3. Hyperbolic Spaces

<u>Def</u> Let V be a vector space. A function $(x,y) \mapsto x \circ y : V \times V \to \mathbb{R}$ is a <u>bilinear form</u> if it is linear in each variable; in other words, $(ax+by)\circ z = a(x\circ z)+b(y\circ z)$ and $x\circ(ay+bz) = a(x\circ y)+b(x\circ z)$ for all $x,y,z \in V$ and all $a,b \in \mathbb{R}$.

<u>Def</u> A bilinear form $(x,y) \mapsto x\circ y : V\times V \to \mathbb{R}$ is <u>symmetric</u> if $x\circ y = y\circ x$ for all $x,y \in V$.

<u>Def</u> A bilinear form $(x,y) \mapsto x\circ y : V\times V \to \mathbb{R}$ is <u>non-degenerate</u> if for every non-zero $x \in V$, there is a $y \in V$ such that $x \circ y \neq 0$.

<u>Def</u> Let $(x,y) \mapsto x\circ y : V\times V \to \mathbb{R}$ be a bilinear form. Then we call the pair $(V,\circ)$ a <u>b.f. space</u>. If the bilinear form is symmetric, we call $(V,\circ)$ a <u>s.b.f. space</u>. If the bilinear form is symmetric and non-degenerate, we call $(V,\circ)$ a <u>n.s.b.f. space</u>.

<u>Example</u> Let r and s be non-negative integers such that $r+s \leq n$. Define $\circ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$x\circ y = x_1y_1 + \cdots + x_ry_r - x_{r+1}y_{r+1} - \cdots - x_{r+s}y_{r+s}$$

for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

Clearly $\circ$ is a symmetric bilinear form.
If $r+s<n$, then $\circ$ is degenerate because $x \circ e_n = 0$ for every $x \in \mathbb{R}^n$. If $r+s=n$, then $\circ$ is non-degenerate; for $0 \neq x = (x_1, \dots, x_n) \in \mathbb{R}^n$, let $\bar{x} = (x_1, \dots, x_r, -x_{r+1}, \dots, -x_n)$. Then $x \circ \bar{x} = \|x\|^2 \neq 0$.
If $r = n-1$ and $s=1$, so that $x \circ y = x_1 y_1 + \cdots + x_{n-1} y_{n-1} - x_n y_n$ then $(\mathbb{R}^n, \circ)$ is called Minkowski n-space and is denoted M^n.

Def Let $(V, \circ)$ be a s.b.f. form space. For $x \in V$, if $x \circ x > 0$, call x a positive element of V; and if $x \circ x < 0$, call x a negative element of V. If every non-zero element of V is positive, then $(V, \circ)$ is said to be positive definite; and if every non-zero element of V is negative, then $(V, \circ)$ is said to be negative definite. If $(V, \circ)$ is non-degenerate and neither positive definite nor negative definite, then $(V, \circ)$ is said to be indefinite. If $x \in V$ and $x \circ y = 0$ for every $y \in V$, then call x a degenerate element of V. Let $D(V)$ denote the set of all degenerate elements of V.

Lemma 3.1 If $(V, \circ)$ is a s.b.f.-space, then $D(V)$ is a vector subspace of V.

Exercise Prove lemma 3.1.

Def Let $(V, \circ)$ be a s.b.f. space. A sequence $x_1, \ldots, x_k$ of elements of V is orthonormal if:

i) $x_i \circ x_j = 0$ for $i \neq j$

ii) either $x_i \circ x_i = \pm 1$ or x_i is degenerate for $1 \leq i \leq k$, and

iii) the subsequence of all degenerate elements of $x_1, \ldots, x_k$ is linearly independent.

An orthonormal sequence in $(V, \circ)$ which also spans V is called an orthonormal basis for $(V, \circ)$.

Lemma 3.2. Every orthonormal sequence in a s.b.f. space is linearly independent.

Proof Let $x_1, \ldots, x_l$ be an orthonormal sequence in the s.b.f. space $(V, \circ)$ ordered so that for some k, $0 \leq k \leq l$, $x_i \circ x_i = \pm 1$ for $1 \leq i \leq k$ and x_i is degenerate for $k+1 \leq i \leq l$. Suppose $\sum_{i=1}^{l} a_i x_i = 0$ for some $a_1, \ldots, a_l \in \mathbb{R}$. Then for $1 \leq j \leq k$

$$0 = x_j \circ 0 = x_j \circ \left(\sum_{i=1}^{l} a_i x_i\right) = a_j (x_j \circ x_j) = \pm a_j.$$

Hence, $a_j = 0$ for $1 \leq j \leq k$. Therefore, $\sum_{i=k+1}^{l} a_i x_i = 0$. Since $x_{k+1}, \ldots, x_l$ are linearly independent, then $a_i = 0$ for $k+1 \leq i \leq l$. This proves $x_1, \ldots, x_l$ are linearly independent. $\square$

Theorem 3.3 Every finite dimensional s.b.f. space has an orthonormal basis. Furthermore, every orthonormal sequence in a finite dimensional s.b.f. space can be enlarged to an orthonormal bases.

