3. Arc Length and Volume

Def For each $x \in \mathbb{H}^n$, the tangent space to $\mathbb{H}^n$ at x is the set $T_x(\mathbb{H}^n) =$

$$\{\gamma'(0) : \gamma : (-\delta, \delta) \to \mathbb{H}^n \text{ is a differentiable map for some } \delta > 0 \text{ and } \gamma(0) = x\}.$$

Lemma 3.1 For each $x \in \mathbb{H}^n$, $T_x(\mathbb{H}^n) = \{v \in \mathbb{M}^{n+1} : x \circ v = 0\}$.

Proof Let $v \in T_x(\mathbb{H}^n)$. Then $v = \gamma'(0)$ where $\gamma : (-\delta, \delta) \to \mathbb{H}^n$ is a differentiable map such that $\gamma(0) = x$. Since $\gamma(t) \circ \gamma(t) = -1$, then

$$0 = \frac{d}{dt}\big(\gamma(t) \circ \gamma(t)\big) = 2\gamma(t) \circ \gamma'(t).$$

Therefore, $0 = \gamma(0) \circ \gamma'(0) = x \circ v$. This proves $T_x(\mathbb{H}^n) \subset \{v \in \mathbb{M}^{n+1} : x \circ v = 0\}$.

Let $v \in \mathbb{M}^{n+1}$ such that $x \circ v = 0$. If $v = 0$, let $\gamma : (-\delta, \delta) \to \{x\}$ be the constant map. Then $\gamma'(0) = 0 = v$. So $v \in T_x(\mathbb{M}^n)$. Now assume $v \neq 0$. Since $x \circ x = -1$, $x \circ v = 0$ and $v \neq 0$, then Corollary 1.3 implies $v \circ v > 0$. Let $u = v/\|v\|$. Then x, u is an orthonormal sequence in $\mathbb{M}^{n+1}$. $\Gamma_{x,u} : \mathbb{R} \to \mathbb{H}^n$ is a differentiable map such that $\Gamma_{x,u}(0) = x$ and $\Gamma_{x,u}'(0) = u$.

Define $\gamma : \mathbb{R} \to \mathbb{H}^n$ by $\gamma(t) = \Gamma_{x,u}(|||v|||t)$. Then γ is differentiable, $\gamma(0) = \Gamma_{xu}(0) = x$ and $\gamma'(0) = |||v||| \, \Gamma_{xu}'(0) = |||v||| \, u = v$. Thus, $v \in T_x(\mathbb{H}^n)$. This proves $\{v \in \mathbb{M}^{n+1} : x \circ v = 0\} \subset T_x(\mathbb{H}^n)$. □

Clearly, $\{v \in \mathbb{M}^{n+1} : x \circ v = 0\}$ is a vector subspace of $\mathbb{M}^{n+1}$. Also if $v \in \mathbb{M}^{n+1}$, $x \circ v = 0$ and $v \neq 0$, then Corollary 1.3 implies $v \circ v > 0$. Hence, the restriction of $\circ$ to $\{v \in \mathbb{M}^{n+1} : x \circ v = 0\}$ is a positive definite symmetric bilinear form (i.e., an inner product). Thus we have:

<u>Corollary 3.2</u>. For $x \in \mathbb{H}^n$, $T_x(\mathbb{H}^n)$ is a vector subspace of $\mathbb{M}^{n+1}$ and $\circ$ restricts to an inner product on $T_x(\mathbb{H}^n)$. □

These properties make $\mathbb{H}^n$ a Riemannian manifold. A <u>Riemannian manifold</u> is a differentiable manifold M in which each tangent space $T_x(M)$ has been equipped with an inner product $\underset{x}{\circ}$ so that $\underset{x}{\circ}$ varies differentiably with x.

10/16 → In a Riemannian manifold M, the length $L(f)$ of a continuously differentiable path $f:[a,b]\to M$ is defined by

$$L(f) = \int_a^b \|f'(t)\|_{f(t)}\, dt$$

where $\| \ \|_x$ is the norm on $T_x(M)$ determined by the inner product $\underset{x}{\circ}$. (Thus, $\|v\|_x = \sqrt{v \underset{x}{\circ} v}$.)
A metric d_M is defined on a Riemannian manifold M by the formula

$$d(p,q) = \inf\{L(f) : f \in \mathcal{P}(p,q)\}$$

where $\mathcal{P}(p,q)$ is the set of all continuously differentiable paths $f:[a,b]\to M$ such that $f(a)=p$ and $f(b)=q$.

For each $x \in H^n$, observe that the norm on $T_x(H^n)$ determined by the inner product $\circ$ is $|||v||| = \sqrt{|v\circ v|} = \sqrt{v\circ v}$ for $v \in T_x(H^n)$ (because $v\circ v \geq 0$ whenever $x \in H^n$ and $v \in T_x(H^n)$). Hence, a ~~path~~ continuously differentiable path $f:[a,b]\to H^n$ has length

$$L(f) = \int_a^b |||f'(t)|||\, dt.$$

We have already define a metric η on $\mathbb{H}^n$ by the formula

$$\eta(p,q) = \cosh^{-1}(-p \circ q).$$

We now show that η coincides with the metric that arises by regarding $\mathbb{H}^n$ as a Riemannian manifold and defining the distance between two points to be the infimum of the lengths of path joining the two points.

<u>Lemma 3.3</u> If $p, q \in \mathbb{H}^n$ and $\Gamma_{uv} : \mathbb{R} \to \mathbb{H}^n$ is a geodesic such that $\Gamma_{uv}(a) = p$ and $\Gamma_{uv}(b) = q$ where $a < b$, then

$$\eta(p,q) = L(\Gamma_{uv} | [a,b]).$$

<u>Proof</u> Since Γ_{uv} is a geodesic, then
$\eta(p,q) = \eta(\Gamma_{uv}(a), \Gamma_{uv}(b)) = |a-b| = b-a.$
Recall that $|||\Gamma_{uv}'(t)||| = 1$. Hence,
$L(\Gamma_{uv} | [a b]) = \int_a^b |||\Gamma_{uv}'(t)||| dt = \int_a^b 1\, dt = b-a.$
Therefore, $\eta(p,q) = L(\Gamma_{uv} | [a b])$. $\square$

Lemma 3.4 If $p, q \in \mathbb{H}^n$ and $f : [a b] \to \mathbb{H}^n$ is a continuously differentiable path such that $f(a) = p$ and $f(b) = q$, then

$$L(f) \geq \eta(p, q).$$

Proof Let $d = \eta(p, q)$. There is a geodesic $g : \mathbb{R} \to \mathbb{H}^n$ such that $g(0) = p = f(a)$ and $g(d) = q = f(b)$.

Since $[a, b]$ is compact, the set $\{\eta(p, f(t)) : t \in [a b]\}$ is bounded. Hence, there is a $c < 0$ such that $\eta(p, f(t)) < -c$ for all $t \in [a b]$. Since $\eta(p, g(c)) = \eta(g(0), g(c)) = -c$, then $g(c) \notin f([a, b])$.

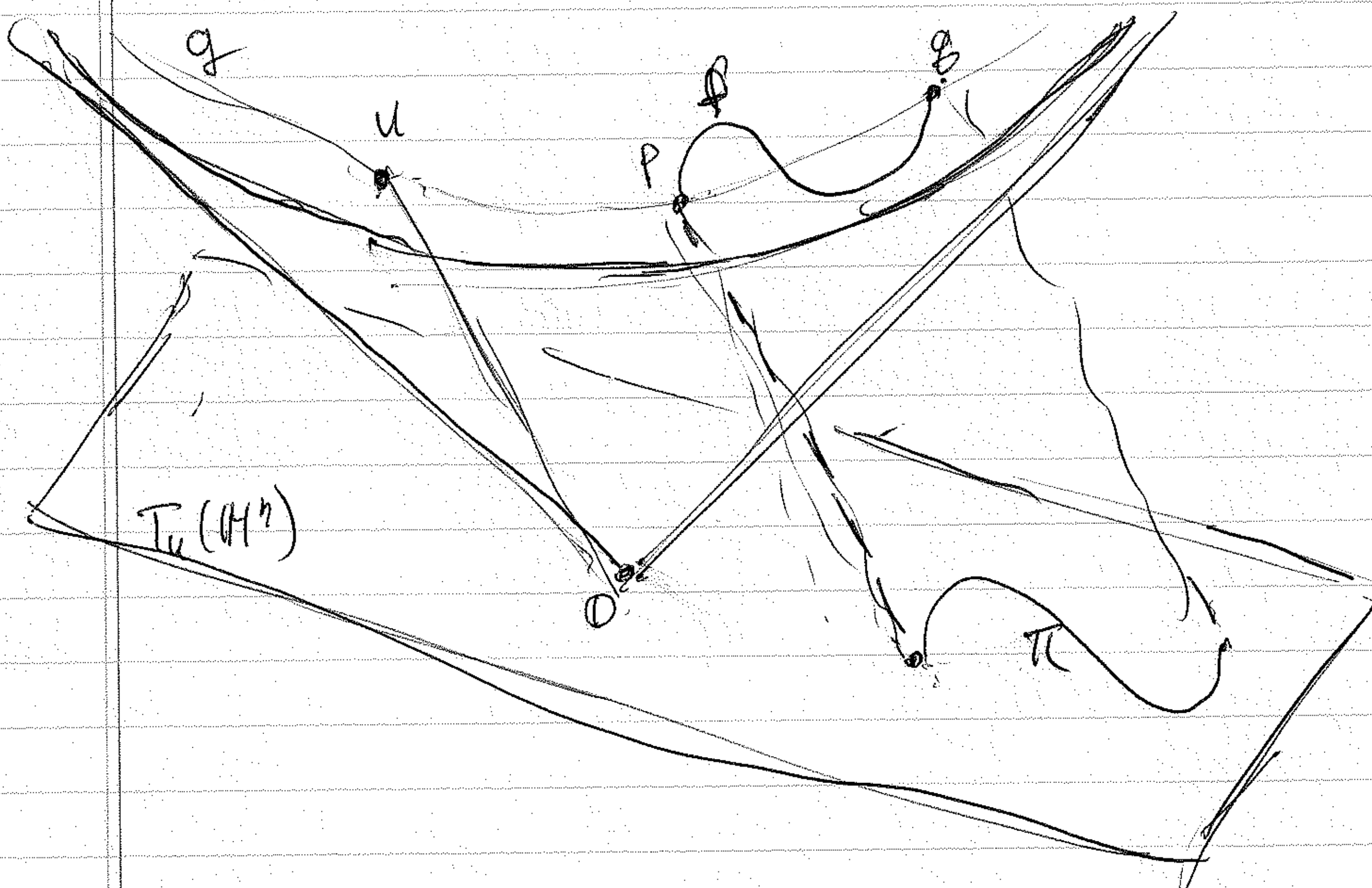
Let $u = g(c)$. Then $u \in \mathbb{H}^n$. Define $\pi : [a, b] \to \mathbb{R}^{n+1}$ by

$$\pi(t) = f(t) + (f(t) \circ u) u$$

Then $\pi(t) \circ u \equiv 0$. Hence, $\pi : [a b] \to T_u(\mathbb{H}^n)$ by Lemma 3.1. (π is the projection of f into $T_u(\mathbb{H}^n)$ parallel to u.)

Observe that $\pi(t) \circ \pi(t) = (f(t) \circ u)^2 - 1$. For $t \in [a, b]$, since $f(t) \neq u$, then $f(t) \circ u < -1$ by the Cauchy Inequality for T^n

(Theorem 1.4). Hence, $\pi(t) \circ \pi(t) > 0$.



Hence, we can define $v : [a,b] \to T_u(\mathbb{H}^n)$ by $v(t) = \pi(t) / |||\pi(t)|||$. Then $v(t) \circ v(t) = 1$ and $v(t) \circ u = 0$.

Now $f(t) = \pi(t) - (f(t) \circ u)\, u =$
$|||\pi(t)|||\, v(t) + (-f(t) \circ u)\, u = \sqrt{(f(t) \circ u)^2 - 1}\; v(t) + (-f(t) \circ u)\, u$.

