0. Background

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f: X \to Y$ is <u>distance preserving</u> if $d_Y(f(x), f(x')) = d_X(x, x')$ for all $x, x' \in X$. A distance preserving onto function from X to Y is called an <u>isometry</u>.

id_X is an isometry from a metric space X to itself. If $f: X \to Y$ and $g: Y \to Z$ are isometries between metric spaces, then $g \circ f: X \to Z$ is an isometry. Distance preserving functions are injective; hence, isometries are bijective. If $f: X \to Y$ is an isometry between metric spaces, then $f^{-1}: Y \to X$ is an isometry.

If X is a metric space, then an isometry from X to itself is called a <u>rigid motion</u> of X. The set of all rigid motions of X is called the <u>isometry group</u> of X and is denoted $\mathcal{I}(X)$. $\mathcal{I}(X)$ is a group with respect to composition.

Endow $\mathbb{R}$ with the standard metric: $d(x,y) = |x-y|$. A connected subset of $\mathbb{R}$ is called an <u>interval</u>. There are three types of intervals:

<u>open intervals</u>: $(-\infty, \infty)$, (a, ∞), $(-\infty, b)$,
<u>closed intervals</u>: $[a\ \infty)$, $(-\infty, b]$; $[ab]$, and
<u>half-open intervals</u>: $[ab)$ and $(a, b]$.

Endow each interval J with a metric by restricting the standard metric on $\mathbb{R}$ to J. A function $f: J \to X$ from an interval J to a metric space X is called a <u>geodesic</u> if it is distance preserving. $f: J \to X$ is called a <u>local geodesic</u> if every point of J is contained in an interval K which is a relatively open subset of J such that $f|K: K \to X$ is a geodesic. A local geodesic $f: J \to X$ is called a <u>geodesic line</u> / <u>geodesic ray</u> / <u>geodesic segment</u> if $J = (-\infty, \infty)$ / $J = [a, \infty)$ or $J = (-\infty, b]$ / $J = [ab]$, respectively. A metric space X is <u>totally geodesic</u> if every pair of points of X lie in a geodesic line in X.

— 0.3 —

Let V be a vector space (over $\mathbb{R}$). A <u>bilinear form</u> on V is a function $(x,y) \mapsto x*y : V\times V \to \mathbb{R}$ that satisfies

$$(ax+by)*z = a(x*z)+b(y*z) \quad \text{and}$$

$$x*(ay+bz) = a(x*y)+b(x*z)$$

for all $x,y,z \in V$ and all $a,b\in\mathbb{R}$.

A bilinear form $*$ on V is <u>symmetric</u> if $x*y = y*x$ for all $x,y\in V$.

A bilinear form $*$ on V is <u>positive definite</u> if for all $x\in V$: $x*x \geq 0$, and $x*x=0$ if and only if $x=0$.

A positive definite symmetric bilinear form on a vector space V is called an <u>inner product</u> or a <u>Euclidean product</u> on V.

<u>Example</u> The <u>dot product</u> $\cdot$ on $\mathbb{R}^n$ is a Eucliden product on $\mathbb{R}^n$ which is defined by

$$x\cdot y = \sum_{i=1}^{n} x_i y_i$$

for $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)\in\mathbb{R}^n$.

If $*$ is a Euclidean product on a vector space V, then the associated Euclidean norm on V is the function $x \mapsto \|x\| : V \to [0, \infty)$ defined by

$$\|x\| = \sqrt{x * x}.$$

Theorem 0.1: The Cauchy Inequality.

Let $*$ be a Euclidean product on a vector space V. Then

$$|x * y| \leq \|x\| \, \|y\|$$

for all $x, y \in V$. Furthermore, for all $x, y \in V$ and $\varepsilon = \pm 1$:

$$x * y = \varepsilon \|x\| \, \|y\| \text{ if and only if } \|y\| x = \varepsilon \|x\| y.$$

Proof Claim A If $u, v \in V$ and $\|u\| = \|v\| = 1$, then $|u * v| \leq 1$.

Proof of Claim A.

$0 \leq (u-v) * (u-v) = 2 - 2u * v$ and
$0 \leq (u+v) * (u+v) = 2 + 2u * v$. Hence,
$-1 \leq u * v \leq 1$. So $|u * v| \leq 1$ $\square$.

Now let $x, y \in V$. If $x = 0$ or $y = 0$, then $|x*y| = 0 = \|x\|\,\|y\|$. So we can assume $x \neq 0 \neq y$. Let

$$u = \left(\frac{1}{\|x\|}\right)x \text{ and } v = \left(\frac{1}{\|y\|}\right)y.$$

Then $\|u\| = \|v\| = 1$. Hence $|u*v| \leq 1$ by Claim A. Therefore

$$\left|\left(\left(\frac{1}{\|x\|}\right)x\right) * \left(\left(\frac{1}{\|y\|}\right)y\right)\right| \leq 1.$$

Consequently $|x*y| \leq \|x\|\,\|y\|$.

