2. Hyperbolic Spaces

<u>Def</u> <u>Hyperbolic n-space</u> is the set

$$\mathbb{H}^n = \{x \in \mathbb{M}^{n+1} : x \circ x = -1 \text{ and } x_{n+1} > 0\}.$$

More generally, for $r>0$, define

$$\mathbb{H}^n_r = \{x \in \mathbb{M}^{n+1} : x \circ x = -r^2 \text{ and } x_{n+1} > 0\}.$$

Recall that $\cosh : \mathbb{R} \to [1,\infty)$ is defined by

$$\cosh(t) = (e^t + e^{-t})/2.$$

Then $\cosh|[0,\infty) : [0,\infty) \to [1,\infty)$ is a bijection. Let $\cosh^{-1} : [1,\infty) \to [0,\infty)$ denote the inverse of $\cosh|[0,\infty) : [0,\infty) \to [1,\infty)$.

Observe that for $x, y \in \mathbb{H}^n$, Theorem 1.4 (The Cauchy Inequality for T^n) implies $x \circ y \leq -|||x|||\,|||y||| = -1$. Thus $-x\circ y \in [1,\infty)$. Hence, the following definition is justified.

<u>Def</u> Define $\eta : \mathbb{H}^n \times \mathbb{H}^n \to [0,\infty)$ by

$$\eta(x,y) = \cosh^{-1}(-x \circ y)$$

for $x, y \in \mathbb{H}^n$.

More generally, observe that for $r>0$ and $x,y \in \mathbb{H}^n_r$, $x \circ y \le -|||x|||\,|||y||| = -r^2$. Thus, $-(x\circ y)/r^2 \in [1,\infty)$. This justifies the following definition.

<u>Def</u> For $r>0$, define $\eta_r : \mathbb{H}^n_r \times \mathbb{H}^n_r \to [0,\infty)$ by $\eta_r(x,y) = r\cosh^{-1}(-x\circ y/r)$ for $x,y \in \mathbb{H}^n_r$.

We will prove η is a metric on $\mathbb{H}^n$. Our proof depends on the Hyperbolic Law of Cosines. To state and prove the Hyperbolic Law of Cosines, we must first define the concept of angle measure in $\mathbb{H}^n$.

9/23 → <u>Def</u> For $x,y,z \in \mathbb{H}^n$ such that $y,z \ne x$, let $p = y + (y\circ x)x$ and $q = z + (z\circ x)x$. Then $p\circ p = (y\circ x)^2 - 1 > 0$ and $q\circ q = (z\circ x)^2 - 1 > 0$.

<u>Proof</u> $p\circ p = y\circ y + 2(y\circ x)^2 + (y\circ x)^2(x\circ x) = (y\circ x)^2 - 1$. Since $x \ne y$, Theorem 1.4 implies $y\circ x < -|||y|||\,|||x||| = -1$. Thus, $(y\circ x)^2 - 1 > 0$. Similarly, $q\circ q = (z\circ x)^2 - 1 > 0$. □

Let $v = p/|||p|||$ and $w = q/|||q|||$. Then $|v\circ w| \le 1$.

~~Proof Since $p\circ x = 0 = q\circ x$, then $v\circ x = 0 = w\circ x$.~~
~~Therefore, $(v \pm w)\circ x = 0$. Since $x\circ x < 0$, then~~

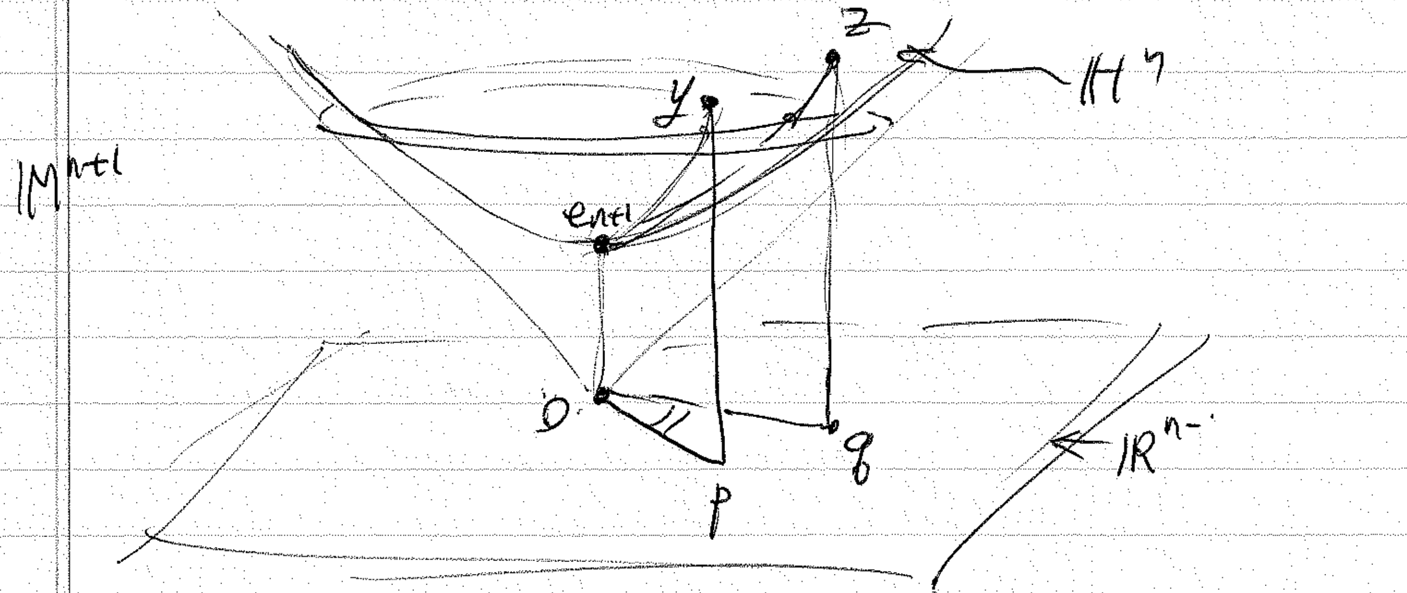
Proof First, if $v = \pm w$, then $|v \circ w| = |v \circ v| = 1$. Now assume $v \neq \pm w$. Thus, $v \pm w \neq 0$. Since $p \circ x = 0 = q \circ x$, then $v \circ x = 0 = w \circ x$. Therefore $(v \pm w) \circ x = 0$. Since $x \circ x < 0$, then Corollary 1.3 implies $(v \pm w) \circ (v \pm w) > 0$. Thus, $2 \pm 2 v \circ w > 0$. So $\pm v \circ w \leq 1$. Thus, $|v \circ w| \leq 1$. $\square$

Since $\cos^{-1} : [-1, 1] \to [0, \pi]$, then we can define

$$m(\angle yxz) = \cos^{-1}(v \circ w).$$

We motivate this definition by considering the special case in which $x = e_{n+1}$. Then $p = y + (y \circ e_{n+1}) e_{n+1}$ and $q = z + (z \circ e_{n+1}) e_{n+1}$. Hence,
$p \circ e_{n+1} = y \circ e_{n+1} - y \circ e_{n+1} = 0$ and $q \circ e_{n+1} = 0$.

Thus, p and q are the orthogonal projections of y and z, respectively, into the hyperplane $\mathbb{R}^n \times \{0\}$. Then $v = p/|||p|||$ and $w = q/|||q|||$ are unit-length renormalizations of p and q in $\mathbb{R}^n \times \{0\}$, and $v \circ w = v \cdot w$. Hence, $\cos^{-1}(v \circ w)$ is the Euclidean angle between v and w in $\mathbb{R}^n \times \{0\}$.



In general, when $x, y, z \in \mathbb{H}^n$ such that $y, z \neq x$, there is an $f \in O^+(\mathbb{M}^n)$ such that $f(x) = e_{n+1}$. (f can taken to be a time preserving linear reflection.) Let $y' = f(y)$ and $z' = f(z)$. Then $m(\angle yxz) = m(\angle y' e_{n+1} z')$.

<u>Proof</u> Let $p = y + (y \circ x)x$, $q = z + (z \circ x)x$, $p' = y' + (y' \circ e_{n+1})e_{n+1}$ and $q' = z' + (z' \circ e_{n+1})e_{n+1}$. Since $f \in O(\mathbb{M}^n)$, then $f(p) = p'$ and $f(q) = q'$. Let $v = p/|||p|||$, $w = q/|||q|||$, $v' = p'/|||p'|||$, $w' = q'/|||q'|||$. Then $f(v) = v'$ and $f(w) = w'$. Hence $v \circ w = v' \circ w'$. Therefore, $m(\angle yxz) = \cos^{-1}(v \circ w) = \cos^{-1}(v' \circ w') = m(\angle y' e_{n+1} x')$.

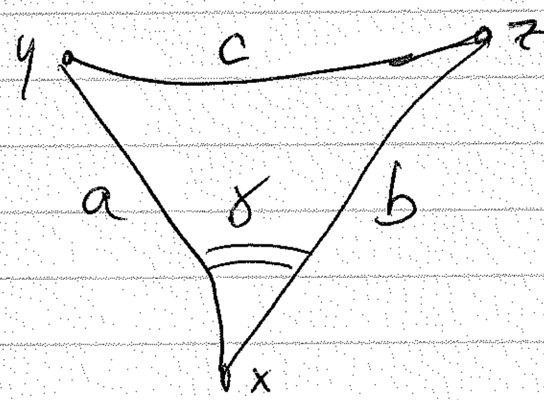
Thus $m(\angle yxz)$ equals the Euclidean measure of the angle obtained by using an element f of $O^+(\mathbb{M}^n)$ to move x to e_{n+1} and projecting $e_{n+1}, f(y), f(z)$ orthogonally into the Euclidean hyperplane $\mathbb{R}^n \times \{0\}$.

Next we state and prove

<u>Theorem 2.1 The Hyperbolic Law of Cosines</u>.

Let $x, y, z \in \mathbb{H}^n$ such that $y, z \neq x$. Let
$a = \eta(x,y)$, $b = \eta(x,z)$, $c = \eta(y,z)$ and $\gamma = m(\angle yxz)$.
Then

$$\cosh(c) = \cosh(a)\cosh(b) - \sinh(a)\sinh(b)\cos(\gamma).$$



<u>Proof</u> Let $p = y + (y \circ x)x$ and $q = z + (z \circ x)x$.
Then (as proved previously) $p \circ p = (y \circ x)^2 - 1 > 0$ and
$q \circ q = (z \circ x)^2 - 1 > 0$. Let $v = p / |||p|||$ and $w = q / |||q|||$.
Then $\gamma = m(\angle yxz) = \cos^{-1}(v \circ w)$. Hence,

$\cosh(a) = -x \circ y$, $\cosh(b) = -x \circ z$, $\cosh(c) = -y \circ z$
and $\cos(\gamma) = v \circ w$.

Therefore, $p = y - \cosh(a)x$, $q = z - \cosh(b)x$,
$p \circ p = \cosh^2(a) - 1 = \sinh^2(a)$ and $q \circ q = \cosh^2(b) - 1 = \sinh^2(b)$.