Since $-f(t) \circ u \geq 1$, we can define $\varphi : [a,b] \to (0, \infty)$ by
$$\varphi(t) = \cosh^{-1}(-f(t) \circ u) = \eta(f(t), u).$$

Then $\cosh\varphi(t) = -f(t)\circ u$. Hence
$\sqrt{(f(t)\circ u)^2 - 1} = \sqrt{\cosh^2(\varphi(t)) - 1} =$
$\sqrt{\sinh^2(\varphi(t))} = \sinh(\varphi(t))$,
because $\varphi(t) > 0$ implies $\sinh\varphi(t) > 0$.
Thus, $f(t) = \sinh(\varphi(t))\, v(t) + \cosh(\varphi(t))\, u$.

<u>Remark</u> If $v(t)$ is a constant v,
then $f(t) = \Gamma_{uv}(\varphi(t))$. Thus
$|||f'(t)||| = |||\Gamma'_{uv}(\varphi(t))\,\varphi'(t)||| = |\varphi'(t)|\cdot 1 = |\varphi'(t)|$
So $L(f) = L(\varphi)$. Since $\varphi: [ab] \to (0\,\infty)$
is a "scalar function", then $L(\varphi) \geq \varphi(b) - \varphi(a)$.
We will see below that $\varphi(b) - \varphi(a) \geq \eta(p,q)$.
This would finish the argument if $v(t)$
were constant. If $v(t)$ is not constant,
then $f'(t)$ has a component parallel to $v'(t)$.
Fortunately we can discard this component.

Now
$$f'(t) = \cosh(\varphi(t))(\varphi'(t))v(t) + \sinh(\varphi(t))v'(t) + \sinh(\varphi(t))\varphi'(t)u$$
Observe that since $v(t)\circ v(t) = 1$, then $v(t)\circ v'(t) = 0$.
Also since $v(t)\circ u = 0$, then $v'(t)\circ u = 0$.
Hence, $v'(t) \in T_u(\mathcal{H}^n)$. Consequently, $v'(t)\circ v'(t) \geq 0$.
Thus,

$f'(t) \circ f'(t) =$

$\cosh^2(\varphi(t))(\varphi'(t))^2(1) + \sinh^2(\varphi(t))|||v'(t)|||^2 + \sinh^2(\varphi(t))(\varphi'(t))^2(-1)$

$= (\cosh^2(\varphi(t)) - \sinh^2(\varphi(t)))(\varphi'(t))^2 + \sinh^2(\varphi(t))|||v'(t)|||^2 =$

$(\varphi'(t))^2 + \sinh^2(\varphi(t))|||v'(t)|||^2 \geq (\varphi'(t))^2.$

Since $f(t) \in \mathbb{H}^n$, then $f'(t) \in T_{f(t)}(\mathbb{H}^n)$.

Hence $f'(t) \circ f'(t) \geq 0$. Thus $|||f'(t)|||^2 = f'(t) \circ f'(t)$.

So $|||f'(t)|||^2 \geq (\varphi'(t))^2$. Therefore, $|||f'(t)||| \geq \varphi'(t)$.

Consequently, $L(f) = \int_a^b |||f'(t)|||dt \geq \int_a^b \varphi'(t)dt =$

$\varphi(b) - \varphi(a) = \eta(f(b), u) - \eta(f(a), u) =$

$\eta(g(d), g(c)) - \eta(g(0), g(c)) = |d-c| - |0-c| =$

$(d-c) - (c) = d = \eta(p,q)$. $\square$

Lemmas 3.3 and 3.4 together imply:

<u>Corollary 3.5.</u> For all $p, q \in \mathbb{H}^n$,

$\eta(p,q) = \inf\{L(f) : f \in \mathcal{P}(p,q)\}.$

<u>Homework Problem 3.1.</u> Recall that the metric on $S^n \subset \mathbb{E}^{n+1}$ is $\theta(x,y) = \cos^{-1}(x \cdot y)$. The *length* of a continuously differentiable curve $f : [a,b] \to S^n$ is $L(f) = \int_a^b \|f'(t)\| dt$. Prove that ~~for all~~ $p, q \in S^n$, $\theta(p,q) = \inf\{L(f) : f : [a,b] \to S^n$ is a continuously differentiable path such that $f(a) = p$ and $f(b) = q\}$.

<u>Def</u> Let V be an n-dimensional inner product space. For $x_1, \dots, x_k \in V$, let

$$P(x_1, \dots x_k) = \left\{ \sum_{i=1}^{k} a_i x_i : a_i \in [0,1] \text{ for } 1 \le i \le k \right\}.$$

$P(x_1, \dots x_k)$ is the <u>k-dimensional parallelopiped determined by $x_1, \dots, x_k$.</u> (If $x_1 \dots x_k$ are linearly dependent then $P(x_1, \dots x_k)$ is actually of dimension $< k$.) We define the <u>k-dimensional volume</u> $V^k(P(x_1 \dots x_k))$ of $P(x_1 \dots x_k)$ as follows. Let $u_1, \dots, u_n$ be an orthonormal basis for V such that $x_1, \dots, x_k \in \operatorname{span}(u_1, \dots, u_k)$. Let A be the $n \times n$ matrix whose (i,j)th entry is

$$A_{ij} = \begin{cases} x_j \cdot u_i & 1 \le j \le k,\ 1 \le i \le n \\ u_j \cdot u_i & k+1 \le j \le n,\ 1 \le i \le n \end{cases}$$

Thus, the jth column of A is

$$A_j = \begin{pmatrix} x_j \cdot u_1 \\ \vdots \\ x_j u_n \end{pmatrix} = \begin{pmatrix} x_j \cdot u_1 \\ \vdots \\ x_j \cdot u_k \\ 0 \\ 1 \\ 0 \end{pmatrix};$$

the entries of A_j are the coordinates of x_j with respect to $u_1, \dots, u_n$. Then we define

$$V^k(P(x_1 \cdots x_k)) = |\det(A)|$$

The following result shows this definition of $V^k(P(x_1 \cdots x_k))$ is independent of the choice of basis $u_1 \dots u_n$ and it also gives a formula for $V^k(P(x_1 \cdots x_k))$ which doesn't rely on the basis.

<u>Lemma 3.6.</u> Let $x_1, \dots, x_k$ be elements of an n-dimensional inner product space V. Let P be the $k \times k$ matrix whose (i,j)th entry is

$$P_{ij} = x_i \cdot x_j .$$

Then $\det(P) \geq 0$ and $V^k(P(x_1 \cdots x_k)) = \sqrt{\det(P)}$

<u>Proof</u> Recall the $n \times n$ matrix A has entries $A_{ij} = \begin{cases} x_j \cdot u_i & 1 \leq j \leq k, \ 1 \leq i \leq n \\ u_j \cdot u_i & k+1 \leq j \leq n, \ 1 \leq i \leq n \end{cases}$

where $x_j \cdot u_i = 0$ for $k+1 \leq i \leq n$. Let B be the

$k \times k$ matrix whose (i,j)th entry is

$$B_{ij} = x_j \cdot u_i.$$

Then A has the form

$$A = \left(\begin{array}{c|c} B & 0 \\ \hline 0 & I \end{array}\right).$$

Hence, $A^T A = \left(\begin{array}{c|c} B^T & 0 \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} B & 0 \\ \hline 0 & I \end{array}\right) = \left(\begin{array}{c|c} B^T B & 0 \\ \hline 0 & I \end{array}\right)$

Thus, $\det(B^T B) = \det(A^T A) = \det(A^T)\det(A)$
$= (\det(A))^2 = \left(V^k(P(x_1 \cdots x_k))\right)^2$

Now the (i,j)th entry of $B^T B$ is

$$(B^T B)_{ij} = (x_i \cdot u_1 \cdots x_i \cdot u_k)\begin{pmatrix} x_j \cdot u_1 \\ \vdots \\ x_j \cdot u_k \end{pmatrix} = \sum_{l=1}^{k} (x_i \cdot u_l)(x_j \cdot u_l) =$$

$$\sum_{l=1}^{n} (x_i \cdot u_l)(x_j \cdot u_l) = \left(\sum_{l=1}^{n} (x_i \cdot u_l) u_l\right) \cdot \left(\sum_{m=1}^{n} (x_j \cdot u_m) u_m\right)$$

$= x_i \cdot x_j = P_{ij}$. Thus $B^T B = P$.

Hence, $\left(V^k(P(x_1 \cdots x_k))\right)^2 = \det(P)$. Thus

$\det(P) \geq 0$ and $V^k(P(x_1 \cdots x_k)) = \sqrt{\det(P)}$. $\square$

Def Let M^n be a Riemannian n-manifold. Let U be an open subset of $\mathbb{R}^k$. Let $f: U \to M^n$ be a continuously differentiable embedding. Observe that for $x = (x_1, \ldots, x_k) \in U$ and $\delta > 0$ such that $(x_1 - \delta, x_1 + \delta) \times \cdots \times (x_k - \delta, x_k + \delta) \subset U$ and for $1 \le i \le k$, the function

$$t \mapsto f(x + te_i) : (-\delta, \delta) \to M$$

is a continuously differentiable curve whose derivative at $t=0$ is an element of $T_{f(x)}(M)$ which is denoted $\frac{\partial f}{\partial x_i}(x)$. Thus $P\left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_k}(x)\right)$ is a k-dimensional parallelopiped in the inner product space $T_{f(x)}(M)$, and $V^k\left(P\left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_k}(x)\right)\right)$ is defined. Now define the k-dimensional volume for $f(U)$ to be

$$V^k(f(U)) = \int_U V^k\left(P\left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_k}(x)\right)\right) dx.$$

Now Lemma 3.6 yields:

<u>Corollary 3.7</u> If M^n is a Riemannian n-manifold, U is an open subset of $\mathbb{R}^k$ and $f: U \to M^n$ is a continuously differentiable embedding, then

$$V^k(f(U)) = \int_U \sqrt{\det(D(x))}\, dx$$

where $D(x)$ is the $k \times k$ matrix whose (i,j)th entry is $D_{ij}(x) = \frac{\partial f}{\partial x_i}(x) \underset{f(x)}{\cdot} \frac{\partial f}{\partial x_j}(x)$ $\square$

10/23 $\longrightarrow$ <u>Def</u> If $f: M^m \to N^n$ is a continuously differentiable map between Riemannian manifolds, then for each $x \in M^m$, a linear map $df_x : T_x(M^m) \to T_{f(x)}(N^n)$ is defined as follows. For $v \in T_x(M^m)$, choose a differentiable map $\gamma: (-\delta, \delta) \to M^m$ such that $\gamma(0) = x$ and $\gamma'(0) = v$. Then $f \circ \gamma : (-\delta, \delta) \to N^n$ is a differentiable map such that $f\circ\gamma(0) = f(x)$, and $df_x(v) \in T_{f(x)}(N)$ is defined to be $(f\circ\gamma)'(0)$. $f: M^m \to N^n$ is a <u>diffeomorphism</u>

if it is bijective and $f^{-1}: N \to M$ is also continuously differentiable. In this case, the Chain Rule implies that for $x \in M$ and $y = f(x) \in N$:

$$id_{T_x(M)} = d(id_M)_x = d(f^{-1} \circ f)_x = d(f^{-1})_{f(x)} \circ df_x = df_y^{-1} \circ df_x$$

and

$$id_{T_y(N)} = d(id_N)_y = d(f \circ f^{-1})_y = df_{g(y)} \circ df_y^{-1} = df_x \circ df_y^{-1} .$$

Thus $df_x : T_x(M) \to T_y(N)$ is a linear isomorphism.

(The Inverse Function Theorem is a local converse to this observation. It says that if $df_x : T_x M \to T_{f(x)} N$ is an isomorphism for some $x \in M$, then there is a neighborhood U of x in M and a nbhd V of $f(x)$ in N such that $f|U : U \to V$ is a diffeomorphism.)

$f: M \to N$ is a <u>local isometry</u> if for every $x \in M$, $df_x : T_x M \to T_{f(x)} N$ is an inner product preserving linear isomorphism. (Thus, $df_x(v) \underset{f(x)}{\circ} df_x(w) = v \underset{x}{\circ} w$ for each $x \in M$ and all $v, w \in T_x(M)$.)

$f: M \to N$ is an <u>isometry</u> if it is a diffeomorphism and a local isometry.

<u>Remark</u>. If $g \in O^+(M^{n+1})$, then $g|H^n : H^n \to H^n$ is an isometry in the sense just defined (as

well as a distance preserving bijection) because for each $x \in \mathbb{H}^n$, $dg_x : T_x(\mathbb{H}^n) \to T_{g(x)}(\mathbb{H}^n)$ is simply $g \mid T_x(\mathbb{H}^n)$ and g preserves the inner product $\circ$ restricted to $T_x(\mathbb{H}^n)$.