<u>Claim B</u> If $u, v \in V$, $\|u\| = \|v\| = 1$ and $u*v = \varepsilon = \pm 1$, then $u = \varepsilon v$.

<u>Proof of Claim B</u>

$(u - \varepsilon v)*(u - \varepsilon v) = 2 - 2\varepsilon(u*v) = 0$.

Hence $u = \varepsilon v$. $\square$

Now let $x, y \in V$ and $\varepsilon = \pm 1$. Suppose $x*y = \varepsilon\|x\|\,\|y\|$. If $x=0$ or $y=0$, then obviously $\|y\|x = 0 = \varepsilon\|x\|y$. So we can assume $x \neq 0 \neq y$. Let

$$u = \left(\frac{1}{\|x\|}\right)x \text{ and } v = \left(\frac{1}{\|y\|}\right)y.$$

Then $\|u\| = \|v\| = 1$ and $u*v = \varepsilon$. Hence $u = \varepsilon v$ by Claim B. Consequently $\|y\|x = \varepsilon\|x\|y$. $\square$

<u>Theorem 0.2</u>. Let $*$ be a Euclidean product on a vector space V, and let $x \mapsto \|x\| : V \to [0, \infty)$ be the associated Euclidean norm. Then:

a) For every $x \in V$, $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$,

b) For every $x \in V$ and $a \in \mathbb{R}$, $\|ax\| = |a| \, \|x\|$, and

c) For all $x, y \in V$, $\|x+y\| \leq \|x\| + \|y\|$.

<u>Homework Problem 0.1</u>. Prove Theorem 0.2.a) and 0.2.b).

<u>Proof of Theorem 0.2.c</u>

By the Cauchy Inequality:

$$\|x+y\|^2 = (x+y)*(x+y) = \|x\|^2 + 2x*y + \|y\|^2$$
$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2.$$

Hence, $\|x+y\| \leq \|x\| + \|y\|$. $\square$

Let $*$ be a Euclidean product on a vector space V and let $x \mapsto \|x\| : V \to [0, \infty)$ be the associated Euclidean norm. Define the associated <u>Euclidean metric</u> on V to be ~~$d_E(x,y) = \|x-y\|$ for all $x, y \in V$.~~ the function $d_E : V \times V \to [0, \infty)$ defined by $d_E(x,y) = \|x-y\|$ for all $x, y \in V$.

<u>Theorem 0.3</u> Let $*$ be a Euclidean product on a vector space V, let $x \mapsto \|x\| : V \to [0\ \infty)$ be the associated Euclidean norm on V, and let d_E be the associated Euclidean metric on V. Then d_E is a metric on V. In other words,

a) $d_E(x,y) \geq 0$, and $d_E(xy) = 0$ if and only if $x = y$, for all $x, y \in V$

b) $d_E(xy) = d_E(yx)$ for all $xy \in V$

c) $d_E(xz) \leq d_E(x,y) + d_E(yz)$ for all $x, y, z \in V$.

<u>Homework Problem 0.2.</u> Prove assertions a) and b) of Theorem 0.3.

<u>Proof of assertion c)</u> For $x, y, z \in V$:

$d_E(x,z) = \|x - z\| = \|(x-y) + (y-z)\| \leq$

$\|x - y\| + \|y - z\| = d_E(xy) + d_E(yz)$. □

Example. $\mathbb{R}^n$ equipped with the Euclidean metric associated with the dot product on $\mathbb{R}^n$ is called Euclidean n-space and is denoted $\mathbb{E}^n$. (Thus, the Euclidean metric d_E on $\mathbb{R}^n$ associated with the dot product satisfies

$$d_E(x,y) = \|x-y\| = \sqrt{(x-y)\cdot(x-y)}.$$

Let $*$ be a Euclidean product on a vector space V. An element u of V is a unit vector if $u*u=1$. A sequence $u_1,\dots,u_k$ in V is orthogonal if $u_i*u_j=0$ for $i\neq j$. An orthogonal sequence of unit vectors in V is called an orthonormal sequence in V. An orthonormal sequence in V which is also a basis for V is called an orthonormal basis for V.

Example The sequence of standard unit vectors $e_1,\dots,e_n$ in $\mathbb{R}^n$ is an orthonormal basis for $\mathbb{E}^n$.

Lemma 0.4. Let $*$ be a Euclidean product on a vector space V. Then every orthonormal sequence in V is linearly independent.