Since $a = \eta(x,y) \geq 0$ and $b = \eta(x,z) \geq 0$,
then $\sinh(a) \geq 0$ and $\sinh(b) \geq 0$.
Therefore, $|||p||| = \sinh(a)$ and $|||q||| = \sinh(b)$.

Thus, $v = \dfrac{y - \cosh(a)x}{\sinh(a)}$ and $w = \dfrac{z - \cosh(b)x}{\sinh(b)}$.

Therefore, $v \circ w =$

$$\frac{y\circ z - \cosh(a)(x\circ z) - \cosh(b)(y\circ z) + \cosh(a)\cosh(b)(x\circ x)}{\sinh(a)\sinh(b)}$$

$$= \frac{-\cosh(c) + 2\cosh(a)\cosh(b) - \cosh(a)\cosh(b)}{\sinh(a)\sinh(b)}$$

$$= \frac{-\cosh(c) + \cosh(a)\cosh(b)}{\sinh(a)\sinh(b)}.$$

Hence, $\cosh(c) = \cosh(a)\cosh(b) - \sinh(a)\sinh(b)\cos(\gamma)$.

$\square$

Theorem 2.2. η is a metric on $\mathbb{H}^n$.

Proof For $x, y \in \mathbb{H}^n$, $\eta(x,y) = \cosh^{-1}(-x\circ y)$.
Since $\circ$ is symmetric, so is η.

Since $\cosh^{-1} : [1,\infty) \to [0,\infty)$, then $\eta(x,y) \geq 0$.
If $x = y$, then $\eta(x,y) = \cosh^{-1}(\cancel{-x}\, x\circ x) = \cosh^{-1}(1) = 0$.

Suppose $\eta(x,y)=0$. Then $-x\circ y = \cosh(0)=1$. So $x\circ y = -1 = -|||x|||\,|||y|||$. Hence, Theorem 1.4.a implies $|||y|||x = |||x|||y$. Thus, $x=y$.

To prove the triangle inequality, let $x,y,z \in \mathbb{H}^n$. We will prove

$$\eta(y,z) \le \eta(y,x) + \eta(x,z).$$

If $y=x$ or $z=x$, this inequality becomes the true statement $\eta(y,z) \le \eta(y,z)$. So we can assume $y,z \ne x$. Let $a=\eta(x,y)$, $b=\eta(x,z)$, $c=\eta(y,z)$ and $\gamma = m(\angle yxz)$. Then the Hyperbolic Law of Cosines (Theorem 2.1) implies

$$\cosh(c) = \cosh(a)\cosh(b) - \sinh(a)\sinh(b)\cos(\gamma)$$

Since $a \ge 0$ and $b \ge 0$, then $\sinh(a) \ge 0$ and $\sinh(b) \ge 0$. Also $-\cos(\gamma) \le 1$. Therefore

$$-\sinh(a)\sinh(b)\cos(\gamma) \le \sinh(a)\sinh(b).$$

Thus,

$$\cosh(c) \le \cosh(a)\cosh(b) + \sinh(a)\sinh(b) = \cosh(a+b).$$

Since $\cosh|[0,\infty)$ is strictly increasing, then $c \le a+b$. $\square$

Homework Problem 2.1 Let $r>0$.

a) For $x, y, z \in \mathbb{H}^n_r$ such that $y, z \neq x$, define $m(\angle yxz)$.

b) State and prove a Hyperbolic Law of Cosines for $\mathbb{H}^n_r$.

c) Prove η_r is a metric on $\mathbb{H}^n_r$.

Recall $\mathcal{I}(\mathbb{H}^n)$ denotes the isometry group of $\mathbb{H}^n$ (with the metric η).

Theorem 2.3 $f \mapsto f|\mathbb{H}^n : O^+(\mathbb{M}^{n+1}) \to \mathcal{I}(\mathbb{H}^n)$ is an isomorphism.

Proof Let $f \in O^+(\mathbb{M}^{n+1})$. For $x \in \mathbb{H}^n$, $f(x) \circ f(x) = x \circ x = -1$. Also $f(\mathbb{H}^n) \subset f(T^n_+) = T^n_+$. Thus, $f(\mathbb{H}^n) \subset \{x \in \mathbb{M}^{n+1} : x \circ x = -1\} \cap T^n_+ = \mathbb{H}^n$. Since $f^{-1} \in O^+(\mathbb{M}^{n+1})$, then $f^{-1}(\mathbb{H}^n) \subset \mathbb{H}^n$. Hence, $\mathbb{H}^n = f \circ f^{-1}(\mathbb{H}^n) \subset f(\mathbb{H}^n)$. Thus, $f(\mathbb{H}^n) = \mathbb{H}^n$. Consequently, $f|\mathbb{H}^n : \mathbb{H}^n \to \mathbb{H}^n$ is surjective.

For $x, y \in \mathbb{H}^n$, $\eta(f(x), f(y)) = \cosh^{-1}(-f(x) \circ f(y)) = \cosh^{-1}(-x \circ y) = \eta(x,y)$. Thus, $f|\mathbb{H}^n : \mathbb{H}^n \to \mathbb{H}^n$ is distance preserving.

We conclude that for each $f \in O^+(\mathbb{M}^{n+1})$, $f|\mathbb{H}^n \in \mathcal{I}(\mathbb{H}^n)$.

For $f, g \in O^+(\mathbb{M}^{n+1})$, since $g \circ f|\mathbb{H}^n = (g|\mathbb{H}^n) \circ (f|\mathbb{H}^n)$, then $f \mapsto f|\mathbb{H}^n : O^+(\mathbb{M}^{n+1}) \to \mathcal{I}(\mathbb{H}^n)$ is a group homomorphism.

To prove that $f \mapsto f|\mathbb{H}^n : O^+(\mathbb{M}^{n+1}) \to \mathcal{I}(\mathbb{H}^n)$ is injective, observe that

$e_1 + \sqrt{2}e_{n+1}, e_2 + \sqrt{2}e_{n+1}, \dots, e_n + \sqrt{2}e_{n+1}, e_{n+1}$

is a basis for $\mathbb{M}^{n+1}$ that lies in $\mathbb{H}^n$.
(If $x = (x_1 \dots x_{n+1}) \in \mathbb{M}^n$, then
$x = \sum_{i=1}^n x_i(e_i + \sqrt{2}e_{n+1}) + (x_{n+1} - (\sum_{i=1}^n x_i)\sqrt{2})e_{n+1}$.
Also $(e_i + \sqrt{2}e_{n+1}) \circ (e_i + \sqrt{2}e_{n+1}) = 1 - 2 = -1$.)
Hence, if $f \in O^+(\mathbb{M}^{n+1})$ and $f|H = id$, then f fixes each element of this basis. Since f is linear, then $f = id$.

9/25 → It remains to prove $f \mapsto f|\mathbb{H}^n : O^+(\mathbb{M}^{n+1}) \to \mathcal{I}(\mathbb{H}^n)$ is surjective. Let $g \in \mathcal{I}(\mathbb{H}^n)$.

First assume $g(e_{n+1}) = e_{n+1}$. In this case, the proof has three steps. <u>Step 1</u>: "Orthogonally project" g to a map $f: \mathbb{R}^n \to \mathbb{R}^n$ and prove $f \in O(\mathbb{R}^n)$.

<u>Step 2</u>: "Extend" f to a map $\bar{f}: M^{n+1} \to M^{n+1}$ defined by $\bar{f}(x_1 \cdots x_n\, x_{n+1}) = (f(x_1 \cdots x_n), x_{n+1})$ and prove $\bar{f} \in O^+(M^{n+1})$. <u>Step 3</u>: Prove $\bar{f} \mid \mathbb{H}^n = g$.

We now explain <u>Step 1. Define</u> $\varphi: \mathbb{R}^n \to [1, \infty)$ by $\varphi(x) = \sqrt{\|x\|^2 + 1}$. (Then $\mathbb{H}^n$ is the graph of φ.) Define $E: \mathbb{R}^n \to \mathbb{H}^n$ by $E(x_1 \cdots x_n) = (x_1 \cdots x_n, \varphi(x_1 \cdots x_n))$, and define $P: \mathbb{H}^n \to \mathbb{E}^n$ by $P(x_1 \cdots x_n\, x_{n+1}) = (x_1 \cdots x_n)$. Then $P \circ E = id_{\mathbb{R}^n}$. Define $f: \mathbb{R}^n \to \mathbb{R}^n$ by $f = P \circ g \circ E$. We will prove $f \in O(\mathbb{R}^n)$.

First we make two observations about g. For $x, y \in \mathbb{H}^n$, since $\eta(x,y) = \eta(g(x), g(y))$, then $-x \circ y = \cosh(\eta(x,y)) = \cosh(\eta(g(x), g(y))) = -g(x) \circ g(y)$. Thus, $g(x) \circ g(y) = x \circ y$ for $x, y \in \mathbb{H}^n$. In other words, g preserves $\circ$. Also, if $x = (x_1 \cdots x_n\, x_{n+1}) \in \mathbb{H}^n$ and $g(x) = y = (y_1 \cdots y_n\, y_{n+1})$, then
$y_{n+1} = -y \circ e_{n+1} = -g(x) \circ g(e_{n+1}) = -x \circ e_{n+1} = x_{n+1}$.
Thus, g preserves $(n+1)$th coordinates.

To prove $f \in O(\mathbb{R}^n)$, ~~let~~ let $x = (x_1 \cdots x_n)$ and $y = (y_1 \cdots y_n) \in \mathbb{R}^n$. Then $E(x) = (x_1 \cdots x_n, \varphi(x))$ and $E(y) = (y_1 \cdots y_n, \varphi(y))$. Since g preserves $(n+1)$th coordinates, then there exist $x' = (x_1' \cdots x_n')$ and $y' = (y_1' \cdots y_n') \in \mathbb{R}^n$ such that $g(E(x)) = (x_1' \cdots x_n', \varphi(x))$ and $g(E(y)) = (y_1' \cdots y_n', \varphi(y))$.

Since g preserves $\circ$, then $g(E(x)) \circ g(E(y)) = E(x) \circ E(y)$. Therefore, $\sum_{i=1}^n x_i' y_i' - \varphi(x)\varphi(y) = \sum_{i=1}^n x_i y_i - \varphi(x)\varphi(y)$. Thus, $\sum_{i=1}^n x_i' y_i' = \sum_{i=1}^n x_i y_i$. Now $f(x) = P \circ g \circ E(x) = (x_1' \cdots x_n')$ and $f(y) = P \circ g \circ E(y) = (y_1' \cdots y_n')$. Hence, $f(x) \cdot f(y) = \sum_{i=1}^n x_i' y_i' = \sum_{i=1}^n x_i y_i = x \cdot y$. This proves $f \in O(\mathbb{R}^n)$.

