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PROPER HEREDITARY SHAPE EQUIVALENCES PRESERVE PROPERTY *C*

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In a recent paper [6], van Mill and Mogilski prove that a proper hereditary shape equivalence preserves property *C*, if its domain is σ-compact. In this note, the same result is established without the hypothesis of σ-compactness.

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cell-like map
hereditary shape equivalence
approximately invertible map
countable dimensional
property *C*

1. Introduction

Consider only metrizable spaces. A space is zero dimensional if each open cover is refined by an open cover whose elements are pairwise disjoint. A space is finite (countable) dimensional if it is the union of a finite (countable) collection of zero dimensional spaces. There is a property (defined below), called property *C*, which was first formulated in [3] and which captures an essential part of the nature of countable dimensionality. All countable dimensional spaces have property *C*. The ingenious construction in [5] has recently provided an example of a compactum which has property *C* but is not countable dimensional. (The observation that this example has property *C* was made by R. Engelking and E. Pol after the appearance of [5].)

A map $f: X \to Y$ is cell-like if it is proper and surjective and if for each $y \in Y$, $f^{-1}(y)$ is a cell-like set (has the shape of a point). One of the major unsolved problems of topology is whether cell-maps necessarily preserve finite dimensionality. In [4], it was recognized that the class of cell-like maps which do preserve finite dimensionality coincides with the class of proper hereditary shape equivalences. However, it is not known whether proper hereditary shape equivalences preserve countable dimensionality. A related but (because of the example in [5]) not quite equivalent question is: do proper hereditary shape equivalences preserve property

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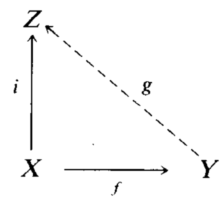
C? This question has recently been answered affirmatively in [6] under the hypothesis that the domain of the map be σ-compact. In this note, it is shown that the answer remains affirmative without the hypothesis of σ-compactness.

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$.

A space has property C if for each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, there is an open cover $\mathcal{V} = \bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ of X such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a pairwise disjoint collection which refines $\mathcal{U}_n$.

A map $f: X \to Y$ is approximately invertible if some closed embedding $i: X \to Z$ into a space Z (hence, every closed embedding $i: X \to Z$ into an ANR Z) has the following property. For every collection $\mathcal{W}$ of open subsets of Z which is refined by $\{i(f^{-1}(y)): y \in Y\}$, there is a map $g: Y \to Z$ such that $g \circ f$ is within $\mathcal{W}$ of i (in other words, $\{\{g \circ f(x), i(x)\}: x \in X\}$ refines $\mathcal{W}$).



Theorem. *Let $f: X \to Y$ be an approximately invertible surjective map such that $f^{-1}(y)$ is compact for each $y \in Y$. If X has property C, then so does Y.*

A proper hereditary shape equivalence is approximately invertible by Theorem 4.5 of [1]. Also a hereditary shape equivalence is necessarily surjective. Thus we have

Corollary. *A proper hereditary shape equivalence preserves property C.*

In [1], it is proved that a cell-like map is a hereditary shape equivalence if its range has property C. Thus we have

Corollary. *Let $f: X \to Y$ be a cell-like map, and suppose X has property C. Then Y has property C if and only if f is a hereditary shape equivalence.*

2. Proof of the theorem

Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open covers of Y. For each $n \in \mathbb{N}$, let $\mathcal{U}_n^*$ be a star refinement of $\mathcal{U}_n$; i.e., for every $U' \in \mathcal{U}_n^*$, there is a $U \in \mathcal{U}_n$ such that $\mathrm{Star}(U', \mathcal{U}_n^*) \subset U$ where

$$\mathrm{Star}(U', \mathcal{U}_n^*) = \bigcup\{U'' \in \mathcal{U}_n^* : U' \cap U'' \neq \emptyset\}.$$

Since X has property C, there is an open cover $\mathcal{V} = \bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ of X such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a pairwise disjoint collection which refines $f^{-1}\mathcal{U}_n^*$. We can assume $\mathcal{V}$ is a locally finite collection; for if it is not, it can be replaced by a "precise" locally finite refinement. (See pages 61 and 62 of [2].)