<u>Corollary 3.8</u>. If $g: M^n \to N^n$ is an isometry between Riemannian manifolds, U is an open subset of $\mathbb{R}^k$ and $f: U \to M^n$ is a continuously differentiable embedding, then $V^k(g \circ f(U)) = V^k(f(U))$.

<u>Proof</u> Corollary 2.7 implies $V^k(f(U)) = \int_U \sqrt{\det(D(x))}\, dx$ and $V^k(g \circ f(U)) = \int_U \sqrt{\det(E(x))}\, dx$ where $D(x)$ and $E(x)$ are $k \times k$ matrices whose (i,j)th entries are

$$\frac{\partial f}{\partial x_i}(x) \underset{f(x)}{\circ} \frac{\partial f}{\partial x_j}(x) \quad \text{and} \quad \frac{\partial (g \circ f)}{\partial x_i}(x) \underset{g \circ f(x)}{\circ} \frac{\partial (g \circ f)}{\partial x_j}(x),$$

respectively. The Chain Rule implies $\frac{\partial (g \circ f)}{\partial x_i}(x) = dg_{f(x)}\left(\frac{\partial f}{\partial x_i}(x)\right)$. Thus

$\frac{\partial(g\circ f)}{\partial x_i}(x) \underset{g\circ f(x)}{\cdot} \frac{\partial(g\circ f)}{\partial x_j}(x) = dg_{f(x)}\left(\frac{\partial f}{\partial x_i}(x)\right) \underset{g\circ f(x)}{\cdot} dg_{f(x)}\left(\frac{\partial f}{\partial x_j}(x)\right)$

$= \frac{\partial f}{\partial x_i}(x) \underset{f(x)}{\cdot} \frac{\partial f}{\partial x_j}(x)$ because $dg_{f(x)} : T_{f(x)}(M) \to T_{g\circ f(x)}(N)$

is inner product preserving. Hence $D(x) = E(x)$.

Therefore, $V^k(g\circ f(U)) = V^k(f(U))$. $\square$

Next we make a useful observation about angle measure.

<u>Lemma 3.7</u> If $f: [0,\infty) \to \mathbb{H}^n$ and $g: [0,\infty) \to \mathbb{H}^n$ are geodesics such that $f(0) = g(0) = x$, then for all $r, s \in (0,\infty)$,
$$m(\angle f(r)\, x\, g(s)) = \cos^{-1}(f'(0) \circ g'(0)).$$

<u>Proof</u> Let $y = f(r)$ and $z = g(s)$. Recall that $f(t) = \Gamma_{xv}(t)$, $g(t) = \Gamma_{x,w}(t)$ and $m(\angle yxz) = \cos^{-1}(v \circ w)$ where v and w are determined as follows. Let $p = y + (x \circ y)x$ and $q = z + (x \circ z)x$. Then $p \circ p = (x \circ y)^2 - 1 > 0$ and $q \circ q = (x \circ z)^2 - 1 > 0$. Let $v = p / |||p|||$ and $w = q / |||q|||$. (See pages 2.2 and 2.3, Theorems 2.5, 2.8 and 2.10.) Since $f'(0) = \Gamma_{xv}'(0) = v$ and $g'(0) = \Gamma_{xw}'(0) = w$, then
$$m(\angle yxz) = \cos^{-1}(f'(0) \circ g'(0)). \quad \square$$

<u>Def</u> If $f : [0\infty) \to \mathbb{E}^n$ is a proper map that has a non-empty set of asymptotic directions, let $f(\infty)$ denote the set of asymptotic directions of f. In this situation, Lemma 2.19 implies there is a non-zero $v \in \mathbb{E}^n$ such that $f(\infty) = \{rv : r > 0\}$. In particular, if $\Gamma_{uv} : \mathbb{R} \to \mathbb{H}^n$ is a geodesic, then

$\Gamma_{uv}|[0\infty)(\infty) = \{r(u+v) : r > 0\}$ by Lemma 2.20.

<u>Def</u> If $f : [0\infty) \to \mathbb{H}^n$ and $g : [0\infty) \to \mathbb{H}^n$ are asymptotic geodesics (so that $f(\infty) = g(\infty)$ by Theorem 2.21) and $\vec{v} = f(\infty) = g(\infty)$, and if $r, s \in [0, \infty)$, then define

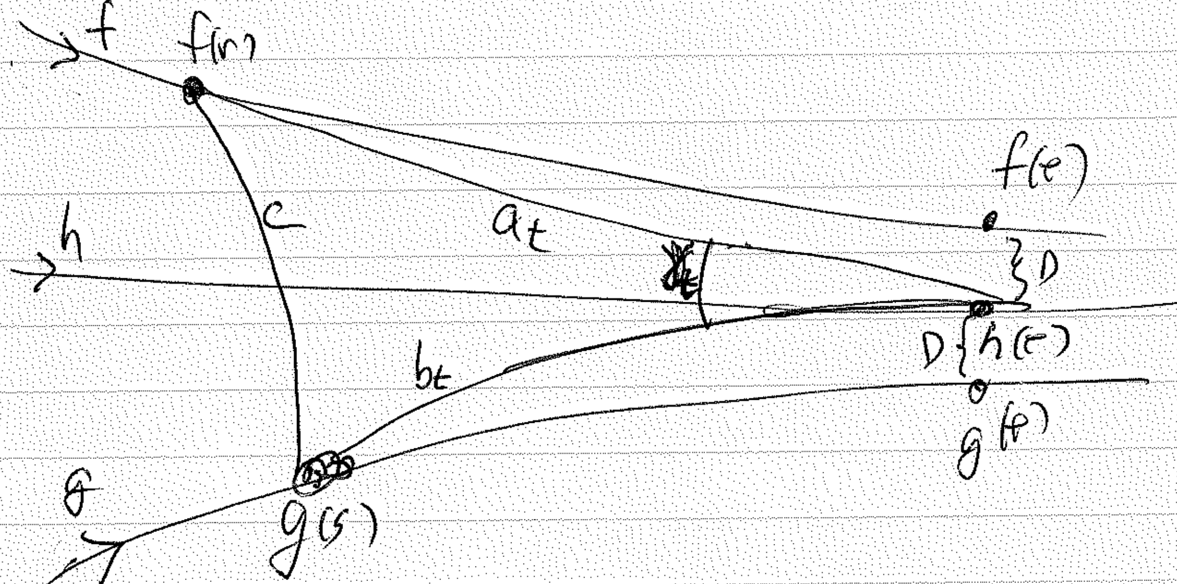
$$m(\angle f(r)\vec{v}g(s)) = 0 .$$

The following lemma serves to motivate this definition.

10/28 → <u>Lemma 3.10</u> If $f : [0\infty) \to \mathbb{H}^n$, $g : [0\infty) \to \mathbb{H}^n$ and $h : [0\infty) \to \mathbb{H}^n$ are asymptotic geodesics and if $r, s \in [0, \infty)$, then

$$\lim_{t \to \infty} m(\angle f(r)h(t)g(s)) = 0 .$$

<u>Proof</u> There is a $D > 0$ such that $\eta(f(t), h(t)) < D$ and $\eta(g(t), h(t)) < D$ for all $t \in [r, \infty)$. Let $a_t = \eta(f(r), h(t))$, $b_t = \eta(g(s), h(t))$, $c = \eta(f(r), g(s))$ and $\gamma_t = m(\angle f(r)\, h(t)\, g(s))$.



The Hyperbolic Law of Cosines implies

$$\cosh(c) = \cosh(a_t)\cosh(b_t) - \sinh(a_t)\sinh(b_t)\cos(\gamma_t).$$

Hence, $$\cos(\gamma_t) = \frac{\cosh(a_t)\cosh(b_t)}{\sinh(a_t)\sinh(b_t)} - \frac{\cosh(c)}{\sinh(a_t)\sinh(b_t)}$$

As $t \to \infty$, $\sinh(t) = \frac{e^t - e^{-t}}{2} \to \infty$

and $$\frac{\cosh(t)}{\sinh(t)} = \frac{e^t + e^{-t}}{e^t - e^{-t}} = \frac{1 + e^{-2t}}{1 - e^{-2t}} \to 1.$$

Observe that $a_t + D \geq \eta(f(r), h(t)) + \eta(h(t), f(t)) \geq \eta(f(r), f(t)) = t - r$. Thus, $a_t \geq t - (r + D)$

Similarly, $b_t \geq t-(s+D)$. Thus, as $t \to \infty$, $a_t \to \infty$ and $b_t \to \infty$. Hence, as $t \to \infty$, $\cos(\gamma_t) \to 1$. Therefore, as $t \to \infty$ $\gamma_t \to 0$. $\square$

<u>Notation</u> If $f: \mathbb{R} \to \mathbb{H}^n$ is a geodesic, let $f_+ = f|[0\infty)$ and define $f_- : [0\infty) \to \mathbb{H}^n$ by $f_-(t) = f(-t)$. Thus, $\Gamma_{u,v-}(t) = \Gamma_{u,-v}(t)$.

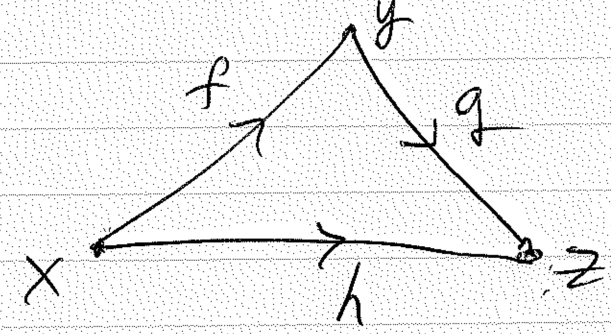
~~Definition~~ Our next goal is to develop a ~~formula for~~ the area of a triangle in $\mathbb{H}^2$. Surprisingly, this goal is made easier if we enlarge the class of triangles to include triangles with ideal vertices where an ideal vertex is a set of asymptotic directions of a geodesic ray. Thus, we will consider triangles in which some or all of the sides are geodesic rays or lines.

<u>Definition</u> For $x, y, z \in \mathbb{H}^n$, the <u>triangle</u> Δxyz is the union

$$f([a_1, a_2]) \cup g([b_1, b_2]) \cup h([c_1, c_2])$$

where $f: [a_1 a_2] \to \mathbb{H}^n$, $g: [b_1 b_2] \to \mathbb{H}^n$ and $h: [c_1 c_2] \to \mathbb{H}^n$ are geodesics such that

$f(a_1)=x$, $f(a_2)=y$, $g(b_1)=y$, $g(b_2)=z$, $h(c_1)=x$, $h(c_2)=z$. Since $x, y, z \in \mathbb{H}^n$,



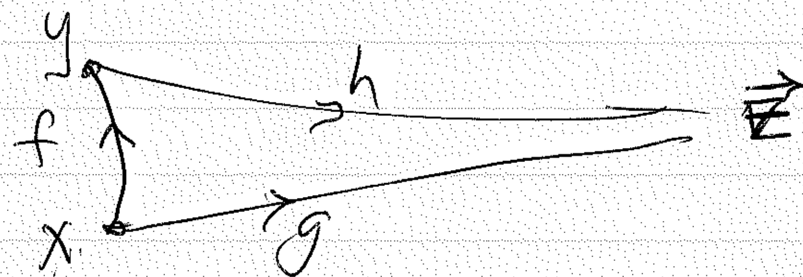
we say that Δxyz has <u>ordinary vertices</u>

Next we describe three types of triangles that have ideal vertices.