Before proving Theorem 3.3, we prove three helpful lemmas.

Lemma 3.4 If $(V, \circ)$ is a s.b.f. space and $D(V) \neq V$, then there is a $u \in V$ such that $u \circ u = \pm 1$.

Proof Since $D(V) \neq V$, there is an $x \in V$ such that $x \notin D(V)$. Hence, there is a $y \in V$ such that $x \circ y \neq 0$. Since

$$(x+y)\circ(x+y) - (x \circ x) - (y \circ y) = 2\, x \circ y \neq 0,$$

then at least one of $(x+y)\circ(x+y)$, $x \circ x$, $y \circ y$ is non-zero. Therefore, there is a $z \in V$ such that $z \circ z \neq 0$. Let

$$u = \frac{z}{\sqrt{|z \circ z|}}.$$

Then $u \circ u = \dfrac{z \circ z}{|z \circ z|} = \pm 1$. $\square$

<u>Def</u> If $(V, \circ)$ is a s.b.f. space and $S \subset V$, let

$$S^{\perp} = \{x \in V : x \circ y = 0 \text{ for every } y \in S\}.$$

<u>Observation</u> If $(V, \circ)$ is a s.b.f space and $S \subset V$, then $S^{\perp}$ is a vector subspace of V.

<u>Exercise</u> Prove this observation.

If W is a vector subspace of $\mathbb{E}^n$, then $W^{\perp}$ has nice properties. In particular, $\mathbb{E}^n = W \oplus W^{\perp}$. However, if W is a vector subspace of a s.b.f. space, $W^{\perp}$ may not behave so nicely. For example, if $\circ$ is the s.b.f defined on $\mathbb{R}^2$ by $x \circ y = x_1 y_1 - x_2 y_2$ and W is the 1-dimensional vector subspace of $\mathbb{R}^2$ spanned by $(1,1)$, then $W^{\perp} = W$. The following lemma describes conditions on a subspace W of a s.b.f. space which make $W^{\perp}$ behave as it does in a Euclidean space.

Lemma 3.5 If $(V, \circ)$ is a s.b.f. space, $u_1, \ldots, u_k$ is an orthonormal sequence in V such that $u_i \circ u_i = \pm 1$ for $1 \leq i \leq k$, and W is the vector subspace of V spanned by $u_1, \ldots, u_k$, then $V = W \oplus W^\perp$. (In other words, $W \cap W^\perp = \{0\}$ and $W + W^\perp = V$.)

Proof Let $x \in W \cap W^\perp$. Since $x \in W$ then $x = \sum_{i=1}^k a_i u_i$ for some $a_1, \ldots, a_k \in \mathbb{R}$. Since $x \in W^\perp$, then for $1 \leq j \leq k$:

$$0 = u_j \circ x = u_j \circ \left(\sum_{i=1}^k a_i u_i\right) = a_j (u_j \circ u_j) = \pm a_j.$$

Thus, $a_j = 0$ for $1 \leq j \leq k$. Hence $x = 0$. This proves $W \cap W^\perp = \{0\}$.

Let $x \in V$. Let $z = x - \sum_{i=1}^k \left(\frac{x \circ u_i}{u_i \circ u_i}\right) u_i$

Then for $1 \leq j \leq k$, $z \circ u_j = x \circ u_j - \left(\frac{x \circ u_j}{u_j \circ u_j}\right)(u_j \circ u_j) = 0$.

Consequently, if $y \in W$, then $y = \sum_{i=1}^k b_i u_i$ for some $b_1, \ldots, b_k \in \mathbb{R}$, from which it follows that

$$y \circ z = \sum_{i=1}^k b_i (u_i \circ z) = \sum_{i=1}^k b_i (0) = 0.$$

This proves $z \in W^\perp$. Hence, $x = \sum_{i=1}^k \left(\frac{x \circ u_i}{u_i \circ u_i}\right) u_i + z$ where $\sum_{i=1}^k \left(\frac{x \circ u_i}{u_i \circ u_i}\right) u_i \in W$ and $z \in W^\perp$. This proves $V = W + W^\perp$. $\square$

In a direct sum splitting of a non-degenerate s.b.f. space, the direct summands may not be non-degenerate. For example, in $\mathbb{R}^2$ with non-degenerate s.b.f $x \circ y = x_1 y_1 - x_2 y_2$, if W and X are the 1-dimensional vector subspaces spanned by $(1,1)$ and $(-1,1)$, respectively, then $\mathbb{R}^2 = W \oplus X$ and $D(W) = W$ and $D(X) = X$. The following lemma describes conditions under which direct summands of a non-degenerate s.b.f. space are non-degenerate.