On to Step 2. Define $\bar{f} : \mathbb{M}^{n+1} \to \mathbb{M}^{n+1}$ by $\bar{f}(x_1 \cdots x_n\ x_{n+1}) = (f(x_1 \cdots x_n), x_{n+1})$. To prove $\bar{f} \in O^+(\mathbb{M}^{n+1})$, let $x = (x_1 \cdots x_{n+1})$ and $y = (y_1 \cdots y_{n+1}) \in \mathbb{M}^{n+1}$. Let $x' = (x_1 \cdots x_n)$ and $y' = (y_1 \cdots y_n) \in \mathbb{R}^n$. Then $\bar{f}(x) = (f(x'), x_{n+1})$ and $\bar{f}(y) = (f(y'), y_{n+1})$. Thus,

$\bar{f}(x) \circ \bar{f}(y) = f(x') \cdot f(y') - x_{n+1} y_{n+1} = x' \cdot y' - x_{n+1} y_{n+1}$
$= x \circ y$. Hence, $\bar{f} \in O(\mathbb{M}^n)$. Since $f \in O(\mathbb{R}^n)$, f is linear. Thus, $f(0) = 0$. Hence, $\bar{f}(e_{n+1}) =$

$(f(0), 1) = (0, 1) = e_{n+1}$. Thus $\bar{f} \in O^+(\mathbb{M}^{n+1})$.

Finally we take Step 3. Let $x = (x_1, \ldots, x_{n+1}) \in \mathbb{H}^n$. Let $x' = (x_1 \ldots x_n) \in \mathbb{R}$. Then $x_{n+1} = \varphi(x')$. So $x = (x', \varphi(x')) = E(x')$. Suppose $g(x) = y = (y_1 \ldots y_{n+1}) \in \mathbb{H}^n$. Let $y' = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Since g preserves $(n+1)$th coordinates, then $y_{n+1} = \varphi(x')$. So $g(x) = (y', \varphi'(x))$. $\bar{f}(x) = (f(x'), \varphi(x'))$ and $f(x') = P \circ g \circ E(x') = P \circ g(x) = P(y', \varphi'(x)) = y'$. Thus, $\bar{f}(x) = (y', \varphi(x')) = g(x)$. This proves $\bar{f} \mid \mathbb{H}^n = g$.

Now drop the assumption that $g(e_{n+1}) = e_{n+1}$. Since $g(e_{n+1}) \in \mathbb{H}^n$, then Corollary 1.14.a implies there is an $h \in O^+(\mathbb{M}^{n+1})$ such that $h(g(e_{n+1})) = e_{n+1}$. (h can be chosen to be a time-preserving reflection of $\mathbb{M}^{n+1}$.) Since $h \in O^+(\mathbb{M}^{n+1})$, then $h(\mathbb{H}^n) = \mathbb{H}^n$ and $\eta(h(x), h(y)) = \cosh^{-1}(-h(x) \circ h(y)) = \cosh^{-1}(-x \circ y) = \eta(x, y)$ for $x, y \in \mathbb{H}^n$. Thus, $h \mid \mathbb{H}^n \in \mathcal{I}(\mathbb{H}^n)$. Therefore, $h \circ g \in \mathcal{I}(\mathbb{H}^n)$ and $h \circ g(e_{n+1}) = e_{n+1}$. The preceding argument shows there is an $f \in O^+(\mathbb{M}^{n+1})$ such that $f \mid \mathbb{H}^n = h \circ g$. Therefore, $h^{-1} \circ f \in O^+(\mathbb{M}^{n+1})$ and $h^{-1} \circ f \mid \mathbb{H} = h^{-1} \circ h \circ g = g$. This completes the proof that $f \mapsto f \mid \mathbb{H}^n$: $O^+(\mathbb{M}^{n+1}) \to \mathcal{I}(\mathbb{H}^n)$ is an isomorphism. $\square$

Def Suppose $u \in \mathbb{H}^n$ and u, v is an orthonormal sequence in $\mathbb{M}^{n+1}$. (Then Corollary 1.3 implies $v \circ v = +1$.) Define $\Gamma_{uv}: \mathbb{R} \to \mathbb{M}^{n+1}$ by

$$\Gamma_{uv}(t) = \cosh(t)u + \sinh(t)v.$$

Theorem 2.4. If $u \in \mathbb{H}^n$ and u, v is an orthonormal sequence in $\mathbb{M}^{n+1}$, then $\Gamma_{uv}(\mathbb{R}) \subset \mathbb{H}^n$, $\Gamma_{uv}(0) = u$, $\Gamma'_{uv}(0) = v$ and $\Gamma_{uv}: \mathbb{R} \to \mathbb{H}^n$ is a geodesic.

Proof For $t \in \mathbb{R}$, $\Gamma_{uv}(t) \circ \Gamma_{uv}(t) = \cosh^2(t)(u \circ u) + \sinh^2(t)(v \circ v) = -(\cosh^2(t) - \sinh^2(t)) = -1$. Clearly, $\Gamma_{uv}(0) = u$. Since $\mathbb{R}$ is connected and Γ_{uv} is continuous, then $\Gamma_{uv}(\mathbb{R})$ is connected. Thus, $\Gamma_{uv}(\mathbb{R})$ lies in the component of $\{x \in \mathbb{M}^{n+1} : x \circ x = -1\}$ that contains u. Hence, $\Gamma_{uv}(\mathbb{R}) \subset \mathbb{H}^n$.

Since $\Gamma'_{uv}(t) = \sinh(t)u + \cosh(t)v$, then $\Gamma'_{uv}(0) = v$.

For $s, t \in \mathbb{R}$, $\eta(\Gamma_{uv}(s), \Gamma_{uv}(t)) =$
$\cosh^{-1}(-\Gamma_{uv}(s) \circ \Gamma_{uv}(t)) =$
$\cosh^{-1}(-(\cosh(s)\cosh(t)(u \circ u) + \sinh(s)\sinh(t)(v \circ v))) =$

$\cosh^{-1}(\cosh(s)\cosh(t) - \sinh(s)\sinh(t)) = \cosh^{-1}(\cosh(s-t)) = \cosh^{-1}(\cosh(|s-t|)) = |s-t|$. Thus, $\Gamma_{u,v}: \mathbb{R} \to \mathbb{H}^n$ is distance preserving and, thus, a geodesic. □

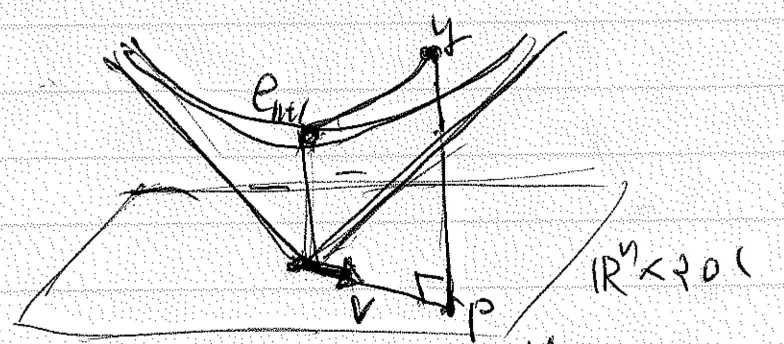
<u>Theorem 2.5</u> (Existence of geodesics) If $x, y \in \mathbb{H}^n$, then there is a $v \in \mathbb{M}^{n+1}$ such that x, v is an orthonormal sequence $\Gamma_{x,v}(0) = x$ and $\Gamma_{x,v}(\eta(x,y)) = y$. (Specifically, $v = p/|||p|||$ where $p = y + (x \circ y)x$.)

<u>Motivation for proof</u>. Since $\Gamma'_{x,v}(0) = v$, the problem is to choose the "direction" v of $\Gamma_{x,v}$ at 0 so that $\Gamma_{x,v}$ passes through y. Consider the special case $x = e_{n+1}$. To find v, project y orthogonally to $\mathbb{R}^n \times \{0\}$ to the point $p = y + (y \circ e_{n+1})e_{n+1}$ and let $v = p/|||p|||$. It remains to verify $\Gamma_{e_{n+1},v}(\eta(e_{n+1}, y)) = y$. Since it is no easier to verify this equation in the special case $x = e_{n+1}$ than in the general case $x \in \mathbb{H}^n$, we now turn to the general case.

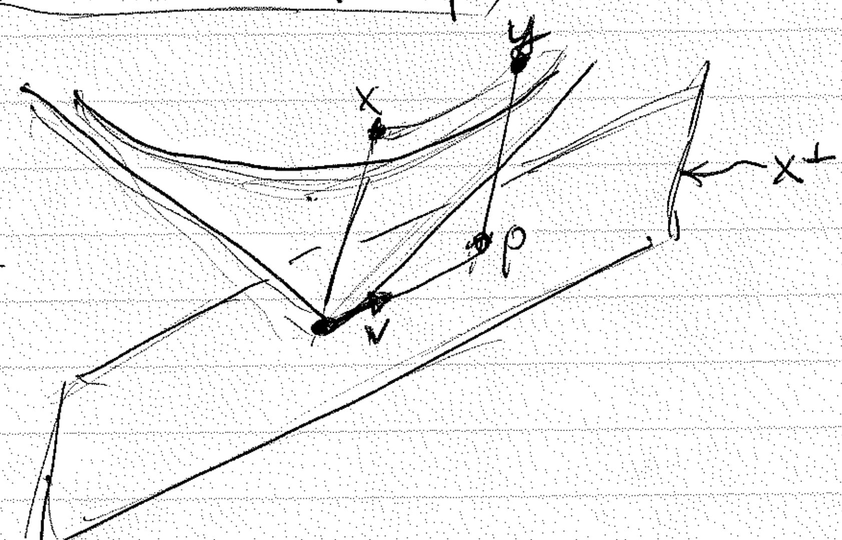
In the general case $x \in \mathbb{H}^n$, there is an $f \in O^+(\mathbb{M}^n)$ such that $f(x) = e_{n+1}$. Let $f(y) = y'$. $p' = y' + (y' \circ e_{n+1})e_{n+1}$ is the orthogonal projection of y' and $v' = p'/|||p'|||$.

The corresponding projection of $y = f^{-1}(y')$ is $p = f^{-1}(p') = f^{-1}(y') + (y' \circ e_{n+1}) f^{-1}(e_{n+1}) = y + (y \circ x)x$. Then $v = f^{-1}(v') = f^{-1}(p') / |||p'||| = p/|||p|||$. Note that if $x \neq e_{n+1}$, then p is <u>not</u> the orthogonal projection of y into $\mathbb{R}^n \times \{0\}$. Instead, p is the orthogonal projection of y into the "orthogonal complement" of x: $x^\perp = \{z \in \mathbb{M}^{n+1} : x \circ z = 0\}$.

Special case:
$x = e_{n+1}$



General case:



<u>Proof</u> If $x = y$, extend x to any orthonormal sequence x, v in $\mathbb{M}^{n+1}$. Then $\Gamma_{xv}(\eta(xy)) = \Gamma_{xv}(0) = x = y$.

Assume $x \neq y$. Let $p = y + (x \circ y)x$. Then $p \circ p = (x \circ y)^2 - 1 > 0$. (See page 2.2.) Thus, $|||p||| = \sqrt{(x \circ y)^2 - 1} > 0$. Let $v = p / |||p|||$. Since $x \circ p = 0$, then $x \circ v = 0$. Also $v \circ v = 1$. Hence, x, v is an orthonormal sequence in $\mathbb{M}^{n+1}$.