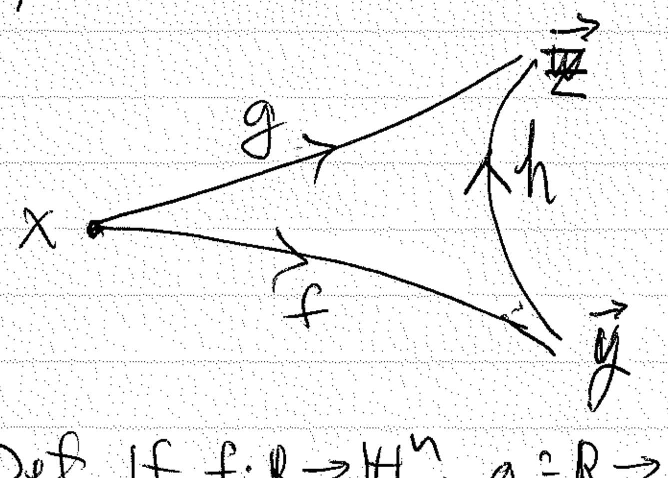
<u>Def</u> If $x, y \in \mathbb{H}^n$, $f:[a,b] \to \mathbb{H}^n$ is a geodesic such that $f(a)=x$ and $f(b)=y$, $g:[0\,\infty) \to \mathbb{H}^n$ and $h:[0\,\infty) \to \mathbb{H}^n$ are geodesics such that $g(0)=x$, $h(0)=y$ and g and h are asymptotic, and $\vec{z}=g(\infty)=h(\infty)$, then $\Delta xy\vec{z}$ is the union

$$f([ab]) \cup g([0\,\infty)) \cup h([0\,\infty))$$

We say that $\Delta xy\vec{z}$ has two ordinary vertices, x and y, and one ideal vertex $\vec{z}$.

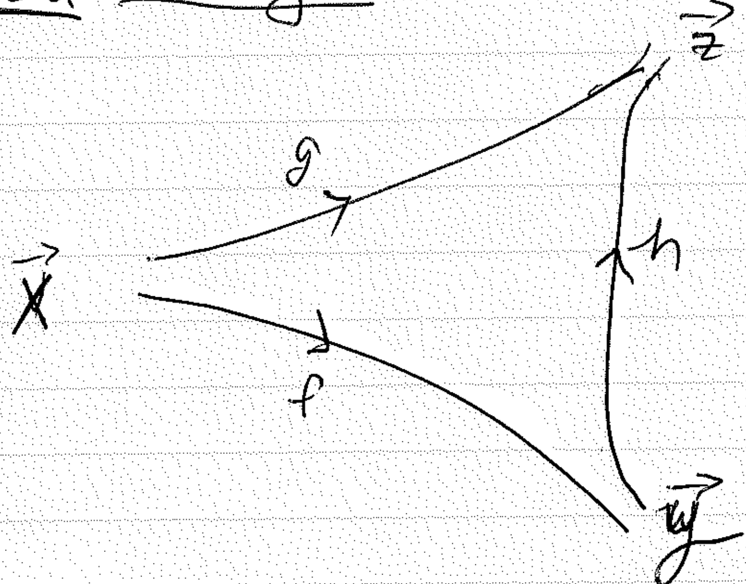


<u>Def</u> If $x \in \mathbb{H}^n$, $f:[0,\infty) \to \mathbb{H}^n$, $g:[0,\infty) \to \mathbb{H}^n$ and $h:\mathbb{R} \to \mathbb{H}^n$ are geodesics such that $f(0)=x=g(0)$, f and h_- are asymptotic and $\vec{y}=f(\infty)=h_-(\infty)$, and g and h_+ are asymptotic and $\vec{z}=g(\infty)=h_+(\infty)$, then $\triangle x\vec{y}\vec{z}$ is the union $f([0,\infty)) \cup g([0,\infty)) \cup h(\mathbb{R})$. We say that $\triangle x\vec{y}\vec{z}$ has one ordinary vertex, x, and two ideal vertices $\vec{y}$ and $\vec{z}$.

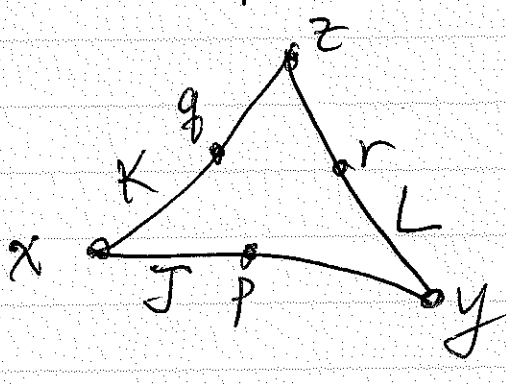


<u>Def</u> If $f:\mathbb{R} \to \mathbb{H}^n$, $g:\mathbb{R} \to \mathbb{H}^n$ and $h:\mathbb{R} \to \mathbb{H}^n$ are geodesics such that f_- and g_- are asymptotic and $\vec{x}=f_-(\infty)=g_-(\infty)$, f_+ and h_- are asymptotic and $\vec{y}=f_+(\infty)=h_-(\infty)$, and g_+ and h_+ are asymptotic and $\vec{z}=g_+(\infty)=h_+(\infty)$, then

$\Delta \vec{x}\vec{y}\vec{z}$ is the union $f(\mathbb{R}) \cup g(\mathbb{R}) \cup h(\mathbb{R})$. We say that $\Delta \vec{x}\vec{y}\vec{z}$ has three ideal vertices, $\vec{x}$, $\vec{y}$ and $\vec{z}$. A triangle with three ideal vertices is called an <u>ideal triangle</u>.

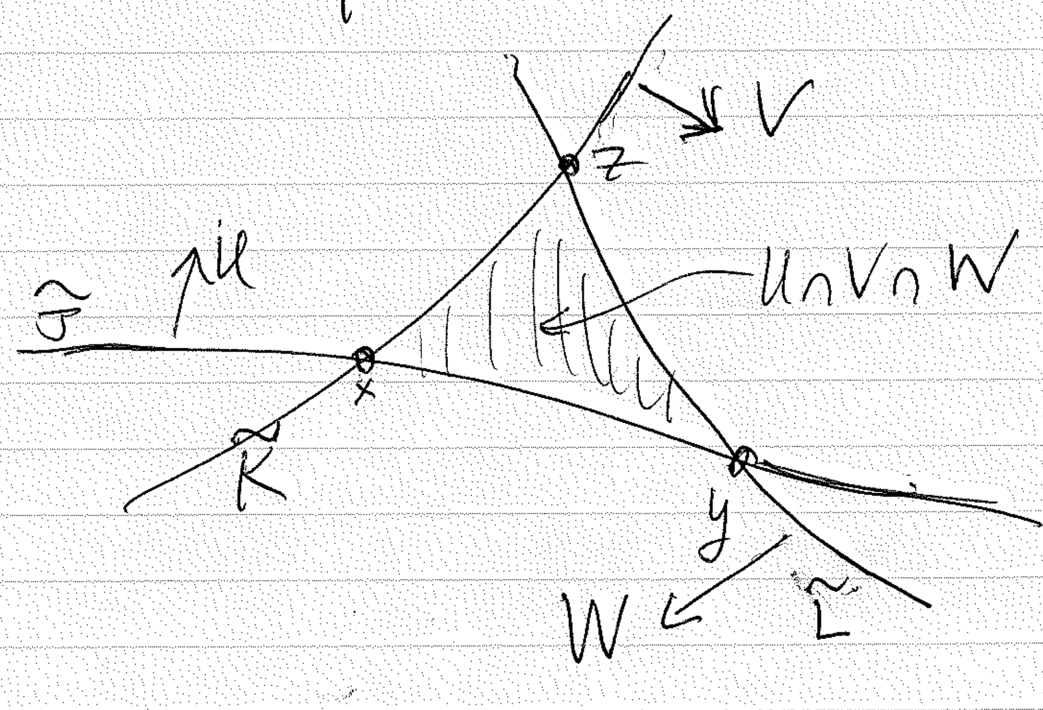


<u>Def</u> Let Δxyz be a triangle in $\mathbb{H}^2$ where x, y and z are either ordinary or ideal vertices. Let J, K and L be geodesic segments or rays or lines so that $\Delta xyz = J \cup K \cup L$, J joins x to y, K joins x to z, and L joins y to z. Let $p \in J - \{x, y\}$, $q \in K - \{x, z\}$ and $r \in L - \{y, z\}$. Define



$m(\angle x) = m(\angle pxq)$, $m(\angle y) = m(\angle pyr)$ and $m(\angle z) = m(\angle qzr)$. Let $\tilde{J}$, $\tilde{K}$ and $\tilde{L}$ be geodesic lines in $\mathbb{H}^2$ such that $J \subset \tilde{J}$, $K \subset \tilde{K}$ and $L \subset \tilde{L}$. Let U, V and W be the components of $\mathbb{H}^2 - J$, $\mathbb{H}^2 - K$ and $\mathbb{H}^2 - L$ such that $z \in U$, $y \in V$ and $x \in W$.

Define Area(Δxyz) to be the 2-dimensional volume of $U \cap V \cap W$. (Another way to define Area(Δxyz) is to declare it to be the 2-dimensional volume of the convex hull of Δxyz.)



Our goal is to prove:

<u>Theorem 3.11.</u> If Δxyz is a triangle in $\mathbb{H}^2$ (where x, y and z are either ordinary or ideal vertices), then

$$\text{Area}(\Delta xyz) = \pi - \big(m(\angle x) + m(\angle y) + m(\angle z)\big).$$

<u>Corollary 3.12.</u> Every ideal triangle in $\mathbb{H}^2$ (all vertices ideal) has area π. □

<u>Corollary 3.13</u> If Δxyz is a triangle in $\mathbb{H}^2$ with ordinary or ideal vertices or both and if Δxyz has positive area, then $m(\angle x) + m(\angle y) + m(\angle z) < \pi$. □

The first and most difficult step of the proof of Theorem 3.11 is to establish the theorem for triangles that have one ordinary vertex and two ideal vertices.

<u>Lemma 3.14.</u> If $\Delta x\vec{y}\vec{z}$ is a triangle in $\mathbb{H}^2$ with ordinary vertex x and ideal vertices $\vec{y}$ and $\vec{z}$, then

$$\text{Area}(\Delta x\vec{y}\vec{z}) = \pi - m(\angle x).$$

<u>Proof</u> $\Delta x\vec{y}\vec{z} = f([0\infty)) \cup g([0\infty)) \cup h(\mathbb{R})$ where $f: [0\infty) \to \mathbb{H}^2$, $g: [0\infty) \to \mathbb{H}^2$ and $h: \mathbb{R} \to \mathbb{H}^2$ are geodesics such that $f(0) = g(0) = x$, $f(\infty) = h_-(\infty) = \vec{y}$ and $g(\infty) = h_+(\infty) = \vec{z}$.

Let $v = f'(0)$ and $w = g'(0) \in T_x(\mathbb{H}^2)$. Then $x \circ v = 0 = x \circ w$.

<u>Claim</u>: $|||v||| = |||w||| = 1$.

<u>Proof</u> Since f is distance preserving, Lemma 3.3 implies

$$t = \eta(f(0), f(t)) = L(f|[0,t]) = \int_0^t |||f'(s)|||ds.$$

Hence, $1 = \frac{d}{dt}\int_0^t |||f'(s)|||ds = |||f'(t)|||$.

Thus $|||v||| = |||f'(0)||| = 1$. Similarly, $|||w||| = 1$. □

Let $\alpha = m(\angle x)$. Then Lemma 3.9 implies $\alpha = \cos^{-1}(f'(0) \circ g'(0)) = \cos^{-1}(v \circ w)$. So $\cos\alpha = v \circ w$.

<u>Claim</u>: The lemma holds if $\alpha = \pi$.

<u>Proof</u> Assume $\alpha = \pi$. Then

$v \circ w = \cos(\pi) = -1 = -\|v\|\,\|w\|$.

Since $\circ$ is positive definite on $T_x(\mathbb{H}^2)$, then the Cauchy Inequality implies $v = -w$.
Thus, $f = \Gamma_{x,-w}|[0,\infty)$ and $g = \Gamma_{x,w}|[0\infty)$.
Also $h = \Gamma_{pu}$ for some $p \in \mathbb{H}^2$ and some orthonormal sequence p, u in $\mathbb{M}^3$.
Furthermore, $f = \Gamma_{x,-w}|[0\infty)$ and $h_- = \Gamma_{p,-u}|[0\infty)$ are asymptotic, and $g = \Gamma_{x,w}|[0\infty)$ and $h_+ = \Gamma_{p,u}|[0\infty)$ are asymptotic. It follows by Homework Problem 2.10a that there is an $a \in \mathbb{R}$ such that $\Gamma_{pu}(t) = \Gamma_{xw}(a+t)$, for $t \in \mathbb{R}$. Hence, if $a \geq 0$, then
$h([0\infty)) = g([0\infty))$ and $h((-\infty,0]) = f([0\infty)) \cup g([0a])$.
On the other hand, if $a < 0$, then
$h([0\infty)) = f([0 -a]) \cup g([0\infty))$ and $h((-\infty,0]) = f([-a,\infty))$
In either case $h(\mathbb{R}) = f([0\infty)) \cup g([0\infty))$.
So $\text{Area}(\Delta \vec{x}\vec{y}\vec{z}) = 0 = \pi - \alpha = \pi - m(\angle x)$. $\square$

<u>Claim</u>: $\alpha \neq 0$.