<u>Lemma 3.6</u> If $(V, \circ)$ is a non-degenerate s.b.f. space, W and X are vector subspaces of V such that $V = W \oplus X$ and $X \subset W^\perp$, then W and X are each non-degenerate with respect to the restriction of $\circ$.

<u>Proof</u> We will prove that if $X \subset W^\perp$, then X is non-degenerate. Since $X \subset W^\perp$ implies $W \subset X^\perp$, then the same argument proves W is non-degenerate.

Let $0 \neq x \in X$. Since V is non-degenerate, there is a $y \in V$ such that $x \circ y \neq 0$. Since $V = W \oplus X$, then $y = w + y'$ where $w \in W$ and

$y' \in X$. Since $x \in X \subset W^\perp$, then $x \circ w = 0$. Hence,

$$x \circ y' = x \circ (y - w) = x \circ y - x \circ w = x \circ y \neq 0.$$

This proves X is non-degenerate. $\square$

The next lemma shows how to "project" any s.b.f. space into a non-degenerate s.b.f space. This lemma will help in the proof of Theorem 3.3.

<u>Lemma 3.7</u> If $(V, \circ)$ is an s.b.f. space, then the s.b.f. $\circ$ on V induces a non-degenerate s.b.f. $\circ'$ on $V/D(V)$ such that for all $x, y \in V$, $x \circ y = \pi(x) \circ' \pi(y)$, where $\pi : V \to V/D(V)$ is the natural projection. Thus, the following diagram commutes.

$$\begin{tikzcd} V \times V \arrow[dr, "\circ"] \arrow[d, "\pi \times \pi"'] & \\ V/D(V) \times V/D(V) \arrow[r, "\circ'"'] & \mathbb{R} \end{tikzcd}$$

<u>Proof</u>: To prove the existence of $\circ' : V/D(V) \times V/D(V) \to \mathbb{R}$ so that this diagram commutes, it suffices to show for all (x, y) and $(x', y') \in V \times V$: $\pi \times \pi(x, y) = \pi \times \pi(x', y')$

implies $x \circ y = x' \circ y'$. To this end, assume (x, y) and $(x', y') \in V \times V$ and $\pi \times \pi(x, y) = \pi \times \pi(x', y')$. Thus, $\pi(x) = \pi(x')$ and $\pi(y) = \pi(y')$. Hence $\pi(x - x') = \pi(x) - \pi(x') = 0$ and $\pi(y - y') = \pi(y) - \pi(y') = 0$. Therefore, $x - x'$ and $y - y' \in D(V)$. Consequently, $(x - x') \circ z = 0 = (y - y') \circ z$ for every $z \in V$. It follows that

$$x \circ y = (x' + (x - x')) \circ (y' + (y - y')) =$$

$$x' \circ y' + x' \circ (y - y') + (x - x') \circ y' + (x - x') \circ (y - y') =$$

$$x' \circ y' + 0 + 0 + 0 = x' \circ y'.$$

We conclude that $\circ' : V/D(V) \times V/D(V) \to \mathbb{R}$ exists and satisfies $x \circ y = \pi(x) \circ' \pi(y)$ for all $x, y \in V$.

Since $\circ$ is bilinear and π is linear, then it follows that $\circ'$ is bilinear. Also, since $\circ$ is symmetric, then $\circ'$ is symmetric.

To prove that $\circ'$ is non-degenerate, suppose $0 \neq \bar{x} \in V/D(X)$. Then there is an $x \in V$ such that $\pi(x) = \bar{x}$ and $x \notin D(V)$. Therefore, there is a $y \in V$ such that $x \circ y \neq 0$. Let $\bar{y} = \pi(y) \in V/D(V)$. Then

$$\bar{x} \circ' \bar{y} = \pi(x) \circ' \pi(y) = x \circ y \neq 0. \quad \square$$

<u>Observation</u> If $(V, \circ)$ is an s.b.f. space, $\bar{u}_1, \ldots, \bar{u}_k$ is an orthonormal sequence in $V/D(V)$ and $u_i \in V$ such that $\pi(u_i) = \bar{u}_i$ for $1 \leq i \leq k$, then $u_1, \ldots, u_k$ is an orthonormal sequence in V such that $u_i \circ u_i = \pm 1$ for $1 \leq i \leq k$.

<u>Exercise</u> Prove this observation.

<u>Proof of Theorem 3.3</u> Let $(V, \circ)$ be a finite dimensional s.b.f. space. We will prove V has an orthonormal basis.

First assume $(V, \circ)$ is non-degenerate. Then $D(V) = 0$. We will induct on $\dim V$.