Now consider the geodesic $\Gamma_{x,v} : \mathbb{R} \to \mathbb{H}^n$. $\Gamma_{x,v}(0) = x$ by Theorem 2.4.

$$\Gamma_{x,v}(\eta(x,y)) = \cosh(\eta(x,y))\,x + \sinh(\eta(x,y))\,v.$$

Since $\eta(x,y) = \cosh^{-1}(-x \circ y)$, then $\cosh(\eta(x,y)) = -x \circ y$. For $r > 0$, since $\cosh^2(r) - \sinh^2(r) = 1$ and $\sinh(r) > 0$, then $\sinh(r) = \sqrt{\cosh^2(r) - 1}$.

Thus, $\sinh(\eta(x,y)) = \sqrt{\cosh^2(\eta(x,y)) - 1} = \sqrt{(-x \circ y)^2 - 1} = \sqrt{(x \circ y)^2 - 1} = |||p|||$.

Combining this information, we have

$\Gamma_{x,y}(\eta(x,y)) = (-x \circ y)x + |||p|||v =$
$-(x \circ y)x + p = -(x \circ y)x + (y + (x \circ y)x) = y.$ $\square$

<u>Theorem 2.6</u> (<u>Reparametrization of geodesics</u>) Suppose $u \in H^n$, u, v is an orthonormal sequence in M^{n+1} and $a \in R$. If $\Gamma_{uv}(a) = w$ and $\Gamma'_{uv}(a) = z$, then $w \in H^n$, w, z is an orthonormal sequence in M^{n+1} and $\Gamma_{uv}(t+a) = \Gamma_{wz}(t)$ for all $t \in R$.

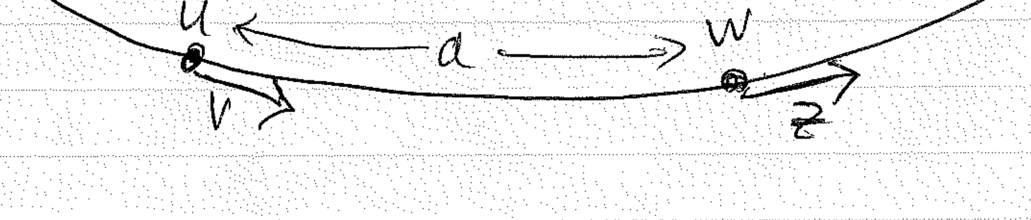
<u>Proof</u> Clearly $w = \Gamma_{uv}(a) \in H^n$, $w = \cosh(a)u + \sinh(a)v$ and $z = \sinh(a)u + \cosh(a)v$. Since $z \circ z = \sinh^2(a)(-1) + \cosh^2(a)(1) = 1$ and $w \circ z = \cosh(a)\sinh(a)(-1) + \sinh(a)\cosh(a)(+1) = 0$, then w, z is an orthonormal sequence in M^{n+1}.

$$\Gamma_{u,v}(t+a) = \cosh(t+a)u + \sinh(t+a)v =$$

$$(\cosh(t)\cosh(a) + \sinh(t)\sinh(a))u$$
$$+ (\sinh(t)\cosh(a) + \cosh(t)\sinh(a))v =$$

$$\cosh(t)(\cosh(a)u + \sinh(a)v)$$
$$+ \sinh(t)(\sinh(a)u + \cosh(a)v) =$$

$$\cosh(t)w + \sinh(t)z = \Gamma_{wz}(t). \quad \square$$



The following result generalizes Theorem 2.5

<u>Corollary 2.7</u> If $x, y \in \mathbb{H}^n$ and $a \in \mathbb{R}$, then there is a $w \in \mathbb{H}^n$ and an orthonormal sequence w, z in $\mathbb{M}^{n+1}$ such that $\Gamma_{wz}(a) = x$ and $\Gamma_{wz}(a + \eta(x, y)) = y$.

<u>Homework Problem 2.2</u>- Prove Corollary 2.7-

9/30 → <u>Theorem 2.8</u> (Uniqueness of $\Gamma_{u,v}$'s.)
If $u, w \in \mathbb{H}^n$, u, v and w, z are orthonormal sequences in $\mathbb{M}^{n+1}$, and $a < a'$ and $b < b'$ such that $\Gamma_{uv}(a) = \Gamma_{wz}(b)$ and $\Gamma_{uv}(a') = \Gamma_{wz}(b')$, then $a' - a = b' - b$ and $\Gamma_{uv}(a + t) = \Gamma_{wz}(b + t)$ for all $t \in \mathbb{R}$. Furthermore, in the special case that $a = b$, then $u = w$ and $v = z$ and $\Gamma_{uv} = \Gamma_{wz}$.

<u>Proof</u> Since Γ_{uv} and Γ_{wz} are distance preserving, then $a' - a = |a - a'| = \eta(\Gamma_{uv}(a), \Gamma_{uv}(a')) = \eta(\Gamma_{wz}(b), \Gamma_{wz}(b')) = |b - b'| = b' - b$. Let $r = a' - a = b' - b$. Since $\Gamma_{uv}(a') = \Gamma_{wz}(b')$, then $\Gamma_{uv}(a + r) = \Gamma_{wz}(b + r)$. Hence,
$$\cosh(a+r)u + \sinh(a+r)v = \cosh(b+r)w + \sinh(b+r)z.$$
Therefore,

$$(\cosh(a)\cosh(r)+\sinh(a)\sinh(r))u$$
$$+(\sinh(a)\cosh(r)+\cosh(a)\sinh(r))v=$$
$$(\cosh(b)\cosh(r)+\sinh(b)\sinh(r))w$$
$$+(\sinh(b)\cosh(r)+\cosh(b)\sinh(r))z.$$

Thus,

$$\cosh(r)(\cosh(a)u+\sinh(a)v)+\sinh(r)(\sinh(a)u+\cosh(a)v)$$
$$=\cosh(r)(\cosh(b)w+\sinh(b)z)+\sinh(r)(\sinh(b)w+\cosh(b)z)$$

Since $\Gamma'_{uv}(t)=\sinh(t)u+\cosh(t)v$, we have

$$\cosh(r)\Gamma_{uv}(a)+\sinh(r)\Gamma'_{uv}(a)=\cosh(r)\Gamma_{wz}(b)+\sinh(r)\Gamma'_{wz}(b).$$

Since $\Gamma_{uv}(a)=\Gamma_{wz}(b)$ and $\sinh(r)>0$,

we obtain $\Gamma'_{uv}(a)=\Gamma'_{wz}(b)$.

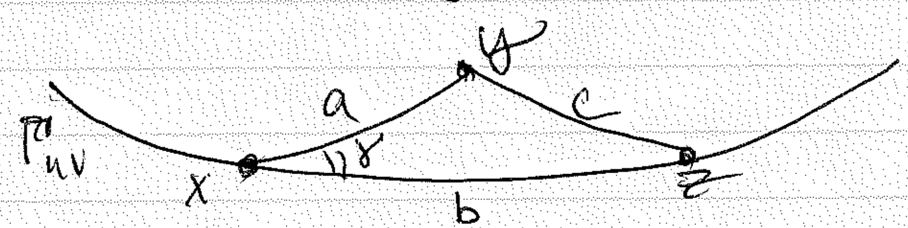
Let $x=\Gamma_{uv}(a)=\Gamma_{wz}(b)$ and $y=\Gamma'_{uv}(a)=\Gamma'_{wz}(b)$.

Then Theorem 2.6 implies $x\in\mathbb{H}^n$, x,y is an orthonormal sequence in $\mathbb{M}^{n+1}$ and $\Gamma_{uv}(a+t)=\Gamma_{xy}(t)=\Gamma_{wz}(b+t)$ for all $t\in\mathbb{R}$.

In the case that $a=b$, we have $\Gamma_{uv}(t)=\Gamma_{wz}(t)$ for all $t\in\mathbb{R}$. Hence, $u=\Gamma_{uv}(0)=\Gamma_{wz}(0)=w$ and $v=\Gamma'_{uv}(0)=\Gamma'_{wz}(0)=z$. $\square$

Our next goal is to prove that the Γ_{uv}'s are the only geodesics on $\mathbb{H}^n$. The next lemma is a crucial step in this proof.

<u>Lemma 2.9</u>. Suppose $u \in \mathbb{H}^n$ and u, v is an orthonormal sequence in $\mathbb{M}^{n+1}$, $r < s$, $\Gamma_{uv}(r) = x$, $\Gamma_{uv}(s) = z$ and $y \in \mathbb{H}^n$ such that $\eta(x,y) + \eta(y z) = \eta(x,z)$. Then
$$\Gamma_{uv}(r + \eta(xy)) = y$$



<u>Proof</u> If $x = y$, the proof is trivial. If $x = z$, then $\eta(xz) = 0$ forces $\eta(xy) = 0$ and the proof is again trivial. So we can assume $x \neq y$ and $x \neq z$.

We recall the definition of $m(\angle yxz)$. Let $p = y + (y \circ x)x$ and $q = z + (z \circ x)x$. Then $p \circ p = (y \circ x)^2 - 1 > 0$ and $q \circ q = (z \circ x)^2 - 1 > 0$ (page 2.2). Let $w = p / |||p|||$ and $w = q / |||q|||$. Then $|w \circ w| \le 1$ (page 2.3), and we defined
$$m(\angle yxz) = \cos^{-1}(w \circ w) \in [0, \pi].$$

2.21

Let $a=\eta(xy)$, $b=\eta(xz)$, $c=\eta(yz)$ and $\gamma=m(\angle yxz)$. Then $a+c=b$. The Hyperbolic Law of Cosines (Theorem 2.1) implies

$$\cosh(c)=\cosh(a)\cosh(b)-\sinh(a)\sinh(b)\cos(\gamma).$$

Also

$$\cosh(c)=\cosh(b-a)=\cosh(a)\cosh(b)-\sinh(a)\sinh(b).$$

Thus, $\sinh(a)\sinh(b)\cos(\gamma)=\sinh(a)\sinh(b)$.
Since $x\neq y$ and $x\neq z$, then $a>0$ and $b>0$.
Hence, $\sinh(a)>0$ and $\sinh(b)>0$.
Therefore $\cos(\gamma)=1$. Thus $r\circ w=1$.

We claim $r=w$.

<u>Proof</u> Since $x\circ p=0=x\circ q$, then $x\circ r=0=x\circ w$. Also $r\circ r=1=w\circ w$. Thus, x, r and x, w are orthonormal sequences in $\mathbb{M}^{n+1}$. We invoke Theorem 1.15 to extend x, r to an orthonormal basis $x, r, r_3, \ldots, r_{n+1}$ for $\mathbb{M}^{n+1}$. Since $x\circ x=-1$, then Theorem 1.7 implies $r_i\circ r_i=+1$ for $3\le i\le n+1$. There are $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\in\mathbb{R}$ such that $w=\alpha_1 x+\alpha_2 r+\sum_{i=3}^{n+1}\alpha_i r_i$.
Then $0=w\circ x=\alpha_1(x\circ x)=-\alpha_1$ and
$1=w\circ r=\alpha_2(r\circ r)=\alpha_2$. Thus $\alpha_1=0$ and $\alpha_2=1$.
So $w=r+\sum_{i=3}^{n+1}\alpha_i r_i$. Therefore,

$1 = w \circ w = r \circ r + \sum_{i=3}^{n+1} \alpha_i^2 (r_i \circ r_i) = 1 + \sum_{i=3}^{n+1} \alpha_i^2$

Hence, $\alpha_i = 0$ for $3 \le i \le n+1$. Thus, $w = r$.