For each $y \in Y$, since $f^{-1}(y)$ is compact and $\mathcal{V}$ is locally finite, $f^{-1}(y)$ has an open neighborhood W_y in X which intersects only a finite number of elements of $\mathcal{V}$. Thus the set

$$N_y = \{n \in \mathbb{N}: W_y \text{ intersects an element of } \mathcal{V}_n\}$$

is finite. We "cut down" W_y further so that for each $n \in N_y$, there is a $U \in \mathcal{U}_n^*$ such that $W_y \subset f^{-1}(U)$.

Let $i: X \to Z$ be a closed embedding into a space Z which witnesses the approximate invertibility of the map $f: X \to Y$. For convenience, we regard X as a closed subset of Z and $i: X \to Z$ as the inclusion. Let ρ be a metric on Z. For each subset S of X, let

$$\tilde{S} = \{z \in Z: \rho(z, S) < \rho(z, X - S)\}.$$

We shall use the following facts whose easy proofs are left to the reader.

1) $\tilde{\emptyset} = \emptyset$, because $\rho(z, \emptyset) = \infty$ for each $z \in Z$.
2) For each $S \subset X$, $\tilde{S}$ is an open subset of Z.
3) If S is an open subset of X, then $\tilde{S} \cap X = S$.
4) If $S \subset T \subset X$, then $\tilde{S} \subset \tilde{T}$.
5) For all $S, T \subset X$, $\tilde{S} \cap \tilde{T} = \widetilde{S \cap T}$.

Let $R = \bigcup\{\tilde{V}: V \in \mathcal{V}\}$. R is an open neighborhood of X in Z. Let $\mathcal{W} = \{\tilde{W}_y \cap R: y \in Y\}$. Then $\mathcal{W}$ is a collection of open subsets of Z which is refined by $\{i(f^{-1}(y)): y \in Y\}$. Since the closed embedding $i: X \to Z$ witnesses the approximate invertibility of $f: X \to Y$, there is a map $g: Y \to Z$ such that $g \circ f$ is within $\mathcal{W}$ of i. Hence $g(Y) \subset R$.

So $\{g^{-1}(\tilde{V}): V \in \mathcal{V}\} = \bigcup\{\{g^{-1}(\tilde{V}): V \in \mathcal{V}_n\}: n \in \mathbb{N}\}$ is an open cover of Y. For each $n \in \mathbb{N}$, since $\{\tilde{V}: V \in \mathcal{V}_n\}$ is a pairwise disjoint collection, so is $\{g^{-1}(\tilde{V}): B \in \mathcal{V}_n\}$.

Let $n \in \mathbb{N}$. It remains to show that $\{g^{-1}(\tilde{V}): V \in \mathcal{V}_n\}$ refines $\mathcal{U}_n$. Let $V \in \mathcal{V}_n$. Then there is a $U' \in \mathcal{U}_n^*$ such that $V \subset f^{-1}(U')$, and there is a $U \in \mathcal{U}_n$ such that $\text{Star}(U', \mathcal{U}_n^*) \subset U$. We shall prove that $g^{-1}(\tilde{V}) \subset U$. Let $y \in g^{-1}(\tilde{V})$. Let $x \in f^{-1}(y)$. Since $g \circ f$ is within $\mathcal{W}$ of i, there is a $z \in Y$ such that $\{x, g(y)\} \subset \tilde{W}_z$. Hence $g(y) \in \tilde{V} \cap \tilde{W}_z = \widetilde{V \cap W_z}$. So $V \cap W_z \neq \emptyset$. Since $V \in \mathcal{V}_n$, it follows that $n \in N_z$. So there is a $U'' \in \mathcal{U}_n^*$ such that $W_z \subset f^{-1}(U'')$. Since $\emptyset \neq V \cap W_z \subset f^{-1}(U') \cap f^{-1}(U'') = f^{-1}(U' \cap U'')$, then $U' \cap U'' \neq \emptyset$. Since $x \in W_z \subset f^{-1}(U'')$, then $y = f(x) \in U''$. Consequently, $y \in \text{Star}(U', \mathcal{U}_n^*)$. So $y \in U$. This proves $g^{-1}(\tilde{V}) \subset U$.

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