightarrow[p]{} & \Gamma\backslash H^n=M \end{array}$$

and the horizontal maps are principal Γ bundles. As a topological space $I_+(H^n)=K\times H^n$ and the Haar measure h_o on $I_+(H^n)$ is the product of the one on K and the volume form Ω_{H^n} . Since h_o is left invariant and $I_+(H^n)\to D(M)$ is a locally trivial Γ-bundle, there is a unique measure h_M on D(M) such that $I_+(H^n)\to D(M)$ is locally measure preserving. Since, locally, h_M is the product of the Haar measure on K and the volume form Ω_M one has

(4.1) $\qquad h_M(D(M))=V(M)$.

One now defines a function

$$\alpha: C^1(\Delta(k),H^n)\to \mathscr{C}_k(M)$$

as follows. Given $\sigma:\Delta(k)\to H^n$ there is a continuous map

$$\varphi_\sigma: D(M)\to C^1(\Delta(k),M)$$

given by

$$\varphi_\sigma(\Gamma g)=pg\sigma \ , \quad g\in I_+(H^n) \ .$$

We let

$$\alpha(\sigma)=\varphi_{\sigma *}(h_M)\in\mathscr{B}_k(M) \quad .$$

It is then easy to check the following properties.

<u>Lemma 4.2</u>

(i) $\alpha(\sigma)=\alpha(g\sigma)$, all $g\in I_+(H^n)$

(ii) $\alpha\left(\sigma^{(i)}\right)=\partial_i\alpha(\sigma)$, $\sigma^{(i)}=i^{\text{th}}$ face of σ

(iii) $\|\alpha(\sigma)\|=V(M)$ if $\sigma\in C^1(\Delta(n),H^n)$

(iv) If $\sigma\in C^1(\Delta(n),H^n)$ then $\langle\alpha(\sigma),\Omega_M\rangle=V(\sigma)V(M)$ where $V(\sigma)=\int_{\Delta(n)}\sigma^*(\Omega_{H^n})$.

In fact (ii) is purely formal, (iii) a restatement of (4.1), (i) a consequence of the right invariance of h_M under $I_+(H^n)$, and (iv) is seen by the following computation

$$\begin{aligned}
&\langle\alpha(\sigma),\Omega_M\rangle = \\
&=\int_{\tau\in C^1(\Delta(n),M)}\left(\int_{\Delta(n)}\tau^*(\Omega_M)\right)d(\varphi_{\sigma *}(h_M)) \\
&=\int_{\Gamma g\in D(M)}\left(\int_{\Delta(n)}\varphi_\sigma(\Gamma g)^*(\Omega_M)\right)dh_M \\
&=\int_{\Gamma g\in D(M)}\left(\int_{\Delta(n)}\sigma^* g^* p^*(\Omega_M)\right)dh_M \\
&=\int_{\Gamma g\in D(M)}\left(\int_{\Delta(n)}\sigma^*(\Omega_{H^n})\right)dh_M \\
&=\left(\int_{\Delta(n)}\sigma^*(\Omega_{H^n})\right)V(M) \quad .
\end{aligned}$$

Note that when $\sigma:\Delta(n)\to H^n$ is affine with image set $\bar{\sigma}\subseteq H^n$ then $V(\sigma)=\pm V(\bar{\sigma})$ where the sign depends on the orientation character of σ .

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It is easily seen

For any affine $\sigma \in C^1(\Delta(n),M)$ let $\zeta(\sigma)=\alpha(\sigma)-\alpha(\sigma_-)\in\mathcal{C}_n(M)$ where $\sigma_- = \sigma$ followed by a reflection in one of $\bar{\sigma}$'s faces. Properties (i) and (ii) above immediately imply that $\zeta(\sigma)$ is a cycle ($\sigma^{(i)}$ and $\sigma_-^{(i)}$ are congruent modulo $I_+(H^n)$ even though σ and σ_- are not). Also $\|\zeta(\sigma)\| = \|\alpha(\sigma)\| + \|\alpha(\sigma_-)\| = 2V(M)$ by (iii) (and because $\alpha(\sigma)$, $\alpha(\sigma_-)$ are disjointly supported). And, because of (iv), $\zeta(\sigma)$ represents $2V(\sigma)[M]$. It follows that $\|M\| \leq V(M)/V(\bar{\sigma})$, and since $V(\bar{\sigma})$ can be chosen arbitrarily close to V_n this implies that $\|M\| \leq V(M)/V_n$.

Q.E.D.

§5 Proof of step 3

Assume that $v_o,v_1,\ldots,v_n \in S_\infty^{n-1}$ span a geodesic simplex of maximal volume, but $\tilde{f}_+(v_o),\ldots,\tilde{f}_+(v_n)$ do not. We may find neighbourhoods U_i of v_i , in H^n , and an $\varepsilon>0$ so that

(5.1) If $v_i \in U_i$ and σ is the geodesic simplex spanned by $v_0,\ldots,v_n$ then $V(s\tilde{f}_+(\sigma)) \leq V_n-\varepsilon$.

Here s is the "straightening" map introduced in section 4. Note that (5.1) deals only with geodesic simplices in H^n, no ideal vertices are involved any more.