Since $x, y, z \in \mathbb{H}^n$, $p = y + (y \circ x)x$, $q = z + (z \circ x)x$, $r = p / |||p|||$ and $w = q / |||q|||$, then $\Gamma_{x,r}(\eta(x,y)) = y$, $\Gamma_{x,w}(\eta(x,z)) = z$ and $\Gamma_{x,w}(0) = x$. (See Theorem 2.5.)
Since $w = r$, then $\Gamma_{x,w}(\eta(x,y)) = y$.

Thus, we have $\Gamma_{uv}(r) = x = \Gamma_{x,w}(0)$ and $\Gamma_{uv}(s) = z = \Gamma_{xw}(\eta(x,z))$. Hence, Theorem 2.8 implies $\Gamma_{uv}(r+t) = \Gamma_{x,w}(t)$ for all $t \in \mathbb{R}$.
In particular, $\Gamma_{uv}(r + \eta(x,y)) = \Gamma_{x,w}(\eta(x,y)) = y$. $\square$

Now we prove the Γ_{uv}'s are the only geodesics in $\mathbb{H}^n$.

Theorem 2.10. If J is an interval in $\mathbb{R}$ and $f : J \to \mathbb{H}^n$ is a geodesic, then there is a $u \in \mathbb{H}^n$ and an orthonormal sequence u, v in $\mathbb{M}^{n+1}$ such that $\Gamma_{uv} | J = f$.

Proof First assume $J = [a, c]$. Let $x = f(a)$ and $z = f(c)$. Then $\eta(x,z) = \eta(f(a), f(c)) = |a-c| = c-a$. So $c = a + \eta(xz)$.

Corollary 2.7 implies there is a $u \in \mathbb{H}^n$ and an orthonormal sequence u, v in $\mathbb{M}^{n+1}$ such that $\Gamma_{uv}(a) = x$ and $\Gamma_{uv}(c) = \Gamma_{uv}(a + \eta(xz)) = z$.

Suppose $a < b < c$ and $y = f(b)$. Then $\eta(x,y) = \eta(f(a), f(b)) = |a-b| = b-a$ and $\eta(y,z) = \eta(f(b), f(c)) = |b-c| = c-b$. Hence, $\eta(xy) + \eta(yz) = (b-a) + (c-b) = c-a = \eta(xz)$. Also $b = a + \eta(xy)$. Therefore, Lemma 2.9 implies $\Gamma_{uv}(b) = \Gamma_{uv}(a + \eta(xy)) = y = f(b)$. This proves $\Gamma_{uv} | J = f$.

Now assume J is an arbitrary interval in $\mathbb{R}$. Choose $a<b$ so that $[a,b] \subset J$. The preceding argument shows there exists a $u \in \mathbb{H}^n$ and an orthonormal sequence u, v in $\mathbb{M}^{n+1}$ such that $\Gamma_{uv}|[a,b] = f|[a,b]$.

We will prove $\Gamma_{uv}|J = f$. Let $t \in J$. There exists $c<d$ such that $[a,b]\cup\{t\} \subset [c,d] \subset J$. The preceding argument shows there is a $w \in \mathbb{H}^n$ and an orthonormal sequence w, z in $\mathbb{M}^{n+1}$ such that $\Gamma_{wz}|[c,d] = f|[c,d]$. Then $\Gamma_{uv}(a) = f(a) = \Gamma_{wz}(a)$ and $\Gamma_{uv}(b) = f(b) = \Gamma_{wz}(b)$. Therefore Theorem 2.8 implies $\Gamma_{uv} = \Gamma_{wz}$. Thus, $f(t) = \Gamma_{wz}(t) = \Gamma_{uv}(t)$. This proves $\Gamma_{uv}|J = f$. $\square$

Related arguments show:

<u>Theorem 2.11.</u> If J is an interval in $\mathbb{R}$ and $f: J \to \mathbb{H}^n$ is a local geodesic, then $f: J \to \mathbb{H}^n$ is a geodesic.

Due 10/7 $\longrightarrow$ <u>Homework Problem 2.3</u> - Prove Theorem 2.11.

10/2 → Theorem 2.12. a) If $u \in \mathbb{H}^n$, u, v is an orthonormal sequence in $\mathbb{M}^{n+1}$ and V is the vector subspace of $\mathbb{M}^{n+1}$ spanned by $\{u, v\}$, then $\dim(V) = 2$ and $V \cap \mathbb{H}^n = \Gamma_{uv}(\mathbb{R})$.

b) Conversely, if V is a 2-dimensional vector subspace of $\mathbb{M}^{n+1}$ such that $V \cap \mathbb{H}^n \neq \emptyset$, then there is a $u \in \mathbb{H}^n$ and an orthonormal sequence u, v in $\mathbb{M}^{n+1}$ such that $V \cap \mathbb{H}^n = \Gamma_{uv}(\mathbb{R})$.

Proof of a). Assume $u \in \mathbb{H}^n$, u, v is an orthonormal sequence in $\mathbb{M}^{n+1}$, and V is the vector subspace of $\mathbb{M}^{n+1}$ spanned by $\{u, v\}$. Lemma 1.6 implies u, v are linearly independent. Hence, $\dim(V) = 2$.

Since V contains all linear combinations of u, v and $\Gamma_{uv}(t) = \cosh(t)u + \sinh(t)v$ for each $t \in \mathbb{R}$, then obviously $\Gamma_{uv}(\mathbb{R}) \subset V \cap \mathbb{H}^n$.

Suppose $x \in V \cap \mathbb{H}^n$. Then $x = au + bv$ for some $a, b \in \mathbb{R}$. Since sinh maps $\mathbb{R}$ onto $\mathbb{R}$, then $b = \sinh(t)$ for some $t \in \mathbb{R}$. Since $x \in \mathbb{H}^n$, then $-1 = x \circ x = -a^2 + b^2$. Thus, $a^2 = b^2 + 1 = \sinh^2(t) + 1 = \cosh^2(t)$. So $a = \pm\cosh(t)$. Since $u, x \in \mathbb{H}^n$, then Theorem 1.4, a implies $-a = u \circ x \leq -|||u|||\,|||x||| = -1$.

Thus $a \geq 1$. So $a = \cosh(t)$. Hence, $x = \cosh(t)u + \sinh(t)v = \Gamma_{uv}(t)$. This proves $V \cap \mathbb{H}^n \subset \Gamma_{uv}(\mathbb{R})$.

b) Assume V is a 2-dimensional vector subspace of $\mathbb{M}^{n+1}$ such that $V \cap \mathbb{H}^n \neq \emptyset$. Let $u \in V \cap \mathbb{H}^n$. Since V is 2-dimensional, there is an $x \in V$ such that x is not a scalar multiple of u. Let $p = x + (x \circ u)u$. Then $p \in V$, $p \neq 0$ and $u \circ p = 0$. Since $u \circ u = -1$, then Corollary 1.3 implies $p \circ p > 0$. Let $v = p / |||p|||$. Then $v \in V$, $u \circ v = 0$ and $v \circ v = 1$. Thus, u, v is an orthonormal sequence in $\mathbb{M}^{n+1}$. Hence, Lemma 1.6 implies u, v are linearly independent. Since $u, v \in V$ and $\dim(V) = 2$, then $\{u, v\}$ spans V. Now part a) of this theorem implies $V \cap \mathbb{H}^n = \Gamma_{uv}(\mathbb{R})$. $\square$

10/7 → Lemma 2.13 If x and y are distinct points of $\mathbb{H}^n$, then x and y are linearly independent, and, hence, the vector subspace spanned by x, y is 2-dimensional.

Proof Suppose $a, b \in \mathbb{R}$ such that $ax+by=0$. Then $0 = x \circ (ax+by) = -a + b(x \circ y)$ and $0 = y \circ (ax+by) = a(x \circ y) - b$. Hence, $a = b(x \circ y)$ and $b = a(x \circ y)$. Thus $a = a(x \circ y)^2$ and $b = b(x \circ y)^2$. If $a \neq 0$, then $(x \circ y)^2 = 1$. Since $x \circ y \leq -1$ by Theorem 1.4.a, then $x \circ y = -1$. Hence, Theorem 1.4.a implies $x = y$, a contradiction. Consequently, $a = 0$. Similarly, $b = 0$. Thus, x and y are linearly independent. □

Theorem 2.14. Suppose x and y are distinct points of $\mathbb{H}^n$ and V is the 2-dimensional vector subspace of $\mathbb{M}^{n+1}$ spanned by x, y.

a) If $f: \mathbb{R} \to \mathbb{H}^n$ is a geodesic that passes through x and y, then $f(\mathbb{R}) = V \cap \mathbb{H}^n$.

b) If $f, g: \mathbb{R} \to \mathbb{H}^n$ are geodesics that pass through x and y, then $f(\mathbb{R}) = g(\mathbb{R})$ and there is an isometry $h: \mathbb{R} \to \mathbb{R}$ such that $g = f \circ h$.

Recall that $h: \mathbb{R} \to \mathbb{R}$ is an isometry if and only if either h is a translation ($h(t) = t + a$ for some $a \in \mathbb{R}$) or h is a reflection ($h(t) = 2a - t$ for some $a \in \mathbb{R}$).

Proof of a). Theorem 2.10 implies there is a $u \in \mathbb{H}^n$ and an orthonormal sequence u, v in $\mathbb{M}^{n+1}$ such that $f = \Gamma_{uv}$. Theorem 2.12.a implies that if W is the 2-dimensional vector subspace of $\mathbb{M}^{n+1}$ spanned by u, v, then $W \cap \mathbb{H}^n = \Gamma_{uv}(\mathbb{R})$. Hence, $W \cap \mathbb{H}^n = f(\mathbb{R})$. So $x, y \in W$. Since x, y are linearly independent by Lemma 2.13 and $\dim(W) = 2$, then x, y span W. So $W = V$. Consequently $V \cap \mathbb{H}^n = f(\mathbb{R})$. $\square$

Proof of b) Part a) of this theorem implies $f(\mathbb{R}) = V \cap \mathbb{H}^n = g(\mathbb{R})$. Define $h: \mathbb{R} \to \mathbb{R}$ by $h = f^{-1} \circ g$. Then h is an isometry and $g = f \circ h$. $\square$

Observation $\mathbb{H}^1 = \{x \in \mathbb{M}^2 : x \circ x = -1$ and $x_2 > 0\}$. Since $\Gamma_{e_2, e_1} : \mathbb{R} \to \mathbb{H}^1$ is an isometry, then $\mathbb{H}^1$ is isometric to $\mathbb{R}$.