Suppose $\dim V = 1$. Lemma 3.4 provides a $u \in V$ such that $u \circ u = \pm 1$. Since $u \neq 0$ and $\dim V = 1$, then u is an orthonormal basis for V.

Now suppose $\dim V > 1$ and assume that if W is a non-degenerate s.b.f. space with $\dim W < \dim V$, then W has an orthonormal basis. Again, Lemma 3.4 provides a $u_1 \in V$ such that $u_1 \circ u_1 = \pm 1$. Let $\mathbb{R}u_1$ denote the vector subspace of V spanned by u_1. Lemma 3.5 implies

$V = (\mathbb{R}u_1) \oplus (\mathbb{R}u_1)^\perp$, and Lemma 3.6 implies $(\mathbb{R}u_1)^\perp$ is a non-degenerate s.b.f. space. Clearly, $\dim(\mathbb{R}u_1)^\perp < \dim V$. So the inductive hypothesis implies $(\mathbb{R}u_1)^\perp$ has an orthonormal basis $u_2, \dots, u_n$. Then clearly $u_1 u_2 \dots u_n$ is an orthonormal basis ~~for~~ for V. This proves the existence of orthonormal bases for non-degenerate s.b.f. spaces.

Now suppose $(V, \circ)$ is a finite dimensional s.b.f. space which may not be non-degenerate. Then Lemma 3.7 implies that $V/D(V)$ is a finite dimensional non-degenerate s.b.f. space. Hence, $V/D(V)$ has an orthonormal basis $\bar{u}_1, \dots \bar{u}_k$. For $1 \le i \le k$, choose $u_i \in V$ such that $\pi(u_i) = \bar{u}_i$. Then $u_1, \dots, u_k$ is an orthonormal sequence in V such that $u_i \circ u_i = \pm 1$ for $1 \le i \le k$. Let $v_1, \dots, v_\ell$ be a basis for $D(V)$. Then clearly, $u_1 \dots u_k v_1 \dots v_\ell$ is an orthonormal sequence in V. Hence, Lemma 3.2 implies $u_1, \dots, u_k, v_1, \dots, v_\ell$ is linearly independent. Since $\pi : V \to V/D(V)$ is a linear surjection with kernel $D(V)$, then $\dim V = \dim(V/D(V)) + \dim D(V) = k + \ell$. Hence, $u_1, \dots, u_k, v_1 \dots v_\ell$ must span V. Consequently, $u_1 \dots u_k v_1 \dots v_\ell$ is an orthonormal bases for V.

This completes the proof that every s.b.f. space has an orthonormal basis.

Next let $(V,\circ)$ be a finite dimensional s.b.f. space and let $u_1, \cdots, u_k$ be an orthonormal sequence in V. We will prove that $u_1, \cdots, u_k$ can be enlarged to an orthonormal basis for V.

First assume $(V,\circ)$ is non-degenerate. Let W be the vector subspace of V spanned by $u_1, \cdots, u_k$. Since each $u_i \neq 0$ and V is non-degenerate, then $u_i \circ u_i = \pm 1$ for $1 \le i \le k$. Hence, Lemma 3.5 implies $V = W \oplus W^\perp$. $W^\perp$ has an orthonormal basis $u_{k+1}, \cdots, u_n$. Therefore, $u_1, \cdots, u_k, u_{k+1}, \cdots, u_n$ is clearly an orthonormal basis for V which enlarges $u_1, \cdots, u_k$.

Now assume V is not non-degenerate. We can assume $u_1, \cdots, u_k$ is ordered so that there is a j, $0 \le j \le k$, such that $u_i \circ u_i = \pm 1$ for $1 \le i \le j$ and u_i is degenerate for $j+1 \le i \le k$. Hence, $u_{j+1}, \cdots, u_k$ is a linearly independent sequence in $D(V)$. Enlarge $u_{j+1}, \cdots, u_k$ to a basis $u_{j+1}, \cdots, u_k, u_{k+1}, \cdots, u_\ell$ for $D(V)$. Thus, $\dim D(V) = \ell - j$. Lemma 3.7

implies that $V/D(V)$ is a finite dimensional non-degenerate s.b.f. space, and $\pi(u_1), \ldots, \pi(u_j)$ is an orthonormal sequence in $V/D(V)$. Enlarge $\pi(u_1), \ldots, \pi(u_j)$ to an orthonormal basis $\pi(u_1), \ldots, \pi(u_j), \bar{u}_{l+1}, \ldots, \bar{u}_m$ for $V/D(V)$. Thus, $\dim V/D(V) = j + (m-l)$. Choose $u_i \in V$ so that $\pi(u_i) = \bar{u}_i$ for $l+1 \le i \le m$. Since, according to Lemma 3.7 π preserves the bilinear forms, then it follows that $u_1, \ldots u_j, u_{l+1}, \ldots, u_m$ is an orthonormal sequence in V such that $u_i \circ u_i = \pm 1$ for $1 \le i \le j$ and $l+1 \le i \le m$. Now clearly $u_1 \cdots u_j \; u_{j+1} \cdots u_l \; u_{l+1} \cdots u_m$ is an orthonormal sequence in V. Hence, $u_1, \ldots, u_m$ is linearly independent by Lemma 3.2. Since $\dim V = \dim D(V) + \dim V/D(V) = (l-j) + j + (m-l) = m$, then it follows that $u_1, \ldots, u_m$ spans V. Hence, $u_1, \ldots, u_m$ is an orthonormal basis for V that enlarges $u_1, \ldots, u_k$. $\square$