For smaller neighbourhoods $V_i (\subseteq U_i \subseteq H^n)$ of v_i consider the condition

(5.2) $\Downarrow v_i \in V_i$, $i=0,1,\ldots,n$
$g(v_i) \in U_i$.

It is easily seen that V_i may be chosen so that

$$D_1(M)=\{\Gamma g\in D(M)\mid g \text{ satisfies } (5.2)\}$$

has measure

(5.3) $\quad h_M(D_1(M))=h_1>0$.

Now choose a positively oriented affine simplex $\sigma_o\in C^1(\Delta(n),M)$ with vertices in the neighbourhoods V_i and with

(5.4) $\quad V(\bar{\sigma}_o)>V_n-\delta$.

By (5.1) and the definition of $D_1(M)$ one has

(5.5) $\quad$ If $\Gamma g\in D_1(M)$ then
$V(s\tilde{f}_+(g\sigma_o))\leq V_n-\varepsilon\leq V(\sigma_o)+\delta-\varepsilon$.

Also

(5.6) $\quad$ If $\Gamma g\notin D_1(M)$ then
$V(s\tilde{f}_+(g\sigma_o))\leq V_n\leq V(\sigma_o)+\delta$.

We go on to compute which multiple of $[N]$ is represented by $S_N f_*\zeta(\sigma_o)$. We have

$$\begin{aligned}
&<S_N f_*\alpha(\sigma_o),\Omega_N>=\\
&=\int_{\tau\in C^1(\Delta(n),N)}\left(\int_{\Delta(n)}\tau^*(\Omega_N)\right)d(\bar{s}_*f_*\varphi_{\sigma_o}{}_*(h_M))\\
&=\int_{\rho\in C^1(\Delta(n),M)}\left(\int_{\Delta(n)}(\bar{s}(f\rho))^*(\Omega_N)\right)d(\varphi_{\sigma_o}{}_*(h_M))\\
&=\int_{\Gamma g\in D(M)}\left(\int_{\Delta(n)}(\bar{s}(fpg\sigma_o))^*(\Omega_N)\right)dh_M\\
&=\int_{\Gamma g\in D(M)}\left(\int_{\Delta(n)}(\bar{s}(p\tilde{f}g\sigma_o))^*(\Omega_N)\right)dh_M\\
&=\int_{\Gamma g\in D(M)}\left(\int_{\Delta(n)}(s(\tilde{f}g\sigma_o))^*p^*(\Omega_N)\right)dh_M\\
&=\int_{\Gamma g\in D(M)}\left(\int_{\Delta(n)}(s(\tilde{f}g\sigma_o))^*(\Omega_{H^n})\right)dh_M
\end{aligned}$$

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Now choose $\delta<\varepsilon h$

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$$=\int_{\Gamma g\in D(M)} V(s(\tilde{f}g\sigma_o))\,dh_M$$

$$\leq h_1(V(\sigma_o)+\delta-\varepsilon)+(V(M)-h_1)(V(\sigma_o)+\delta)$$

$$=(V(\sigma_o)+\delta)V(M)-\varepsilon h_1 .$$

Now choose $\delta<\varepsilon h_1/V(M)$. Then one gets

$$<S_N f_*\alpha(\sigma_o),\Omega_N><V(\sigma_o)V(M) .$$

Similarly

$$-<S_N f_*\alpha(\sigma_{o-}),\Omega_N><-V(\sigma_{o-})V(M)=V(\sigma_o)V(M) .$$

Also $V(M)=V(N)$, because $f_*([M])=[N]$ and $\|f_*[M]\|\leq\|M\|$. Hence it follows that $S_N f_*(\zeta(\sigma_o))$ represents a multiple $A[N]$ with $A<2V(\sigma_o)$ and this contradicts the fact that $\zeta(\sigma_o)$ represents $2V(\sigma_o)[M]$.

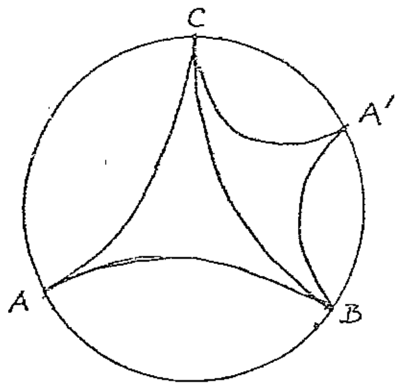
§6 Proof of step 4

It is in this part that theorem 1 enters the picture. It permits one to translate the result of step 3 into

(6.1) If $v_o,v_1,\ldots,v_n\in S_\infty^{n-1}$ span an ideal, regular, geodesic n-simplex in H^n then so do $\tilde{f}_+(v_o),\ldots,\tilde{f}_+(v_n)$.

The rest of the argument is conveniently illustrated in the Poincaré (unit disc) model.

We may compose $\tilde{f}_+$ with an isometry h to obtain that $h_+\tilde{f}_+$ fixes all vertices of some regular, ideal n-simplex, say ABC.



But then it must also fix the reflection of each vertex in the opposite face, such as A' (because, for n>2, there are only two ideal regular n-simplices containing the given face, and $\tilde{f}_+$ is injective). Repeating this procedure ad infinitum we see that $h_+\tilde{f}_+$ fixes a dense subset of S_∞^{n-1}. By continuity $h_+\tilde{f}_+=id$, i.e. the original $\tilde{f}_+=h_+^{-1}$.

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