Theorem 2.15. Let $T \subset \mathbb{H}^n$. The following statements are equivalent.

a) T is a totally geodesic subset of $\mathbb{H}^n$.

b) T is isometric to $\mathbb{H}^m$ for some m, $1 \le m \le n$.

c) There is a vector subspace V of $\mathbb{M}^{n+1}$ such that $\dim(V) \ge 2$ and $V \cap \mathbb{H}^n = T$.

Proof that c) implies b). Assume V is a vector subspace of $\mathbb{M}^{n+1}$, $\dim(V) \ge 2$ and $V \cap \mathbb{H}^n = T$. Choose $u_0 \in T$. Let $W = \{v \in V : u_0 \circ v = 0\}$. Then W is a vector subspace of V.

We claim $W \ne \{0\}$. Since $\dim(V) \ge 2$, then there is an $x \in V$ which is not a scalar multiple of u_0. Let $p = x + (x \circ u_0) u_0$. Then $p \in V$, $p \ne 0$ and $u_0 \circ p = 0$. Thus $p \in W$, proving $W \ne \{0\}$.

If $q \in W$ and $q \ne 0$, then Corollary 1.3

implies $q \circ q > 0$ because $u_0 \circ u_1 = -1$ and $u_0 \circ q = 0$. Thus, $\circ$ restricts to a positive definite symmetric bilinear form — or inner product — on W. Thus, W has an orthonormal basis $w_1, \ldots, w_m$ for some $m \geq 1$ and $w_i \circ w_i = +1$ for $1 \leq i \leq m$. Hence, $w_1, \ldots, w_m, u_0$ is an orthonormal sequence in M^{n+1}. Therefore, $w_1, \ldots, w_m, u_0$ is linearly independent, by Lemma 1.6. Hence $m+1 \leq n+1$. So $m \leq n$.

We claim that $w_1, \ldots, w_m, u_0$ is an orthonormal basis for V. It remains to show $w_1, \ldots, w_m, u_0$ spans V. To this end, let $x \in V$. Let $p = x + (x \circ u_0) u_0$. Then $p \in V$ and $p \circ u_0 = 0$. Thus, $p \in W$. Therefore, $p = \sum_{i=1}^m a_i w_i$ for some $a_1, \ldots, a_m \in R$. Then $x = -(x \circ u_0) u_0 + \sum_{i=1}^m a_i w_i$. This proves that $w_1 \ldots w_m, u_0$ spans V.

Define $f : M^{m+1} \to V$ by $f(x_1 \ldots x_{m+1}) = \sum_{i=1}^m x_i w_i + x_{m+1} u_0$. Then f is clearly a linear isomorphism and f preserves $\circ$. In other words, for $x = (x_1 \ldots x_{m+1})$ and $y = (y_1 \ldots y_{m+1}) \in M^{m+1}$, $f(x) \circ f(y) = \left(\sum_{i=1}^m x_i w_i + x_{m+1} u_0\right) \circ \left(\sum_{i=1}^m y_i w_i + y_{m+1} u_0\right)$
$= \sum_{i=1}^M x_i y_i - x_{m+1} y_{m+1} = x \circ y$.

We claim that $f|\mathbb{H}^m$ maps $\mathbb{H}^m$ isometrically onto T.

If $x \in \mathbb{H}^m$, then $f(x) \circ f(x) = x \circ x = -1$. Thus, $f(\mathbb{H}^m) \subset \{y \in \mathbb{M}^{n+1} : y \circ y = -1\}$. Also $f(e_{m+1}) = u_0 \in \mathbb{H}^m$. Since $\mathbb{H}^m$ is connected, then $f(\mathbb{H}^m)$ is contained in the component of $\{y \in \mathbb{M}^{n+1} : y \circ y = -1\}$ that contains u_0. Thus $f(\mathbb{H}^m) \subset \mathbb{H}^n$. Hence, $f(\mathbb{H}^m) \subset V \cap \mathbb{H}^n = T$.

Suppose $y \in T = V \cap \mathbb{H}^n$. Then $y = \sum_{i=1}^n x_i w_i + x_{m+1} u_0$ for some $x_1, \dots, x_{m+1} \in \mathbb{R}$. Let $x = (x_1, \dots, x_{m+1}) \in \mathbb{M}^{m+1}$. Since $y \in \mathbb{H}^n$, then $-1 = y \circ y = \sum_{i=1}^n x_i^2 - x_{m+1}^2 = x \circ x$. Also since $u_0, y \in \mathbb{H}^n$, then Theorem 1.4.a implies $-x_{m+1} = u_0 \circ y \le -|||u_0|||\,|||y||| = -1$. Thus, $x_{m+1} \geq 1$. It follows that $x \in \mathbb{H}^m$. Clearly $f(x) = y$. This proves $T \subset f(\mathbb{H}^m)$.

Since $f : \mathbb{M}^{m+1} \to V$ is a linear isomorphism, then $f|\mathbb{H}^m : \mathbb{H}^m \to T$ is a bijection.

For $x, y \in \mathbb{H}^m$, $\eta(f(x), f(y)) = \cosh^{-1}(-f(x) \circ f(y)) = \cosh^{-1}(-x \circ y) = \eta(x, y)$. We have proved $f|\mathbb{H}^m : \mathbb{H}^m \to T$ is an isometry. $\square$

<u>Proof that b) implies a)</u>. Assume there is an isometry $g : \mathbb{H}^m \to T$. Let $x, y \in T$. Let $x' = g^{-1}(x)$ and $y' = g^{-1}(y) \in \mathbb{H}^m$. Theorem 2.5 implies there is a geodesic $f : \mathbb{R} \to \mathbb{H}^m$ such that $x', y' \in f(\mathbb{R})$. Since g is an isometry, then $g \circ f : \mathbb{R} \to T$ is a geodesic and $x = g(x')$ and $y = g(y') \in g \circ f(\mathbb{R})$. This proves T is a totally geodesic subset of $\mathbb{H}^n$. $\square$

<u>Proof that a) implies c)</u>. Assume T is a totally geodesic subset of $\mathbb{H}^n$. Let V be the set of all linear combinations of elements of T. V is obviously a vector subspace of $\mathbb{M}^{n+1}$ and $T \subset V$. Hence, $T \subset V \cap \mathbb{H}^n$. We will prove $T = V \cap \mathbb{H}^n$.

Let $M^{n+1}_+ = \{x \in M^{n+1} : x_{n+1} > 0\}$ and define the <u>radial retraction</u> $R : M^{n+1}_+ \to \mathbb{R}^n \times \{1\}$ by

$$R(x) = \left(\frac{1}{x_{n+1}}\right) x$$

for $x = (x_1, \dots x_{n+1}) \in M^{n+1}_+$. Then $R | \mathbb{R}^n \times \{1\} = \mathrm{id}$.
Let $U^n = \{x \in \mathbb{R}^n : \|x\| < 1\}$.

<u>Claim A.</u> $R | \mathbb{H}^n : \mathbb{H}^n \to U^n \times \{1\}$ is a bijection.

<u>Proof of Claim A.</u> Let $x = (x_1, \dots x_{n+1}) \in \mathbb{H}^n$. Let $y = (x_1, \dots x_n)$. Then $-1 = x \circ x = \|y\|^2 - x_{n+1}^2$. Thus, $x_{n+1} = \sqrt{\|y\|^2 + 1} \geq 1$ and $\frac{\|y\|^2}{x_{n+1}^2} = 1 - \frac{1}{x_{n+1}^2} < 1$.
Hence, $\frac{y}{x_{n+1}} \in U^n$. So $R(x) = \left(\frac{y}{x_{n+1}}, 1\right) \in U^n \times \{1\}$.
This proves $R(\mathbb{H}^n) \subset U^n \times \{1\}$.

Let $y \in U^n$. Then $\|y\| < 1$. Let $t = \frac{1}{\sqrt{1 - \|y\|^2}}$. Let $x = t(y, 1) \in M^{n+1}_+$.
Then $x \circ x = t^2(y^2 - 1) = -1$. So $x \in \mathbb{H}^n$.
Also $R(x) = (y, 1)$. This proves $R(\mathbb{H}^n) \supset U^n \times \{1\}$.

Suppose x and $z \in \mathbb{H}^n$ and $R(x) = R(z)$. Then $\left(\frac{1}{x_{n+1}}\right)x = \left(\frac{1}{z_{n+1}}\right)z$.

Therefore, x and z are linearly dependent. Hence, Lemma 2.13 implies $x = z$. This proves $R|\mathbb{H}^n$ is injective.

We conclude that $R|\mathbb{H}^n : \mathbb{H}^n \to U^n \times \{1\}$ is a bijection. $\square$

To prove a) implies c), we will work with $R(T)$ rather than T because the geometry of $R(T)$ is "flatter" and, therefore, simpler than that of T. The key property of $R(T)$, established in the next lemma, is that it is a "relatively affine" subspace of $U^n \times \{1\}$.

<u>Lemma 2.16</u> If $y_1, \ldots, y_m \in R(T)$ and $a_1, \ldots, a_m \in \mathbb{R}$ such that $\sum_{i=1}^m a_i = 1$ and $\sum_{i=1}^m a_i y_i \in U^n \times \{1\}$, then $\sum_{i=1}^m a_i y_i \in R(T)$.

<u>Proof of Lemma 2.16</u>. We will prove Lemma 2.16 by induction on m. The lemma is obviously valid for $m = 1$.

We need to prove the $m=2$ case of the lemma. First, however, we introduce some terminology and establish a claim.

For $y_1 \neq y_2$ in M^{n+1}, let

$$L(y_1, y_2) = \{a_1 y_1 + a_2 y_2 : a_1, a_2 \in \mathbb{R} \text{ and } a_1 + a_2 = 1\}.$$

Thus, $L(y_1, y_2)$ is the straight line through y_1 and y_2. Observe that if $y_1, y_2 \in \mathbb{R}^n \times \{1\}$, then $L(y_1, y_2) \subset \mathbb{R}^n \times \{1\}$ and $R \mid L(y_1, y_2) = id$.

For $x_1, x_2 \in M^{n+1}$, let $W(x_1, x_2)$ denote the vector subspace of M^{n+1} spanned by x_1, x_2. In other words, $W(x_1, x_2)$ is the set of all linear combinations of x_1, x_2.

Claim B. If $y_1 \neq y_2 \in \mathbb{R}^n \times \{1\}$ and $x_1, x_2 \in M^{n+1}_+$ such that $R(x_1) = y_1$ and $R(x_2) = y_2$, then $R^{-1}(L(y_1, y_2)) \subset W(x_1, x_2)$.

10/9 → Proof of Claim B. For $i = 1, 2$, let t_i be the $(n+1)$th coordinate of x_i. Then $y_i = (1/t_i) x_i$ for $i = 1, 2$. Let $x \in R^{-1}(L(y_1, y_2))$. Then there are $a_1, a_2 \in \mathbb{R}$ such that $a_1 + a_2 = 1$ and $R(x) = a_1 y_1 + a_2 y_2$.