Homework Problem 3.1.

Define the non-degenerate s.b.f. $\circ$ on $\mathbb{R}^4$ by $x \circ y = x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4$ for $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$.
Let $u_1 = e_1 + e_2 + e_4$ and $u_2 = e_2 + e_3 + e_4$. Observe that u_1, u_2 is an orthonormal sequence in $(\mathbb{R}^4, \circ)$. Enlarge u_1, u_2 to an orthonormal basis for $(\mathbb{R}^4, \circ)$.

<u>Def</u> Let $(V, \circ)$ be a s.b.f. space. Let $u_1, \dots, u_n$ be an orthonormal basis for V. If $u_1, \dots, u_n$ has $r \geq 0$ positive elements and $s \geq 0$ negative elements, then the ordered pair (r, s) is called the <u>signature</u> of the orthonormal basis $u_1, \dots, u_n$.

<u>Theorem 3.8.</u> If $(V, \circ)$ is a finite dimensional s.b.f. space, then every orthonormal basis for V has the same signature.

We will give two proofs of this theorem. The first proof is more algebraic; the second is more geometric.

<u>Lemma 3.9</u> If $u_1, \dots, u_n$ is an orthonormal basis for a s.b.f. space $(V, \circ)$ and $u_1, \dots, u_n$ has signature (r, s), then $r + s = n - \dim D(V)$.

<u>Proof</u> Reorder $u_1, \dots, u_n$ if necessary so that $u_i \circ u_i = \pm 1$ for $1 \leq i \leq r+s$ and u_i is degenerate for $r+s+1 \leq i \leq n$. Thus, $u_{r+s+1}, \dots, u_n$ is a linearly independent subset of $D(V)$. Let $x \in D(V)$. Then $x = \sum_{i=1}^{n} a_i u_i$ for some $a_1, \dots, a_n \in R$. Since $x \in D(V)$, then $x \circ u_i = 0$ for $1 \leq i \leq n$. Hence, for $1 \leq j \leq r+s$,

$$0 = u_j \circ x = u_j \circ \left(\Sigma_{i=1}^{n} a_i u_i\right) = a_j (u_j \circ u_j) = \pm a_j.$$

Thus, $a_j = 0$ for $1 \le j \le r+s$. Hence, $x = \Sigma_{i=r+s+1}^{n} a_i u_i$. This proves $u_{r+s+1}, \ldots, u_n$ spans $D(V)$. Hence $u_{r+s+1}, \ldots, u_n$ is a basis for $D(V)$. Therefore, $\dim D(V) = n-(r+s)$. $\square$

<u>First Proof of Theorem 3.8</u> First assume $(V, \circ)$ is non-degenerate. Then $D(V) = 0$. Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are orthonormal bases for $(V, \circ)$ with signatures (p, q) and (r, s), respectively. Lemma 3.7 implies $p+q = n = r+s$. We can suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are ordered so that

$$u_i \circ u_i = 1 \text{ for } 1 \le i \le p, \quad u_i \circ u_i = -1 \text{ for } p+1 \le i \le n,$$
$$v_j \circ v_j = 1 \text{ for } 1 \le j \le r, \quad v_j \circ v_j = -1 \text{ for } r+1 \le j \le n.$$

Assume $p \ne r$. Say $p > r$. Then

$$p + (n-r) = p+s > r+s = n.$$

Hence, $u_1, \ldots, u_p, v_{r+1}, \ldots, v_n$ is a linearly dependent sequence in V. Thus, there are real numbers $a_1, \ldots, a_p, b_{r+1}, \ldots, b_n$, not all 0, such that

$$\sum_{i=1}^{p} a_i u_i + \sum_{j=r+1}^{n} b_j v_j = 0$$

Therefore,

$$\sum_{i=1}^{p} a_i u_i = \sum_{j=r+1}^{n} (-b_j) v_j .$$

Hence,

$$\left(\sum_{i=1}^{p} a_i u_i\right) \circ \left(\sum_{i=1}^{p} a_i u_i\right) = \left(\sum_{j=r+1}^{n} (-b_j) v_j\right) \circ \left(\sum_{j=r+1}^{n} (-b_j) v_j\right)$$

Then $\sum_{i=1}^{p} a_i^2 (u_i \circ u_i) = \sum_{j=r+1}^{n} b_j^2 (v_j \circ v_j)$

Thus, $\sum_{i=1}^{p} a_i^2 = - \sum_{j=r+1}^{n} b_j^2 .$

It follows that $\sum_{i=1}^{p} a_i^2 = 0 = \sum_{j=r+1}^{n} b_j^2$. Therefore, $a_1 = \cdots = a_p = 0 = b_{r+1} = \cdots = b_n$. We've reached a contradiction. We conclude that $p = r$. Hence $q = n - p = n - r = s$.