<u>Proof</u> Assume $\alpha = 0$. Then

$$v \circ w = \cos(0) = 1 = |||v|||\ |||w|||.$$

Hence, the Cauchy Inequality implies $v = w$. Therefore $f = \Gamma_{xv}|[0\infty) = \Gamma_{xw}|[0\infty) = g$. Thus $\vec{y} = f(\infty) = g(\infty) = \vec{z}$. Then $h_-(\infty) = \vec{y} = \vec{z} = h_+(\infty)$. Also $h = \Gamma_{p,u}$ where $p \in \mathbb{H}^2$ and p, u is an orthonormal sequence in $\mathbb{M}^3$. Then $p - u \in h_-(\infty)$ and $p + u \in h_+(\infty)$. Since $h_-(\infty) = h_+(\infty)$, then $p - u = r(p+u)$ for some $r > 0$. Therefore

$$-1 = (p-u) \circ u = r(p+u) \circ u = r$$

(because $p \circ u = 0$ and $u \circ u = 1$). We've reached a contradiction. So $\alpha > 0$.

Now we can assume $0 < \alpha < \pi$. Next we prove there is a $v^\perp \in T_x(\mathbb{H}^2)$ such that $x, v, v^\perp$ is an orthonormal basis for $\mathbb{M}^3$ and $w = (\cos\alpha)v + (\sin\alpha)v^\perp$. Since $0 < \alpha < \pi$, then $-1 < v \circ w = \cos\alpha < 1$.

Let $q = w - (v \circ w)v$. Then $x \circ q = 0 = v \circ q$ and $|||q|||^2 = 1 - (v \circ w)^2 > 0$. Let $v^\perp = q / |||q|||$. Then $x \circ v^\perp = v \circ v^\perp = 0$ and $v^\perp \circ v^\perp = 1$. So $x, v, v^\perp$ is an orthonormal basis for $\mathbb{M}^3$. Since

$$w \circ v^\perp = (w \circ q)/|||q||| = (1 - (v \circ w)^2)/|||q||| = |||q|||,$$

then

$$w = (w \circ v)v + q = (w \circ v)v + |||q|||\, v^\perp = (w \circ v)v + (w \circ v^\perp)v^\perp$$

Therefore, $1 = w \circ w = (w \circ v)^2 + (w \circ v^\perp)^2$.

So $(w \circ v^\perp)^2 = 1 - (w \circ v)^2 = 1 - \cos^2\alpha = \operatorname{sen}^2\alpha$

Since $w \circ v^\perp = |||q||| > 0$ and $\sin\alpha > 0$ (because $0 < \alpha < \pi$), then $w \circ v^\perp = \operatorname{sen}\alpha$. Hence, $w = (\cos\alpha)v + (\operatorname{sen}\alpha)v^\perp$.

Let $u = (\cos(\alpha/2), -\sin(\alpha/2), 0)$ and $u^\perp = (\sin(\frac{\alpha}{2}), \cos(\frac{\alpha}{2}), 0)$. Then $e_3, u, u^\perp$ is an orthonormal basis for $\mathbb{M}^3$. Since x and $e_3 \in \mathbb{H}^2$, then there is a $\phi \in O^+(\mathbb{M}^3)$ such that $\phi(x) = e_3$, $\phi(v) = u$ and $\phi(v^\perp) = u^\perp$.

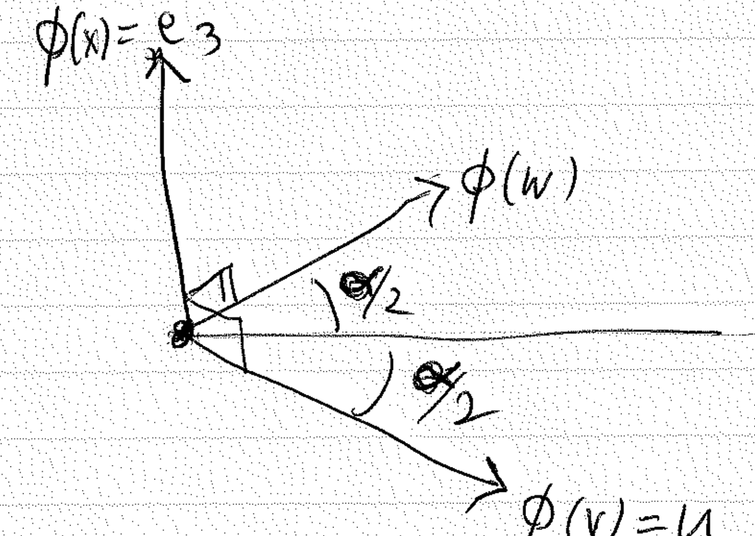
Observe that

$\phi(w) = (\cos\alpha)\phi(v) + (\sin\alpha)\phi(v^\perp) =$

$(\cos\alpha)u + (\sin\alpha)u^\perp =$

$(\cos\alpha\cos(\alpha/2) + \sin\alpha\sin(\alpha/2), \sin\alpha\cos(\alpha/2) - \cos\alpha\sin(\alpha/2), 0)$

$= (\cos(\alpha/2), \sin(\alpha/2), 0)$.



Since $\phi|\mathbb{H}^2 : \mathbb{H}^2 \to \mathbb{H}^2$ is an isometry, then $\phi\circ f$, $\phi\circ g$ and $\phi\circ h$ are geodesics such that $\phi\circ f(0) = \phi\circ g(0) = e_3$ and $\phi\circ f$ and $\phi\circ h_-$ are asymptotic and $\phi\circ g$ and $\phi\circ h_+$ are asymptotic

Since ϕ is linear, then

$(\phi\circ f)'(0) = \phi(f'(0)) = \phi(v) = u = (\cos\alpha/2, -\sin(\alpha/2), 0)$

and $(\phi\circ g)'(0) = \phi(g'(0)) = \phi(w) = (\cos(\alpha/2), \sin(\alpha/2), 0)$.

Thus, $(\phi\circ f)'(0) \circ (\phi\circ g)'(0) = \cos^2(\alpha/2) - \sin^2(\alpha/2) = \cos\alpha$.

Observe that if $y \in f(\infty)$, then there is an $a:(0\,\infty) \to (0\,\infty)$ such that $f(t)/a(t) \to y$ as $t \to \infty$. Since ϕ is continuous and linear, then $(\phi\circ f)(t)/a(t) = \phi(f(t)/a(t)) \to \phi(y)$ as $t \to \infty$. Thus, $\phi(y) \in \phi\circ f(\infty)$. Since ϕ maps positive scalar multiples of y to positive scalar multiples of $\phi(y)$, it follows that $\phi(f(\infty)) = (\phi\circ f)(\infty)$. Similarly:

$\phi(g(\infty)) = (\phi\circ g)(\infty)$, $\phi(h_+(\infty)) = (\phi\circ h_+)(\infty)$ and

$\phi(h_-(\infty)) = (\phi\circ h_-)(\infty)$. Thus,

$\phi\circ f(\infty) = \phi\circ h_-(\infty) = \phi(\vec{y})$ and

$\phi\circ g(\infty) = \phi\circ h_+(\infty) = \phi(\vec{z})$.

Thus, we have a triangle

$\Delta e_3\,\phi(\vec{y})\,\phi(\vec{z}) = \phi\circ f([0\,\infty)) \cup \phi\circ g([0\,\infty)) \cup \phi\circ h(\mathbb{R}) =$

$\phi(\Delta x\vec{y}\vec{z})$. Since ϕ is an isometry, Corollary 3.8 implies $\text{Area}(\Delta e_3\phi(\vec{y})\phi(\vec{z})) = \text{Area}(\Delta x\vec{y}\vec{z})$.

Also Lemma 3.9 implies

$$m(\angle e_3) = \cos^{-1}((\phi\circ f)'(0)\cdot(\phi\circ g)'(0)) = \alpha.$$

So it suffices to prove

$$\text{Area}(\Delta e_3 \phi(\vec{y})\, \phi(\vec{z})) = \pi - m(\angle e_3).$$

In other words, we can assume $x = e_3$, $v = f'(0) = (\cos(\theta/2), -\sin(\theta/2), 0)$ and $w = g'(0) = (\cos(\theta/2), \sin(\theta/2), 0)$.

Recall the radial retraction $R : \mathbb{M}^{n+1}_+ \to \mathbb{R}^n \times \{1\}$ is defined by the formula

$$R(x) = x / x_{n+1}$$

Also recall that $R|\mathbb{H}^n : \mathbb{H}^n \to U^n \times \{1\}$ is a bijection where $U^n = \{x \in \mathbb{E}^n : \|x\| < 1\}$.
Observe that the inverse of $R|\mathbb{H}^n$ has the formula

$$(R|\mathbb{H}^n)^{-1}(x, 1) = \frac{1}{\sqrt{1 - \|x\|^2}}\,(x, 1).$$

Since both $R|\mathbb{H}^n : \mathbb{H}^n \to U^n \times \{1\}$ and $(R|\mathbb{H}^n)^{-1} : U^n \times \{1\} \to \mathbb{H}^n$ are continuous, then $(R|\mathbb{H}^n) : \mathbb{H}^n \to U^n \times \{1\}$ is a homeomorphism. Let $S^{n-1} = \{x \in \mathbb{E}^n : \|x\| = 1\}$.

Lemma 3.15.

a) If $f: J \to \mathbb{H}^n$ is a geodesic, then $R(f(J))$ is a straight line segment.

b) If $f: [0\,\infty) \to \mathbb{H}^n$ is a geodesic, then $f(\infty) \subset \mathbb{M}^{n+1}_+$, $R(f(\infty))$ is a single point in $S^{n-1} \times \{1\}$, and $R(f([0\,\infty)))$ is a straight line segment in $U^n \times \{1\}$ with endpoints $R(f(0))$ and $R(f(\infty))$.

c) If $g: \mathbb{R} \to \mathbb{H}^n$ is a geodesic, then $R(g(\mathbb{R}))$ is a straight line segment in $U^n \times \{1\}$ with endpoints $R(g_-(\infty))$ and $R(g_+(\infty))$.

d) If $f, g: [0\,\infty) \to \mathbb{H}^n$ are geodesics, then f is asymptotic to g if and only if $R(f(\infty)) = R(g(\infty))$.

Proof of a). Theorem 2.12 provides a 2-dimensional vector subspace V of $\mathbb{M}^{n+1}$ such that $f(J) \subset V$. Choose distinct points $x, y \in f(J)$. Then Lemma 2.13 implies x and y are linearly independent.

Hence, $\{x, y\}$ spans V. Let L be the line in $\mathbb{R}^n \times \{1\}$ determined by $R(x)$ and $R(y)$:

$$L = \{rR(x) + sR(y) : r+s=1\}.$$

<u>Claim</u>: $R(V \cap \mathbb{M}^{n+1}_+) \subseteq L$.

<u>Proof</u> Let $z \in V \cap \mathbb{M}^{n+1}_+$. Then $z = ax + by$ for some $a, b \in \mathbb{R}$. Let

$$r = \frac{a x_{n+1}}{a x_{n+1} + b y_{n+1}} \text{ and } s = \frac{b y_{n+1}}{a x_{n+1} + b y_{n+1}}$$

Then $r+s=1$ and

$$R(z) = \frac{z}{z_{n+1}} = \frac{ax+by}{a x_{n+1} + b y_{n+1}} =$$

$$\frac{a x_{n+1}\left(\frac{x}{x_{n+1}}\right) + b y_{n+1}\left(\frac{y}{y_{n+1}}\right)}{a x_{n+1} + b y_{n+1}} = rR(x) + sR(y) \in L. \square$$

Hence, $R(f(J)) \subset L$. Since $R \circ f$ is continuous and J is connected, then $R(f(J))$ is a connected subset of L. Thus, $R(f(J))$ is a straight line segment. $\square$

Proof of b) Theorem 2.10 implies $f = \Gamma_{uv} | [0 \infty)$ where u, v is an orthonormal sequence in M^{n+1} and $u \in H^n$. Lemma 2.20 implies $u + v \in f(\infty)$.