Let t be the $(n+1)$th coordinate of x. Then $t > 0$ (because $x \in \mathbb{M}^{n+1}_t$) and

$$\left(\frac{1}{t}\right)x = R(x) = a_1 y_1 + a_2 y_2 .$$

Thus $x = t a_1 y_1 + t a_2 y_2 = \left(\frac{t a_1}{t_1}\right)x_1 + \left(\frac{t a_2}{t_2}\right)x_2$.

Hence, $x \in W(x_1, x_2)$. We conclude that $R^{-1}(L(y_1, y_2)) \subset W(x_1, x_2)$. $\square$

<u>Homework Problem 2.4.</u> Prove: if $y_1 \neq y_2 \in \mathbb{R}^n \times \{1\}$ and $x_1, x_2 \in \mathbb{M}^{n+1}_+$ such that $R(x_1) = y_1$ and $R(x_2) = y_2$, then $R^{-1}(L(y_1, y_2)) = W(x_1, x_2) \cap \mathbb{M}^{n+1}_+$.

We now prove the $m=2$ case of the lemma. Suppose $y_1, y_2 \in R(T)$, $a_1, a_2 \in \mathbb{R}$, $a_1 + a_2 = 1$ and $a_1 y_1 + a_2 y_2 \in \mathbb{U}^n \times \{1\}$. We may assume $y_1 \neq y_2$. Let $y = a_1 y_1 + a_2 y_2$. Then $y \in L(y_1, y_2) \cap (\mathbb{U}^n \times \{1\})$.

There are $x_1, x_2 \in T$ such that $R(x_1) = y_1$ and $R(x_2) = y_2$. Since $y_1 \neq y_2$, then $x_1 \neq x_2$. Hence, x_1 and x_2 are linearly independent by Lemma 2.13. Therefore, $W(x_1, x_2)$ is a 2-dimensional vector space of $\mathbb{M}^{n+1}$.

Since $R(\mathbb{H}^n) = \mathbb{U}^n \times \{1\}$, then there is an $x \in \mathbb{H}^n$ such that $R(x) = y$. Hence, $x \in R^{-1}(L(y_1, y_2))$. So Claim B implies $x \in W(x_1, x_2)$. Therefore, $x \in W(x_1,x_2) \cap \mathbb{H}^n$. Since $x_1, x_2 \in T$ and T is a totally geodesic space, then there is a geodesic $f : \mathbb{R} \to \mathbb{H}^n$ such that $x_1, x_2 \in f(\mathbb{R}) \subset T$. Theorem 2.14.a implies $f(\mathbb{R}) = W(x_1, x_2) \cap \mathbb{H}^n$. Thus, $x \in f(\mathbb{R}) \subset T$. Hence, $y = R(x) \in R(T)$. This completes the proof of the $m=2$ case.

The following fact helps with the proof of the general induction step.

<u>Claim C</u>. If $y_1, \ldots, y_m \in U^n \times \{1\}$, $a_1, \ldots, a_m \in [0, \infty)$ and $\sum_{i=1}^m a_i = 1$, then $\sum_{i=1}^m a_i y_i \in U^n \times \{1\}$. (This claim is simply the assertion that $U^n \times \{1\}$ is convex.)

<u>Proof of Claim C</u> Let $z_i \in U^n$ such that $y_i = (z_i, 1)$, for $1 \le i \le m$. Then each $\|z_i\| < 1$. Also $\sum_{i=1}^m a_i y_i = (\sum_{i=1}^m a_i z_i, 1)$.
$\| \sum_{i=1}^m a_i z_i \| \le \sum_{i=1}^m a_i \|z_i\| < \sum_{i=1}^m a_i = 1$.
Hence, $\sum_{i=1}^m a_i z_i \in U^n$. Therefore, $\sum_{i=1}^m a_i y_i \in U^n \times \{1\}$. $\square$

Let $m \geq 2$ and assume the lemma holds for all sequences $y_1, \ldots, y_k \in R(T)$ and $a_1, \ldots, a_k \in \mathbb{R}$ of length $k \leq m$.

Let $y_1, \ldots, y_{m+1} \in R(T)$ and $a_1, \ldots, a_{m+1} \in \mathbb{R}$ such that $\sum_{i=1}^{m+1} a_i = 1$ and $\sum_{i=1}^{m+1} a_i y_i \in U^n \times \{1\}$.

We may assume $a_i \neq 0$ for $1 \leq i \leq m+1$.

First consider the case in which all $a_i > 0$. Let $z = \sum_{i=1}^{m} \left(\frac{a_i}{1-a_{m+1}}\right) y_i$. Since each $\frac{a_i}{1-a_{m+1}} > 0$ and $\sum_{i=1}^{m} \frac{a_i}{1-a_{m+1}} = 1$, then $z \in U^n \times \{1\}$ by Claim C. Hence, the inductive hypothesis implies $z \in R(T)$. Since $(1-a_{m+1})z + a_{m+1}y_{m+1} = \sum_{i=1}^{m+1} a_i y_i \in U^n \times \{1\}$, then the $m=2$ case of the lemma implies $(1-a_{m+1})z + a_{m+1}y_{m+1} \in R(T)$. Thus, $\sum_{i=1}^{m+1} a_i y_i \in R(T)$.

Now consider the remaining case in which some $a_i < 0$. (Since $\sum_{i=1}^{m+1} a_i = 1$, then some $a_i > 0$.) Reorder $y_1, \ldots, y_{m+1}$ and $a_1, \ldots, a_{m+1}$ if necessary so that there is an integer k such that $1 \le k \le m$, $a_i > 0$ for $1 \le i \le k$ and $a_i < 0$ for $k+1 \le i \le m+1$. Let $b = \sum_{i=1}^{k} a_i$ and $c = \sum_{i=k+1}^{m+1} a_i$. Then $\frac{a_i}{b} > 0$ for $1 \le i \le k$ and $\sum_{i=1}^{k} \frac{a_i}{b} = 1$; and $\frac{a_i}{c} > 0$ for $k+1 \le i \le m+1$ and $\sum_{i=k+1}^{m+1} \frac{a_i}{c} = 1$. Hence, $\sum_{i=1}^{k} \left(\frac{a_i}{b}\right) y_i$ and $\sum_{i=k+1}^{m+1} \left(\frac{a_i}{c}\right) y_i \in U^n \times \{1\}$ by Claim C. Since $k \le m$ and $m+1-k \le m$, then the inductive hypothesis implies $\sum_{i=1}^{k} \left(\frac{a_i}{b}\right) y_i$ and $\sum_{i=k+1}^{m+1} \left(\frac{a_i}{c}\right) y_i \in R(T)$. Let $z = \sum_{i=1}^{k} \left(\frac{a_i}{b}\right) y_i$ and $w = \sum_{i=k+1}^{m+1} \left(\frac{a_i}{c}\right) y_i$. Since $b + c = \sum_{i=1}^{m+1} a_i = 1$ and $bz + cw = \sum_{i=1}^{m+1} a_i y_i \in U^n \times \{1\}$, then the $m=2$ case of the lemma implies $bz + cz \in R(T)$. Thus, $\sum_{i=1}^{m+1} a_i y_i \in R(T)$. This completes the proof of Lemma 2.16. $\square$

We now complete the proof of Theorem 2.15: a) implies c). We must show that $V \cap \mathbb{H}^n \subset T$. Let $x \in V \cap \mathbb{H}^n$. Then there exist $x_1, \dots, x_m \in T$ and $a_1, \dots, a_m \in \mathbb{R}$ such that $x = \sum_{i=1}^m a_i x_i$. For $1 \le i \le m$, let t_i be the $(n+1)$th coordinate of x_i, and let t be the $(n+1)$th coordinate of x. Then $t = \sum_{i=1}^m a_i t_i$. $R(x_i) = \left(\frac{1}{t_i}\right) x_i$ for $1 \le i \le m$. Also

$$R(x) = \left(\frac{1}{t}\right) x = \sum_{i=1}^m \left(\frac{a_i t_i}{t}\right) R(x_i).$$

Since $R(x_i) \in R(T)$ for $1 \le i \le m$, $\sum_{i=1}^m \left(\frac{a_i t_i}{t}\right) = 1$ and $R(x) \in R(\mathbb{H}^n) = U^n \times \{1\}$, then Lemma 2.16 implies $R(x) \in R(T)$. Since $x \in \mathbb{H}^n$, $T \subset \mathbb{H}^n$ and $R|\mathbb{H}^n$ is injective, then it follows that $x \in T$. This proves $V \cap \mathbb{H}^n \subset T$. We conclude that $T = V \cap \mathbb{H}^n$. $\square$

<u>Theorem 2.17.</u> The metric η and the restriction of the Euclidean metric on $\mathbb{E}^{n+1}$ to $\mathbb{H}^n$ induce the same topology on $\mathbb{H}^n$. In other words, the inclusion of $\mathbb{H}^n$ in $\mathbb{E}^{n+1}$ is a topological embedding.

<u>Proof</u> For $x \in \mathbb{H}^n$ and $r > 0$, let $N_H(x,r) = \{y \in \mathbb{H}^n : \eta(x,y) < r\}$; and for $x \in \mathbb{E}^{n+1}$ and $r > 0$, let $N_E(x,r) = \{y \in \mathbb{E}^{n+1} : \|x - y\| < r\}$. We must prove that for each $x \in \mathbb{H}^n$ and each $\varepsilon > 0$, there is a $\delta > 0$ such that $N_E(x,\delta) \cap \mathbb{H}^n \subset N_H(x,\varepsilon)$ and $N_H(x,\delta) \subset N_E(x,\varepsilon)$.

Since $\eta(x,y) = \cosh^{-1}(-x \circ y)$ for $x, y \in \mathbb{H}^n$, then $\eta : \mathbb{H}^n \times \mathbb{H}^n \to [0,\infty)$ is clearly continuous with respect to the restriction of the Euclidean metric to $\mathbb{H}^n$. Thus, for $x \in \mathbb{H}^n$ and $\varepsilon > 0$, since $\eta(x,x) = 0$, then there is a $\delta > 0$ such that if $y \in N_E(x,\delta) \cap \mathbb{H}^n$, then $\eta(x,y) < \varepsilon$. Thus, $N_E(x,\delta) \cap \mathbb{H}^n \subset N_H(x,\varepsilon)$.

Let $x \in \mathbb{H}^n$ and $\varepsilon > 0$. Corollary 1.14 provides a $g \in O^+(\mathbb{M}^{n+1})$ such that $g(e_{n+1}) = x$. Since g is linear and, therefore, continuous, there is an $\varepsilon' > 0$ such that $g(N_E(e_{n+1},\varepsilon')) \subset N_E(x,\varepsilon)$. Since

$t \mapsto \sqrt{2}\sqrt{\cosh(t) - 1}$ is a continuous function that equals 0 at $t = 0$, then there is a $\delta > 0$ such that $|t| < \delta$ implies $\sqrt{2}\sqrt{\cosh(t)-1} < \varepsilon'$. We claim that $N_H(e_{n+1}, \delta) \subset N_E(e_{n+1}, \varepsilon')$. To prove this claim, let $y \in N_H(e_{n+1}, \delta)$. Then $y \in H^n$. Hence, $-1 = y \circ y = \sum_{i=1}^n y_i^2 - y_{n+1}^2$. Therefore, $\sum_{i=1}^n y_i^2 = y_{n+1}^2 - 1$. Consequently,

$$\|y - e_{n+1}\|^2 = \sum_{i=1}^n y_i^2 + (y_{n+1} - 1)^2 = (y_{n+1}^2 - 1) - (y_{n+1} - 1)^2$$
$$= 2(y_{n+1} - 1) = 2((-e_{n+1} \circ y) - 1) = 2(\cosh(\eta(e_{n+1}, y)) - 1).$$

Thus, $\|y - e_{n+1}\| = \sqrt{2}\sqrt{\cosh(\eta(e_{n+1}, y)) - 1}$.