Now assume V is not non-degenerate. Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are orthonormal bases for $(V, \circ)$ with signatures (p, q) and (r, s), respectively. Lemma 3.9 implies $p+q = n - \dim D(V) = r+s$. We can assume $u_1, \ldots, u_n$ and $v_1 \ldots v_n$ are ordered so that $u_i \circ u_i = \pm 1$ and $v_i \circ v_i = \pm 1$ for $1 \leq i \leq p+q$, and u_i and v_i are degenerate for $p+q+1 \leq i \leq n$. Lemma 3.7 implies $V/D(V)$ is a finite dimensional non-degenerate s.b.f. space and $\pi(u_1), \ldots, \pi(u_{p+q})$ and

$\pi(v_1), \ldots, \pi(v_{r+s})$ are orthonormal sequences in $V/D(V)$. Hence, $\pi(u_1), \ldots, \pi(u_{p+q})$ and $\pi(v_1), \ldots, \pi(v_{r+s})$ are linearly independent sequences by Lemma 3.2. Since $\dim(V/D(V)) = \dim V - \dim D(V) = n - \dim D(V) = p+q = r+s$, then it follows that $\pi(u_1), \ldots, \pi(u_{p+q})$ and $\pi(v_1), \ldots, \pi(v_{r+s})$ are orthonormal basis for $V/D(V)$. Since $\pi(u_i) \circ' \pi(u_i) = u_i \circ u_i$ and $\pi(v_i) \circ' \pi(v_i) = v_i \circ v_i$ for $1 \leq i \leq p+q$, then $\pi(u_1), \ldots, \pi(u_{p+q})$ has signature (p,q), and $\pi(v_1), \ldots, \pi(v_{r+s})$ has signature (r,s). Since $V/D(V)$ is non-degenerate, then the argument in the first paragraph of this proof implies $(p,q) = (r,s)$. $\square$

Def If $(V, \circ)$ is a s.b.f. space, let $O(V, \circ)$ denote the set of all linear isomorphisms $f: V \to V$ such that $f(x) \circ f(y) = x \circ y$ for all $x, y \in V$. An element of $O(V, \circ)$ is called an orthogonal map of $(V, \circ)$.

Def If $(V, \circ)$ is a s.b.f. space and $u \in V$ such that $u \circ u = \pm 1$, call u a unit vector.

Def Let $(V, \circ)$ be a s.b.f. space and suppose u is a unit vector in V. Define

$$Z_u : V \to V$$

by

$$Z_u(x) = x - 2\left(\frac{x \circ u}{u \circ u}\right) u.$$

Call Z_u a linear reflection of $(V, \circ)$.

Theorem 3.10. Let $(V, \circ)$ be a s.b.f. space and let u be a unit vector in V. Then

a) $Z_u \in O(V, \circ)$

b) $Z_u^{-1} = Z_u$

c) For $x \in V$, $Z_u(x) = x$ if and only if $x \circ u = 0$.

d) If $y, z \in V$ such that $y \circ y = z \circ z$ and $(y-z) \circ (y-z) \neq 0$, and if $u = y-z / \sqrt{|(y-z) \circ (y-z)|}$, then u is a unit vector and $Z_u(y) = z$.

Proof a) It is obvious from the equation defining Z_u that it is linear. To see that Z_u preserves the bilinear form, let $x, y \in V$. Then

$$Z_u(x) \circ Z_u(y) = \left(x - 2\left(\frac{x \circ u}{u \circ u}\right) u\right) \circ \left(y - 2\left(\frac{y \circ u}{u \circ u}\right) u\right) =$$

$$x \circ y - 4\,\frac{(x \circ u)(y \circ u)}{(u \circ u)} + 4\,\frac{(x \circ u)(y \circ u)(u \circ u)}{(u \circ u)^2} = x \circ y.$$

b) We prove $Z_u \circ Z_u = id_V$. Let $x \in V$. Then

$$Z_u \circ Z_u(x) = Z_u(x) - 2\left(\frac{Z_u(x) \circ u}{u \circ u}\right) u =$$

$$\left(x - 2\left(\frac{x \circ u}{u \circ u}\right) u\right) - 2\left(\left(x - 2\left(\frac{x \circ u}{u \circ u}\right) u\right) \circ u\right) \frac{u}{(u \circ u)} =$$

$$x - 2\left(\frac{x \circ u}{u \circ u}\right) u - 2\left(\frac{x \circ u}{u \circ u}\right) u + 4\left(\frac{(x \circ u)(u \circ u)}{(u \circ u)^2}\right) u = x.$$

c) Clearly, the following statements are equivalent:

$Z_u(x) = x$, $x - 2\left(\frac{x \circ u}{u \circ u}\right) u = x$, and $x \circ u = 0$.

d) Suppose $y, z \in V$ such that $y \circ y = z \circ z$ and $(y-z)\circ(y-z) \neq 0$. Let

$$u = \frac{y-z}{\sqrt{|(y-z)\circ(y-z)|}}.$$

Then $u \circ u = \dfrac{(y-z)\circ(y-z)}{|(y-z)\circ(y-z)|} = \pm 1$. Thus, u is a unit vector.

Let $\alpha = |(y-z)\circ(y-z)|$. Then $u = \dfrac{y-z}{\sqrt{\alpha}}$

Since $y \circ y = z \circ z$, then

$(y-z)\circ(y-z) = y\circ y - 2y\circ z + z \circ z = 2 y\circ y - 2 y \circ z = 2y\circ(y-z)$.

Observe that $y \circ u = \dfrac{y\circ(y-z)}{\sqrt{\alpha}}$ and

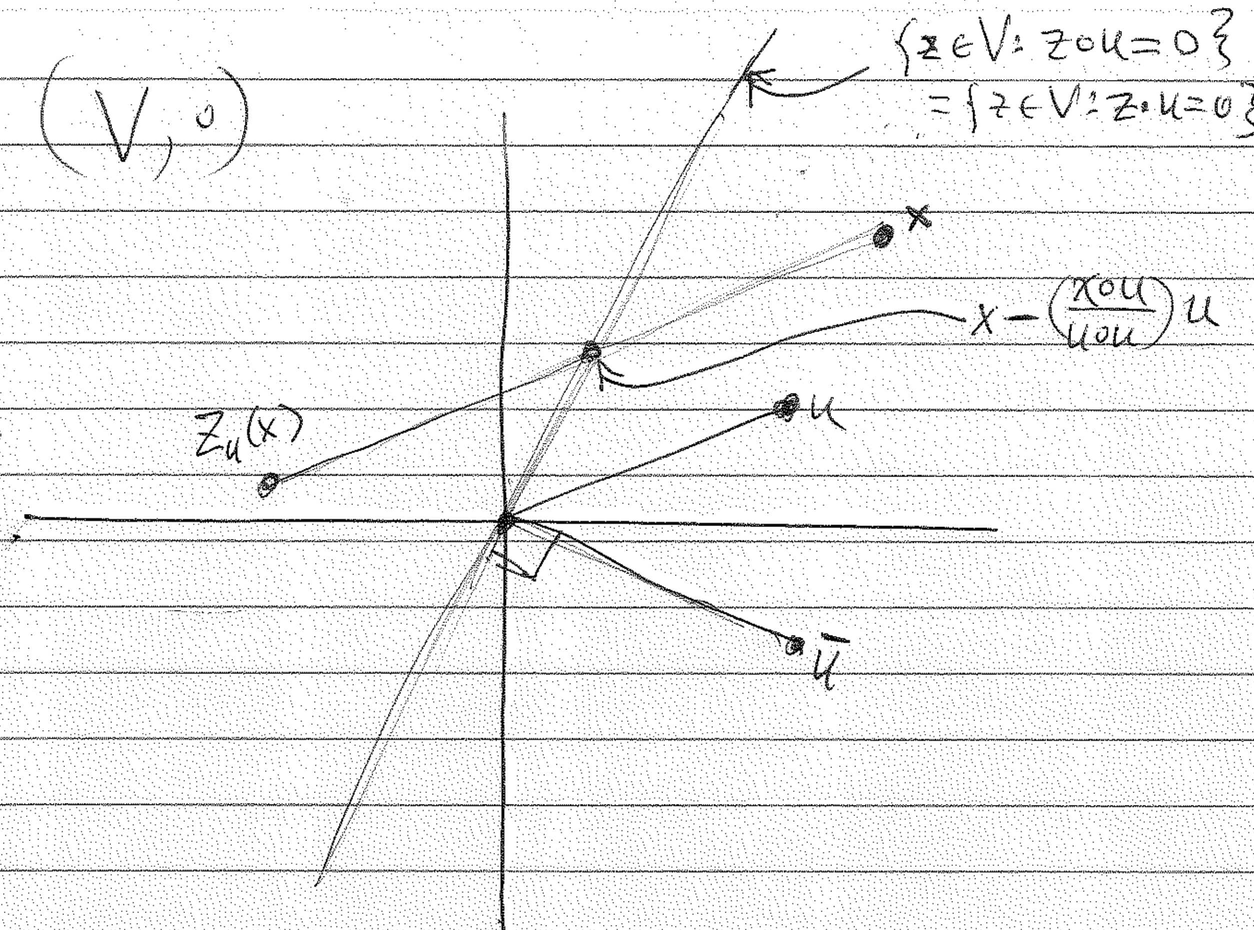
$u \circ u = \dfrac{(y-z)\circ(y-z)}{\alpha} = \dfrac{2y\circ(y-z)}{\alpha}$.