Claim: $u_{n+1} + v_{n+1} > 0$

Proof Write $u = (u_1 \cdots u_{n+1})$ and $v = (v_1 \cdots v_{n+1})$ and let $\bar{u} = (u_1 \cdots u_n)$ and $\bar{v} = (v_1 \cdots v_n)$. Then

$-1 = u \circ u = \|\bar{u}\|^2 - u_{n+1}^2$,

$+1 = v \circ v = \|\bar{v}\|^2 - v_{n+1}^2$ and

$0 = u \circ v = \bar{u} \cdot \bar{v} - u_{n+1} v_{n+1}$

Thus, $\|\bar{u}\|^2 = u_{n+1}^2 - 1$, $\|\bar{v}\|^2 = v_{n+1}^2 + 1$ and $(u_{n+1} v_{n+1})^2 = (\bar{u} \cdot \bar{v})^2 \leq \|\bar{u}\|^2 \|\bar{v}\|^2$ by the Euclidean Cauchy Inequality.

Hence, $(u_{n+1} v_{n+1})^2 \leq (u_{n+1}^2 - 1)(v_{n+1}^2 + 1) = (u_{n+1} v_{n+1})^2 + u_{n+1}^2 - v_{n+1}^2 - 1$.

So $v_{n+1}^2 \leq u_{n+1}^2 - 1 < u_{n+1}^2$.

Thus $-v_{n+1} \leq |v_{n+1}| < |u_{n+1}| = u_{n+1}$ because $u_{n+1} \geq 1$. Therefore, $0 < u_{n+1} + v_{n+1}$. $\square$

Let $x \in f(\infty)$. Then $x = r(u+v)$ for some $r > 0$. Hence, $x_{n+1} = r(u_{n+1}+v_{n+1}) > 0$. So $x \in \mathbb{M}^{n+1}_{+}$. This proves $f(\infty) \subset \mathbb{M}^{n+1}_{+}$.

Let $x = R(u+v) = \dfrac{u+v}{u_{n+1}+v_{n+1}}$, Then

$R(f(\infty)) = \{x\}$ and $x \in \mathbb{R}^n \times \{1\}$.

So we can write $x = (x_1 \cdots x_n\ 1)$.

Let $\bar{x} = (x_1 \cdots x_n)$. Observe that

$(u+v)\circ(u+v) = u\circ u + 2u\circ v + v\circ v = (-1) + 2\cdot 0 + 1 = 0$.

Thus $x \circ x = \dfrac{(u+v)\circ(u+v)}{(u_{n+1}+v_{n+1})^2} = 0$.

Also $x\circ x = \|\bar{x}\|^2 - 1$. Hence $\|\bar{x}\|^2 - 1 = 0$.

So $\|\bar{x}\|^2 = 1$. Therefore $\bar{x} \in S^{n-1}$.

Thus, $x \in S^{n-1} \times \{1\}$. This proves $R(f(\infty)) \in S^{n-1} \times \{1\}$.

$R \circ f$ is a homeomorphism from $[0,\infty)$ to its image in $U^n \times \{1\}$. Hence, by part a), $R\circ f([0,\infty))$ is a straight line segment in $U^n \times \{1\}$ with one endpoint at $R(f(0))$

and the other endpoint at $\lim_{t\to\infty} R(f(t))$

if the latter limit exists. Observe that $\lim_{t\to\infty} R(f(t)) = \lim_{t\to\infty} \frac{f(t)}{f_{n+1}(t)}$.

Suppose $v \in f(\infty)$. Then there is a map $a : (0\ \infty) \to (0\ \infty)$ such that $\lim_{t\to\infty} \frac{f(t)}{a(t)} = v$

Hence, $\lim_{t\to\infty} \frac{f_{n+1}(t)}{a(t)} = v_{n+1}$. Since

$f(\infty) \subset M_+^{n+1}$, then $v_{n+1} > 0$. Thus,

$\lim_{t\to\infty} \frac{a(t)}{f_{n+1}(t)} = \frac{1}{v_{n+1}}$. So

$$\lim_{t\to\infty} R(f(t)) = \lim_{t\to\infty} \frac{f(t)}{f_{n+1}(t)} = \lim_{t\to\infty} \frac{f(t)}{a(t)} \frac{a(t)}{f_{n+1}(t)} =$$

$$\left(\lim_{t\to\infty} \frac{f(t)}{a(t)}\right)\left(\lim_{t\to\infty} \frac{a(t)}{f_{n+1}(t)}\right) = \frac{v}{v_{n+1}} = R(v)$$

Since $R(f(\infty)) = \{R(v)\}$, it follows that the other endpoint of $R(f([0\ \infty))$ is $R(f(\infty))$.

Proof of c) $R\circ g$ maps $\mathbb{R}$ homeomorphically onto its image which, by part a), is a straight line segment in $U^n \times \{1\}$. Using part b), we conclude that the endpoints of this straight line segment are:

$$\lim_{t\to\infty} R(g(t)) = \lim_{t\to\infty} R(g_+(t)) = R(g_+(\infty))$$

and

$$\lim_{t\to-\infty} R(g(t)) = \lim_{t\to\infty} R(g_-(t)) = R(g_-(\infty)).$$

Proof of d) First assume f and g are asymptotic. Then $f(\infty) = g(\infty)$ by Theorem 2.21. Hence, $R(f(\infty)) = R(g(\infty))$.

Second assume $R(f(\infty)) = R(g(\infty))$. Let $v \in f(\infty)$ and $w \in g(\infty)$. Then $R(f(\infty)) = \{R(v)\} = \{v/v_{n+1}\}$ and $R(g(\infty)) = \{R(w)\} = \{w/w_{n+1}\}$.

So $v/v_{n+1} = w/w_{n+1}$. Thus,

$$v = \left(\frac{v_{n+1}}{w_{n+1}}\right) w \in g(\infty) \text{ and } w = \left(\frac{w_{n+1}}{v_{n+1}}\right) v \in f(\infty).$$

Hence, $f(\infty) = g(\infty)$, so Theorem 2.21 implies f is asymptotic to g. □

Recall that $\Delta e_3 \vec{y}\vec{z} = f([0\,\infty)) \cup g([0\,\infty)) \cup h(\mathbb{R})$ where $f(0) = g(0) = e_3$, $f(\infty) = h_-(\infty) = \vec{y}$, $g(\infty) = h_+(\infty) = \vec{z}$, $f'(0) = v = (\cos(\alpha/2), -\sin(\alpha/2), 0)$ and $g'(0) = w = (\cos(\alpha/2), \sin(\alpha/2), 0)$.

We now apply R to $\Delta e_3\vec{y}\vec{z}$. Lemma 3.15 implies that R maps $\Delta e_3\vec{y}\vec{z}$ to the triangle $Rf([0\,\infty)) \cup Rg([0\,\infty)) \cup Rh(\mathbb{R})$ in $U^n \times \{1\}$, where $Rf([0\,\infty))$, $Rg([0\,\infty))$ and $Rh(\mathbb{R})$ are straight line segments, $Rf([0\,\infty))$ joins $R(e_3) = e_3$ to $R(\vec{y})$, $Rg([0\,\infty))$ joins $R(e) = e_3$ to $R(\vec{z})$, and $Rh(\mathbb{R})$ joins $R(\vec{y})$ to $R(\vec{z})$. Since $Rf([0\,\infty))$ and $Rg([0\,\infty))$ are straight line segments joining e_3 to points $Rf(\infty)$ and $Rg(\infty) \in S^n \times \{1\}$, then $Rf([0,\infty))$ and $Rg([0\,\infty))$ are determined by $(Rf)'(0)$ and $(Rg)'(0)$. $(R\circ f)'(0) = dR_{e_3}(f'(0)) = dR_{e_3}(v)$ and

$(R\circ g)'(0) = dR_{e_3}(g'(0)) = dR_{e_3}(w)$.

3.40

We now calculate dR_{e_3}. For $u \in \mathbb{R}^{n+1}$,

$$dR_{e_3}(u) = \lim_{t \to 0} \frac{R(e_3+tu)-R(e_3)}{t} =$$

$$\lim_{t\to 0} \frac{1}{t}\left(\frac{e_3+tu}{1+tu_3} - e_3\right) = \lim_{t\to 0} \frac{(e_3+tu)-(e_3+tu_3e_3)}{t(1+tu_3)} =$$

$$\lim_{t\to 0} \frac{t(u+u_3e_3)}{t(1+tu_3)} = u+u_3e_3. \text{ Thus,}$$

if $u_3=0$, then $dR_{e_3}(u)=u$.

Since $v \equiv (\cos(\alpha/2), -\sin(\alpha/2), 0)$ and $w=(\cos(\alpha/2), \sin(\alpha/2), 0)$, then $\|v\|=\|w\|=1$ and $dR_3(v)=v$ and $dR_3(w)=w$. So $(R\circ f)'(0)=v$ and $(R\circ g)'(0)=w$. Hence,

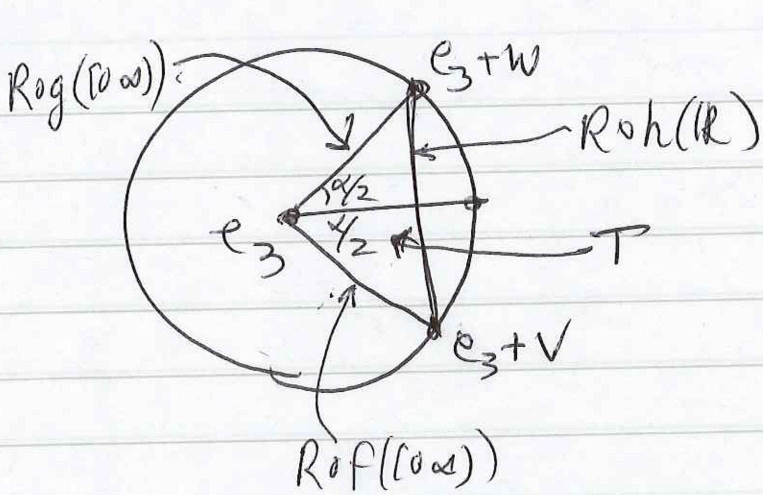
$R\circ f([0\,\infty)) = \{e_3+tv : 0 \le t < 1\}$ and

$R\circ g([0\,\infty)) = \{e_3+tw : 0\le t<1\}$. Thus

$R\circ f(\infty) = e_3+v$ and $R\circ g(\infty) = e_3+w$,

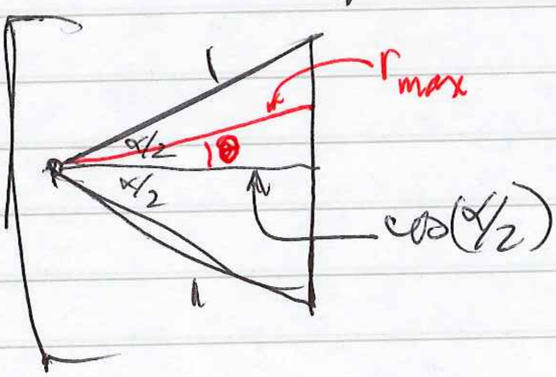
So $R\circ f([0\,\infty))$ joins e_3 to e_3+v, $R\circ g([0\,\infty))$ joins e_3 to e_3+w, and $R\circ h(\mathbb{R}))$ joins e_3+v to e_3+w.

Let $T = \Delta e_3, e_3+v, e_3+w \subset \mathbb{R}^2 \times \{1\}$.



R maps $\mathrm{int}(\triangle e_3 \vec{y}\, \vec{z})$ diffeomorphically onto $\mathrm{int}\, T$. Let $\sigma = (R|\mathbb{H}^2)^{-1} : U^2 \times \{1\} \to \mathbb{H}^2$.