Since $y \in N_H(e_{n+1}, \delta)$, then $\eta(e_{n+1}, y) < \delta$. Hence, $\|y - e_{n+1}\| < \varepsilon'$. So $y \in N_E(e_{n+1}, \varepsilon')$. This proves the claim. To finish this proof we show that $N_H(x, \delta) \subset N_E(x, \varepsilon)$. Let $z \in N_H(x, \delta)$. Since $g \in O^+(\mathbb{M}^{n+1})$, then $g^{-1} \in O^+(\mathbb{M}^{n+1})$ and $g^{-1}|H^n \in \mathcal{I}(S^n)$ by Theorem 2.3. Therefore, $g^{-1} N_H(x, \delta) = N_H(e_{n+1}, \delta)$. Thus, $g^{-1}(z) \in N_H(e_{n+1}, \delta)$. So $g^{-1}(z) \in N_E(e_{n+1}, \varepsilon')$. Therefore, $z \in g(N_E(e_{n+1}, \varepsilon')) \subset N_E(x, \varepsilon)$. This proves $N_H(x, \delta) \subset N_E(x, \varepsilon)$. $\square$

Def A map $f: X \to Y$ is <u>proper</u> if $f^{-1}(C)$ is a compact subset of X whenever C is a compact subset of Y.

<u>Lemma 2.18.</u> a) A (continuous) map $f: [0,\infty) \to \mathbb{E}^n$ is proper if and only if $\lim_{t\to\infty} \|f(t)\| = \infty$.

b) If $f: [0,\infty) \to \mathbb{H}^n$ is a geodesic, then f is proper.

<u>Homework Problem 2.5.</u> Prove Lemma 2.18.

Def Let $f: [0,\infty) \to \mathbb{E}^n$ be a proper map and let $0 \neq v \in \mathbb{E}^n$. v is an <u>asymptotic direction</u> of f if there is a map $a: (0,\infty) \to (0,\infty)$ such that

$$\lim_{t\to\infty} \frac{f(t)}{a(t)} = v .$$

<u>Lemma 2.19</u> Let $f: [0,\infty) \to \mathbb{E}^n$ be a proper map. Then the set of asymptotic directions of f is either empty or a ray of the form $\{rv : r > 0\}$ for some non-zero $v \in \mathbb{E}^n$.

Homework Problem 2.6 Prove Lemma 2.19.

Lemma 2.20 If $u \in \mathbb{H}^n$ and u, v is an orthonormal sequence in $\mathbb{M}^{n+1}$, then $u+v$ is an asymptotic direction of $\Gamma_{uv} | [0,\infty)$.

Homework Problem 2.7. Prove Lemma 2.20.

Observation Since $\Gamma_{u,v}(-t) = \Gamma_{u,-v}(t)$, then $u-v$ is an asymptotic direction of $t \mapsto \Gamma_{uv}(-t) : [0,\infty) \to \mathbb{H}^n$.

Def Two geodesics $f : [0,\infty) \to \mathbb{H}^n$ and $g : [0,\infty) \to \mathbb{H}^n$ are asymptotic if the set $\{\eta(f(t), g(t)) : t \in [0,\infty)\}$ is bounded.

Theorem 2.21. Two geodesics $f : [0,\infty) \to \mathbb{H}^n$ and $g : [0,\infty) \to \mathbb{H}^n$ are asymptotic if and only if they have equal asymptotic directions.

The following homework problem breaks the proof of Theorem 2.21 into several steps.

Due Tuesday 10/21

Homework Problem 2.8

a) Prove: if $u \in \mathbb{H}^n$ and u, v is an orthonormal sequence in $\mathbb{M}^{n+1}$, then $u_{n+1} + v_{n+1} > 0$.

b) Prove: if x and y are null vectors in $\mathbb{M}^{n+1}$ and $x_{n+1} > 0$ and $y_{n+1} > 0$, then $x \circ y \leq 0$.

c) Prove: if x and y are orthogonal null vectors in $\mathbb{M}^{n+1}$, then $y_{n+1} x = x_{n+1} y$.

d) Prove: if $u \in \mathbb{H}^n$ and u, v is an orthonormal sequence in $\mathbb{M}^{n+1}$, then $u+v$ is null.

e) Suppose $u, w \in \mathbb{H}^n$ and u, v and w, z are orthonormal sequences in $\mathbb{M}^{n+1}$. Let
$a = (u+v) \circ (w+z)$, $b = (u+v) \circ (w-z)$,
$c = (u-v) \circ (w+z)$ and $d = (u-v) \circ (w-z)$.
Prove:

$$\Gamma_{uv}(s) \circ \Gamma_{wz}(t) = \left(a e^{s+t} + b e^{s-t} + c e^{-s+t} + d e^{-s-t}\right)/4.$$

f) Prove Theorem 2.21.

<u>Homework Problem 2.9</u>. Let $\mathbb{N}^n = \{x \in \mathbb{M}^{n+1} : x \circ x = 0\}$ and call $\mathbb{N}^n$ the <u>null cone</u> of $\mathbb{M}^{n+1}$.

a) Suppose $u \in \mathbb{H}^n$; u, v is an orthonormal sequence in $\mathbb{M}^{n+1}$, and V is the 2-dimensional vector subspace of $\mathbb{M}^{n+1}$ spanned by $\{u, v\}$. If $w \in V \cap \mathbb{N}^n$ and $w_{n+1} > 0$, then w is an asymptotic direction of either $\Gamma_{uv}|[0,\infty)$ or $\Gamma_{u,-v}|[0,\infty)$.

b) Suppose $u, w \in \mathbb{H}^n$; u, v and w, z are orthonormal sequences in $\mathbb{M}^{n+1}$, V and W are the 2-dimensional vector subspaces of $\mathbb{M}^{n+1}$ spanned by $\{u,v\}$ and $\{w,z\}$, respectively, and $L = V \cap W$ is a 1-dimensional vector subspace of $\mathbb{M}^{n+1}$. Then $L \subset \mathbb{N}^n$ if and only if one of the following pairs of geodesics is asymptotic: $\Gamma_{uv}|[0,\infty)$ and $\Gamma_{wz}|[0,\infty)$, $\Gamma_{uv}|[0,\infty)$ and $\Gamma_{w,-z}|[0,\infty)$, $\Gamma_{u,-v}|[0,\infty)$ and $\Gamma_{w,z}|[0,\infty)$, or $\Gamma_{u,-v}|[0,\infty)$ and $\Gamma_{w,-z}|[0,\infty)$

Homework Problem 2.10 Let $u, w \in \mathbb{H}^n$ and let u, v and w, z be orthonormal sequences in $\mathbb{M}^{n+1}$.

a) Prove that if $\Gamma_{uv}|[0,\infty)$ and $\Gamma_{wz}|[0\infty)$ are asymptotic and $\Gamma_{u,-v}|[0\infty)$ and $\Gamma_{w,-z}|[0\infty)$ are also asymptotic, then there is an $a \in \mathbb{R}$ such that $\Gamma_{uv}(t) = \Gamma_{wz}(a+t)$ for all $t \in \mathbb{R}$.

b) Formulate and prove a result like a) for the situation in which $\Gamma_{uv}|[0\infty)$ and $\Gamma_{w,-z}|[0\infty)$ are asymptotic and $\Gamma_{u,-v}|[0\infty)$ and $\Gamma_{w,z}|[0\infty)$ are asymptotic.

Homework Problem 2.10. Let $u, w \in \mathbb{H}^n$ and let u, v and w, z be orthonormal sequences in $\mathbb{M}^{n+1}$. Prove that if $\Gamma_{uv}|[0\infty)$ and $\Gamma_{wz}|[0\infty)$ are asymptotic, but $\Gamma_{u,-v}|[0\infty)$ and $\Gamma_{w,-z}|[0\infty)$ are not asymptotic, then as $t \to \infty$, $\eta(\Gamma_{uv}(t), \Gamma_{wz}(t))$ is strictly decreasing to a positive limit. Express the value of this limit in terms of u, v, w and z.

<u>Homework Problem 2.12</u> - Let $u, w \in \mathbb{H}^n$ and let u, v and w, z be orthonormal sequences in $\mathbb{M}^{n+1}$. Prove that if $\Gamma_{uv}|[0\infty)$ and $\Gamma_{wz}|[0\infty)$ are not asymptotic and $\Gamma_{u,-v}|[0\infty)$ and $\Gamma_{w,-z}|[0\infty)$ are not asymptotic, then

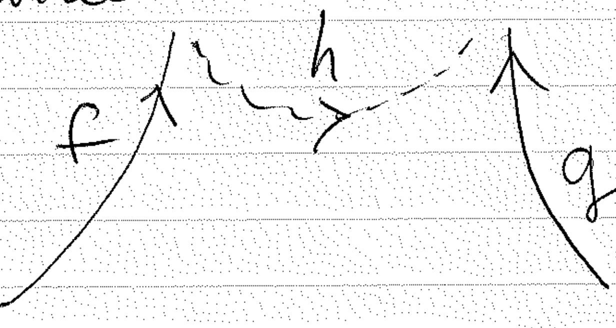
there is a point $(s_0, t_0) \in \mathbb{R}\times\mathbb{R}$ such that

$\eta(\Gamma_{uv}(s_0), \Gamma_{wz}(t_0))$ is the unique minimum

value of $\{\eta(\Gamma_{uv}(s), \Gamma_{wz}(t)) : (s,t) \in \mathbb{R}\times\mathbb{R}\}$.

Express (s_0, t_0) in terms of u, v, w, z.

<u>Definition</u> A metric space X is called a <u>visibility space</u> if whenever $f, g : \mathbb{R} \to X$ are geodesics such that $f|[0\infty)$ and $g|[0\infty)$ are not asymptotic, then there is a geodesic $h : \mathbb{R} \to X$ such that $f|[0\infty)$ and $t \mapsto h(-t) : [0\infty) \to X$ are asymptotic and $h|[0\infty)$ and $g|[0\infty)$ are asymptotic.



Theorem 2.22. $\mathbb{H}^n$ is a visibility space.

Homework Problem 2.13. Prove Theorem 2.22.

$\mathbb{E}^n$ is not a visibility space. In fact, a more general result holds:

Theorem 2.23 If a metric space X contains a subset that is isometric to $\mathbb{E}^2$, then X is not a visibility space.

Homework Problem 2.14. Prove Theorem 2.23.