Proof. Let $u_1, \ldots, u_k$ be an orthonormal sequence in V. Suppose $a_1, \ldots, a_k \in \mathbb{R}$ and $\sum_{i=1}^k a_i u_i = \theta$. Then for $1 \le j \le k$,

$$0 = \theta * u_j = \left(\sum_{i=1}^k a_i u_i\right) * u_j = \sum_{i=1}^k a_i (u_i * u_j) = a_j(1)$$

Thus, $a_j = 0$ for $1 \le j \le k$. $\square$

Theorem 0.5 If $*$ is a Euclidean product on a finite dimensional vector space V, then V has an orthonormal basis.

Sketch of proof. V has a finite basis $v_1, \ldots, v_n$. We perform the Gram Schmidt Orthogonalization Process inductively on $v_1, \ldots, v_n$ to convert it to an orthonormal basis for V.

To begin, let $u_1 = \left(\frac{1}{\|v_1\|}\right) v_1$.

Then u_1 is a unit vector that spans the same vector subspace of V that v_1 spans.

Now let $1 \le k < n$ and assume we have constructed an orthonormal sequence $u_1, \ldots, u_k$ in V that spans the same vector subspace of V that $v_1, \ldots, v_k$ spans. Let

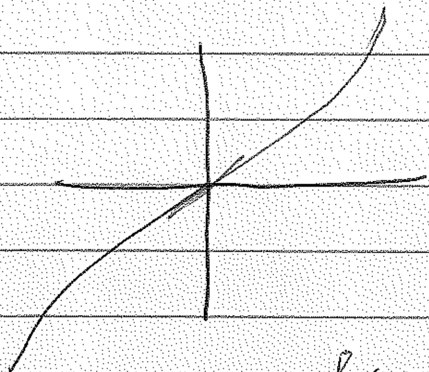
$$y_{k+1} = v_{k+1} - \sum_{i=1}^{k} (v_{k+1} * u_i) u_i .$$

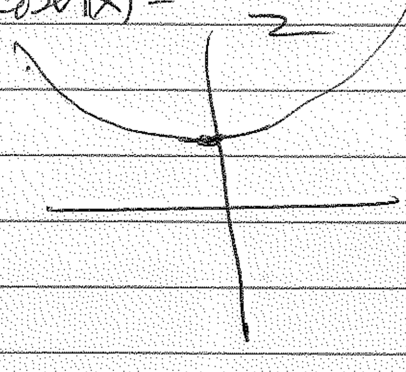
Then $y_{k+1} \neq 0$, $y_{k+1} * u_j = 0$ for $1 \le j \le k$, and $u_1, \ldots, u_k, y_{k+1}$ spans the same vector subspace of V that $v_1, \ldots, v_k, v_{k+1}$ spans. Let $u_{k+1} = \left(\frac{1}{\|y_{k+1}\|}\right) y_{k+1}$. Then $u_1, \ldots, u_k, u_{k+1}$ is an orthonormal sequence in V that spans the same vector subspace of V that $v_1, \ldots, v_k, v_{k+1}$ spans.

When this process terminates, we have an orthonormal sequence $u_1, \ldots, u_n$ in V that spans the same vector subspace of V that $v_1 \ldots v_n$ spans. Hence, $u_1, \ldots, u_n$ spans V. Thus, $u_1, \ldots, u_n$ is an orthonormal basis for V. $\square$

Hyperbolic Functions

Def $\sinh(x) = \frac{e^x - e^{-x}}{2}$, $\cosh(x) = \frac{e^x + e^{-x}}{2}$





$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$ $\coth(x) = \frac{1}{\tanh(x)}$

$\text{sech}(x) = \frac{1}{\cosh(x)}$ $\text{csch}(x) = \frac{1}{\sinh(x)}$

Identities

$\cosh^2(x) - \sinh^2(x) = 1$

$\text{sech}^2(x) = 1 - \tanh^2(x)$

$\text{coch}^2(x) = \coth^2(x) - 1$

$\sinh(-x) = -\sinh(x)$, $\cosh(-x) = \cosh(x)$

$\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$

$\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \cosh(x)\sinh(y)$

Derivatives

$\frac{d}{dx}\sinh(x) = \cosh(x)$ $\frac{d}{dx}\cosh(x) = \sinh(x)$

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$\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x) \qquad \frac{d}{dx}\coth(x) = -\operatorname{csch}^2(x)$

$\frac{d}{dx}\operatorname{sech}(x) = -\operatorname{sech}(x)\tanh(x)$

$\frac{d}{dx}\operatorname{csch}(x) = -\operatorname{csch}(x)\coth(x)$

<u>Power Series</u>

$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \qquad \cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$