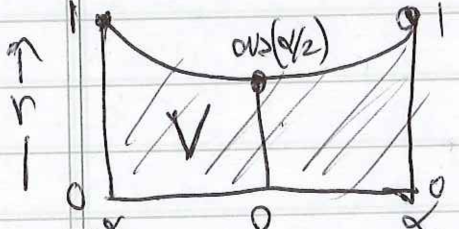
Then $\sigma(x, 1) = \dfrac{(x, 1)}{\sqrt{1 - \|x\|^2}}$ and σ maps $\mathrm{int}(T)$ diffeomorphically onto $\mathrm{int}(\triangle e_3 \vec{y}\, \vec{z})$.



$$\cos\theta = \frac{\cos(\alpha/2)}{r_{max}}$$

$$\therefore r_{max} = \frac{\cos(\alpha/2)}{\cos\theta}$$

Define $V = \left\{ (\theta, r) \in \mathbb{R}^2 : -\frac{\alpha}{2} \le \theta \le \frac{\alpha}{2} \text{ and } 0 < r < \frac{\cos(\alpha/2)}{\cos\theta} \right\}$



Define $\tau : V \to \mathrm{int}(T)$ by $\tau(\theta, r) = (r\cos\theta, r\sin\theta, 1)$.

Then τ maps V diffeomorphically onto $\mathrm{int}(T)$.

Define the diffeomorphism $\psi : V \to \operatorname{int}(\Delta e_3 \vec{y} \vec{z})$ by $\psi = \sigma \circ \tau$. It follows that

$$\operatorname{Area}(\Delta e_3 \vec{y} \vec{z}) = \int_V \sqrt{\det D(\theta, r)}\; d(\theta, r)$$

where $D(\theta,r) = \begin{pmatrix} \frac{\partial \psi}{\partial \theta} \circ \frac{\partial \psi}{\partial \theta} & \frac{\partial \psi}{\partial \theta} \circ \frac{\partial \psi}{\partial r} \\ \frac{\partial \psi}{\partial r} \circ \frac{\partial \psi}{\partial \theta} & \frac{\partial \psi}{\partial r} \circ \frac{\partial \psi}{\partial r} \end{pmatrix}$

Now $\psi(\theta r) = \sigma(\tau(\theta r)) = \sigma(r\cos\theta, r\sin\theta, 1)$
$= \dfrac{(r\cos\theta, r\sin\theta, 1)}{\sqrt{1-r^2}}$. Observe that

$$\frac{\partial}{\partial r}\left(\frac{r}{\sqrt{1-r^2}}\right) = \frac{1}{(1-r^2)^{3/2}} \text{ and } \frac{\partial}{\partial r}\left(\frac{1}{\sqrt{1-r^2}}\right) = \frac{r}{(1-r^2)^{3/2}}.$$

Hence, $\dfrac{\partial \psi}{\partial \theta} = \dfrac{r}{\sqrt{1-r^2}}(-\sin\theta, \cos\theta, 0)$ and

$$\frac{\partial \psi}{\partial r} = \frac{1}{(1-r^2)^{3/2}}(\cos\theta, \sin\theta, r)$$

Thus, $\dfrac{\partial \psi}{\partial \theta} \circ \dfrac{\partial \psi}{\partial \theta} = \dfrac{r^2}{1-r^2}$, $\dfrac{\partial \psi}{\partial \theta} \circ \dfrac{\partial \psi}{\partial r} = 0$ and

$$\frac{\partial \psi}{\partial r} \circ \frac{\partial \psi}{\partial r} = \frac{1-r^2}{(1-r^2)^3} = \frac{1}{(1-r^2)^2}$$

Therefore, $D(\theta, r) = \begin{pmatrix} \frac{r^2}{1-r^2} & 0 \\ 0 & \frac{1}{(1-r^2)^2} \end{pmatrix}$, and

$$\sqrt{\det D(\theta, r)} = \sqrt{\frac{r^2}{(1-r^2)^3}} = \frac{r}{(1-r^2)^{3/2}}.$$

Hence, $\text{Area}(\Delta e_3 \vec{y} \vec{z}) = \int_V \frac{r}{(1-r^2)^{3/2}} \, d(r\theta) =$

$$\int_{-\frac{\alpha}{2}}^{\alpha/2} \left[\int_0^{\frac{\cos(\alpha/2)}{\cos\theta}} \frac{r}{(1-r^2)^{3/2}} \, dr \right] d\theta$$

Observe that $\int \frac{r}{(1-r^2)^{3/2}} \, dr = \frac{1}{\sqrt{1-r^2}} + C$.

11/4 Thus, $\text{Area}(\Delta e_3 \vec{y} \vec{z}) = \int_{-\alpha/2}^{\alpha/2} \left[\frac{1}{\sqrt{1-r^2}} \Big|_{r=0}^{r=\frac{\cos(\alpha/2)}{\cos\theta}} \right] d\theta =$

$$\int_{-\alpha/2}^{\alpha/2} \left(\frac{1}{\sqrt{1 - \left(\frac{\cos(\alpha/2)}{\cos\theta}\right)^2}} - 1 \right) d\theta =$$

$$\int_{-\alpha/2}^{\alpha/2} \frac{\cos\theta}{\sqrt{\cos^2\theta - \cos^2(\alpha/2)}} \, d\theta - \int_{-\alpha/2}^{\alpha/2} 1 \, d\theta =$$

$$\int_{-\alpha/2}^{\alpha/2} \frac{\cos\theta}{\sqrt{\sin^2(\alpha/2) - \sin^2\theta}} \, d\theta - \left(\frac{\alpha}{2} - \left(-\frac{\alpha}{2} \right) \right)$$

$$= \int_{-\alpha/2}^{\alpha/2} \frac{\left(\frac{\cos\theta}{\sin(\alpha/2)}\right)}{\sqrt{1-\left(\frac{\sin\theta}{\sin(\alpha/2)}\right)^2}}\, d\theta - \alpha =$$

$$\int_{-1}^{1} \frac{du}{\sqrt{1-u^2}} - \alpha$$

(using the substitution $u = \frac{\sin\theta}{\sin(\alpha/2)}$, $du = \frac{\cos\theta}{\sin(\alpha/2)} d\theta$, $\theta = -\alpha/2 \Rightarrow u = -1$, and $\theta = \alpha/2 \Rightarrow u = +1$.)

$$= \int_{-\pi/2}^{\pi/2} \frac{\cos\varphi \, d\varphi}{\cos\varphi} - \alpha$$

(using the substitution $u = \sin\varphi$, $du = \cos\varphi \, d\varphi$, $\varphi = -\pi/2 \Rightarrow u = -1$, and $\varphi = \pi/2 \Rightarrow u = +1$)

$$= \int_{-\pi/2}^{\pi/2} 1 \, d\varphi - \alpha = \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) - \alpha = \pi - \alpha.$$

Thus, $\text{Area}(\Delta e_3 \vec{y} \vec{z}) = \pi - \alpha = \pi - m(\angle e_3)$.

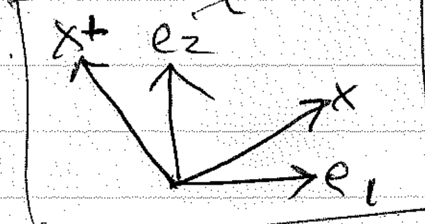
This completes the proof of Lemma 3.14. $\square$

The following homework problems helps with angle measure issues that arise in subsequent theorems.

<u>Homework Problem 3.2</u>- Let V be a 2-dimensional inner product space with orthonormal basis e_1, e_2. (Thus, $x = (x\cdot e_1)e_1 + (x\cdot e_2)e_2$ for each $x \in V$.)

Define the function $x \mapsto x^\perp : V \to V$ by

$$x^\perp = -(x\cdot e_2)e_1 + (x\cdot e_1)e_2$$



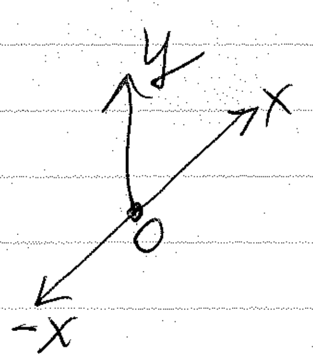
a) Prove $x \mapsto x^\perp : V \to V$ is a linear isometry such that $x \cdot x^\perp = 0$ and $x^{\perp\perp} = -x$ for every $x \in V$.

For $x, y \in V - \{0\}$, define

$$m(\angle xOy) = \cos^{-1}\left(\frac{x}{\|x\|}\cdot\frac{y}{\|y\|}\right)$$

b) For $x, y \in V - \{0\}$, prove

$$m(\angle xOy) + m(\angle yO(-x)) = \pi$$



c) ~~Let~~ $x, y, z \in V - \{0\}$ such that x and y are linearly independent. Prove that the following three statements are equivalent.

i) $z = ax + by$ where $a > 0$ and $b > 0$.

ii) $(y \cdot x^{\perp})(z \cdot x^{\perp}) > 0$ and $(x \cdot y^{\perp})(z \cdot y^{\perp}) > 0$.

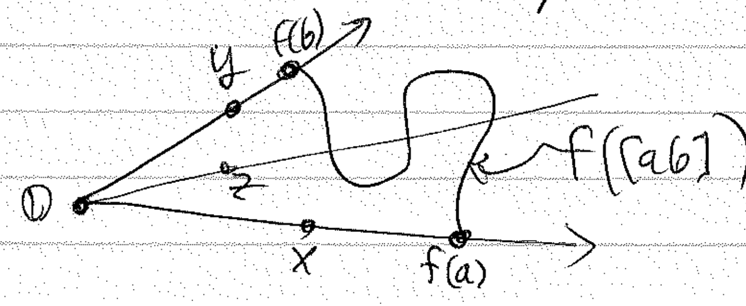
iii) There is a continuous function $f : [a,b] \to V$ such that

$f(a) \in (0,\infty)x = \{tx : t > 0\}$,

$f((a,b]) \cap \mathbb{R}x = \phi$ where $\mathbb{R}x = \{tx : t \in \mathbb{R}\}$,

$f(b) \in (0\infty)y$ and $f([a,b)) \cap \mathbb{R}y = \phi$,

and $f((a,b)) \cap (0\infty)z \neq \phi$

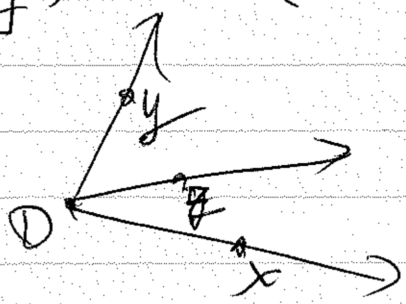


~~Def~~ If $x, y, z \in V - \{0\}$ satisfy i), ii) or iii), we say z _lies between x and y_.

d) Prove that if z lies between x and y, then $(x \cdot z^{\perp})(y \cdot z^{\perp}) < 0$.

e) Prove that if $x, y, z \in V - \{0\}$ and z lies between x and y, then

$$m(\angle xOy) = m(\angle xOz) + m(\angle zOy).$$



f) Let $x, y, z \in V - \{0\}$ so that no two of x, y, z are linearly dependent. Prove that if no one of x, y, z lies between the other two, then either:

- x lies between y and z, or
- y lies between x and z, or
- z lies between x and y.

g) Let $x, y, z \in V - \{0\}$ so that no two of x, y, z are linearly dependent. Prove that if no one of x, y, z lies between the other two, then

$$m(\angle xOy) + m(\angle yOz) + m(\angle xOz) = 2\pi.$$

Homework Problem 3.3 Let $x \in \mathbb{H}^n$.

Recall that $T_x(\mathbb{H}^n) = \{v \in \mathbb{M}^{n+1} : x \circ v = 0\}$.

Define $\pi_x : \mathbb{H}^n \to \mathbb{M}^{n+1}$ by

$$\pi_x(y) = y + (x \circ y)x.$$

Define $\rho_x : T_x(\mathbb{H}^n) \to \mathbb{M}_{n+1}$ by

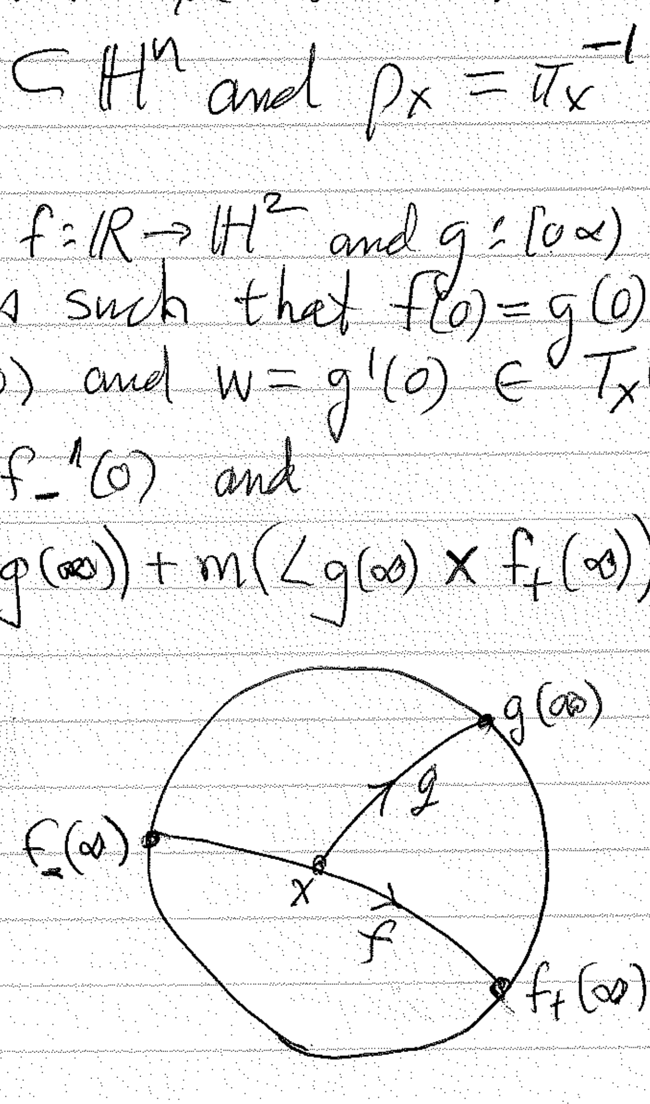
$$\rho_x(v) = \sqrt{|||v|||^2 + 1}\, x + v.$$

a) Prove that $\pi_x(\mathbb{H}^n) \subset T_x(\mathbb{H}^n)$, $\rho_x(T_x(\mathbb{H}^n)) \subset \mathbb{H}^n$ and $\rho_x = \pi_x^{-1}$.

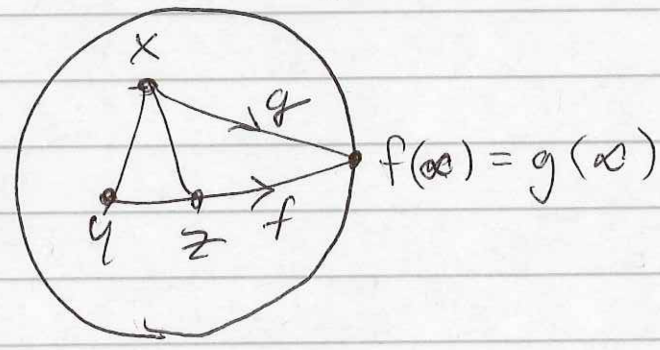
b) Suppose $f : \mathbb{R} \to \mathbb{H}^2$ and $g : [0, \infty) \to \mathbb{H}^2$ are geodesics such that $f(0) = g(0) = x$. Let $v = f'(0)$ and $w = g'(0) \in T_x(\mathbb{H}^2)$.

Prove $-v = f_-'(0)$ and

$$m(\angle f_-(\infty)\, x\, g(\infty)) + m(\angle g(\infty)\, x\, f_+(\infty)) = \pi$$



c) Suppose $f, g : [0, \infty) \to \mathbb{H}^2$ are geodesics such that $g(0) = x$, $f(0) = y$ and $z \in f((0,\infty))$ and f and g are asymptotic.

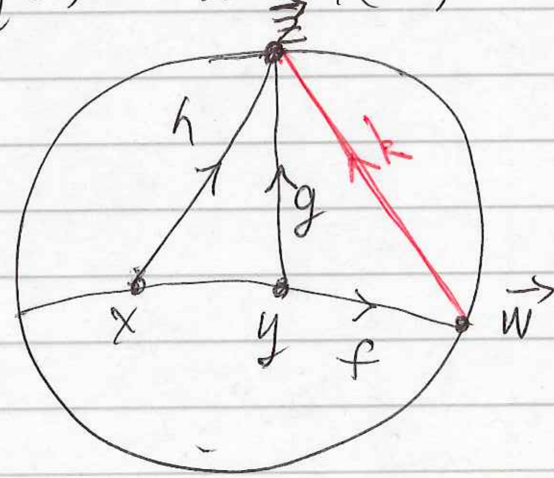


Then prove

$$m(\angle yxz) + m(\angle zxg(\infty)) = m(\angle yxg(\infty)).$$

<u>Lemma 3.16</u> If $\Delta xy\vec{z}$ is a triangle in $\mathbb{H}^2$ with ordinary vertices x and y and ideal vertex $\vec{z}$, then

$$\text{Area}(\Delta xy\vec{z}) = \pi - m(\angle x) - m(\angle y)$$



<u>Proof</u> Let $f: \mathbb{R} \to \mathbb{H}^2$, $g, h: [0\infty) \to \mathbb{H}^2$ be geodesics such that $f(-\eta(xy)) = x$, $f(0) = y$, $g(0) = y$, $g(\infty) = \vec{z}$, $h(0) = x$, $h(\infty) = \vec{z}$.
Let $\vec{w} = f_+(\infty)$. Since $\mathbb{H}^2$ is a visibility space, by Theorem 2-22, then there is a geodesic $k: \mathbb{R} \to \mathbb{H}^2$ such that $k_-(\infty) = \vec{w}$ and $k_+(\infty) = \vec{z}$. Lemma 3.14 implies

$$\text{Area}(\Delta x\vec{w}\vec{z}) = \pi - m(\angle x)$$

and

$$\text{Area}(\Delta y\vec{w}\vec{z}) = \pi - m(\angle \vec{w}y\vec{z})$$

Clearly

$$\text{Area}(\Delta xy\vec{z}) = \text{Area}(\Delta x\vec{w}\vec{z}) - \text{Area}(\Delta y\vec{w}\vec{z})$$

Also Homework Problem 3.3, b implies

$$m(\angle y) + m(\angle \vec{w}y\vec{z}) = \pi.$$

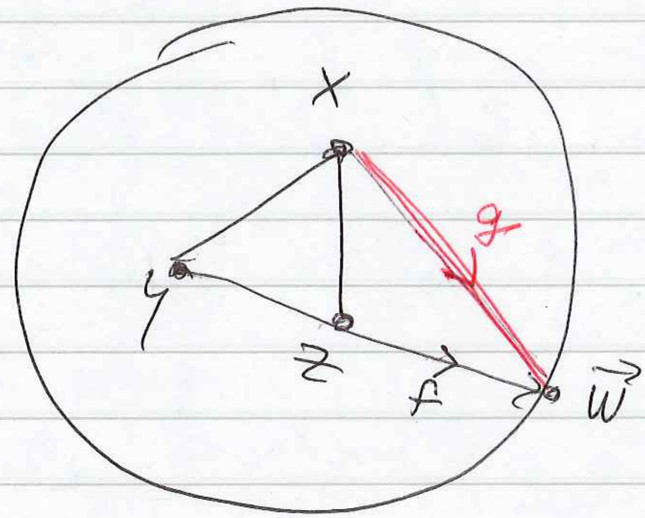
Hence, $\text{Area}(\Delta y\vec{w}\vec{z}) = m(\angle y)$.
Therefore

$$\text{Area}(\Delta xy\vec{z}) = \pi - m(\angle x) - m(\angle y). \quad \square$$

<u>Lemma 3.17</u> If Δxyz is a triangle in $\mathbb{H}^2$ with three ordinary vertices, then

$$\text{Area}(\Delta xyz) = \pi - m(\angle x) - m(\angle y) - m(\angle z).$$

<u>Proof</u>



There is a geodesic $f : [0, \infty) \to \mathbb{H}^2$ such that $f(0) = y$ and $f(r(yz)) = z$. Let $\vec{w} = f(\infty)$.
Homework Problem 3.4 below implies there is a geodesic $g : [0, \infty) \to \mathbb{H}^2$ such that $g(0) = x$ and $g(\infty) = \vec{w}$.

Clearly

$$\text{Area}(\Delta xyz) = \text{Area}(\Delta xy\vec{w}) - \text{Area}(\Delta xz\vec{w}).$$

Lemma 3.16 implies

$$\text{Area}(\Delta xy\vec{w}) = \pi - m(\angle yx\vec{w}) - m(\angle y)$$

and

$$\text{Area}(\Delta xz\vec{w}) = \pi - m(\angle zx\vec{w}) - m(\angle xz\vec{w}).$$

Therefore

$$\text{Area}(\Delta xyz) = -m(\angle yx\vec{w}) - m(\angle y) + m(\angle zx\vec{w}) + m(\angle xz\vec{w})$$

$$= m(\angle xz\vec{w}) - \left(m(\angle yx\vec{w}) - m(\angle zx\vec{w})\right) - m(\angle y).$$

Homework Problem 3.3.b and c imply

$$m(\angle xz\vec{w}) = \pi - m(\angle z)$$

and

$$m(\angle yx\vec{w}) - m(\angle zx\vec{w}) = m(\angle x).$$

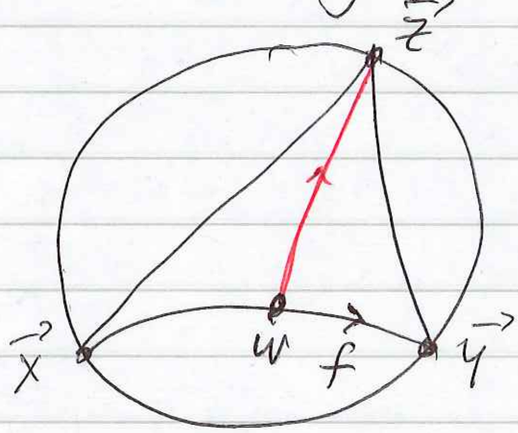
Thus,

$$\text{Area}(\Delta xyz) = \left(\pi - m(\angle z)\right) - m(\angle x) - m(\angle y), \quad \square$$

<u>Homework Problem 3.4</u> Prove that if $x \in \mathbb{H}^n$, $z \in \mathbb{M}^{n+1}_+$ and $z \circ z = 0$, then there is a geodesic $f : [0, \infty) \to \mathbb{H}^n$ such that $f(0) = x$ and $f(\infty) = \vec{z}$ where $\vec{z} = \{rz : r > 0\}$.

Lemma 3.~~4~~8 If $\Delta\vec{x}\vec{y}\vec{z}$ is a triangle in $\mathbb{H}^2$ with three ideal vertices, then $\text{Area}(\Delta\vec{x}\vec{y}\vec{z}) = \pi$.

Proof Let $f:\mathbb{R}\to\mathbb{H}^2$ be a geodesic such that $f_-(\infty)=x$ and $f_+(\infty)=y$. Let $w=f(0)$. Homework Problem 3.4 provides a geodesic ray joining w to $\vec{z}$



Clearly

$$\text{Area}(\Delta\vec{x}\vec{y}\vec{z}) = \text{Area}(\Delta\vec{x}w\vec{z}) + \text{Area}(\Delta\vec{y}w\vec{z})$$

Lemma 3.4 implies

$$\text{Area}(\Delta\vec{x}w\vec{z}) = \pi - m(\angle\vec{x}w\vec{z})$$

and

$$\text{Area}(\Delta\vec{y}w\vec{z}) = \pi - m(\angle\vec{y}wz).$$

Homework Problem 3.3.b implies

$$m(\angle\vec{x}w\vec{z}) + m(\angle\vec{y}w\vec{z}) = \pi$$

Adding these equations we obtain

$$\text{Area}(\Delta \vec{x}\vec{y}\vec{z}) = \pi + \pi - \pi = \pi. \square$$

Use the techniques described in this chapter for computing arc lengths and volumes to solve the following problems.

Homework Problem 3.5.

a) Find the length of a circle of radius R in $\mathbb{H}^2$.

b) Find the area of a disk of radius R in $\mathbb{H}^2$.

Homework Problems 3.6

Let $n \geq 3$ and let $Vol(S^{n-1})$ denote the $(n-1)$-dimensional volume of a sphere of radius 1 in $\mathbb{E}^n$. Let $R > 0$.

a) Prove that the $(n-1)$-dimensional volume of a sphere of radius R in $\mathbb{H}^n$ is $Vol(S^{n-1}) \sinh^{n-1}(R)$.

b) Prove that the n-dimensional volume of a ball of radius R in $\mathbb{H}^n$ is

$$Vol(S^{n-1}) \left(\int_0^R \sinh^{n-1}(t)\, dt \right).$$