Thus,

$$2\left(\frac{y\circ u}{u \circ u}\right)u = 2\left(\frac{y\circ(y-z)/\sqrt{\alpha}}{2y\circ(y-z)/\alpha}\right)\frac{y-z}{\sqrt{\alpha}} = y - z.$$

Therefore,

$$Z_u(y) = y - 2\left(\frac{y\circ u}{u\circ u}\right)u = y-(y-z) = z. \quad \square$$



Z_u is a non-orthogonal reflection in the hyperplane $\{z \in V : z \circ u = 0\} = P(\bar{u}, 0)$ that moves points parallel to u.

Corollary 3.11 If $(V, \circ)$ is a s.b.f. space and $y, z \in V$ such that $y \circ y = z \circ z \neq 0$, then there is an $f \in O(V, \circ)$ such that $f(y) = z$.

Proof Since $y \circ y \neq 0$, $y \circ y$ is not equal to both $y \circ z$ and $-y \circ z$. Hence, at least one of the expressions

$$(y-z)\circ(y-z) = 2(y\circ y - y\circ z)$$

$$(y+z)\circ(y+z) = 2(y\circ y + yz)$$

is non-zero.

<u>Case 1: $(y-z)\circ(y-z) \neq 0$</u>

In this case let $u = \dfrac{y-z}{\sqrt{|(y-z)\circ(y-z)|}}$.

Then Theorem 3.10 implies $Z_u \in O(V,\circ)$ and $Z_u(y) = z$.

<u>Case 2:</u> $(y+z)\circ(y+z) \neq 0$.

In this ~~case~~ case, let $u = \dfrac{y+z}{\sqrt{|(y+z)\circ(y+z)|}}$.

Then Theorem 3.10 implies $Z_u \in O(V,\circ)$ and $Z_u(y) = -z$. Define $f: V \to V$ by $f(x) = -Z_u(x)$. f is clearly linear, and for $w, x \in V$,

$$f(w)\circ f(x) = (-Z_u(w))\circ(-Z_u(x)) = Z_u(w)\circ Z_u(x) = w\circ x.$$

Thus, $f \in O(V,\circ)$. Also $f(y) = -Z_u(y) = -(-z) = z$.

$\square$

Second Proof of Theorem 3.8.

Suppose $(V, \circ)$ is a finite dimensional s.b.f. space and $u_1, \dots, u_n$ and $v_1, \dots, v_n$ are orthonormal bases for $(V, \circ)$ of signatures (p, q) and (r, s), respectively. Then Lemma 3.9 implies $p+q = n - \dim D(V) = r+s$.

We prove: $p=0$ implies $r=0$. For assume $p=0$ and $r>0$. Then $u_i \circ u_i \leq 0$ for $1 \leq i \leq n$, and we can assume $v_1 \circ v_1 = +1$. Since $u_1, \dots, u_n$ is a basis for V, then $v_1 = \sum_{i=1}^n a_i u_i$ where $a_1, \dots, a_n \in \mathbb{R}$. Hence,

$$+1 = v_1 \circ v_1 = \left(\sum_{i=1}^n a_i u_i\right) \circ \left(\sum_{i=1}^n a_i u_i\right) = \sum_{i=1}^n a_i^2 (u_i \circ u_i) \leq 0.$$

We have reached a contradiction. This proves: $p=0$ implies $r=0$. A similar argument shows: $r=0$ implies $p=0$. Thus: $p=0$ if and only if $r=0$. A slight variant of this proof shows $q=0$ if and only if $s=0$.

We now proceed by induction on $\dim V$.

If $\dim V = 1$, then (p,q) and (r,s) $\in \{(0,0), (1,0), (0,1)\}$. Since $p=0$ iff $r=0$ and $q=0$ iff $s=0$, it follows that $(p,q) = (r,s)$.

Now assume $\dim V > 1$ and suppose that Theorem 3.8 holds for every finite dimensional s.b.f. space of dimension $< \dim V$.

If $p = 0$, then $r = 0$. Hence, $q = p + q = n - \dim D(V) = r + s = s$. Thus, $(p, q) \overset{k}{=} (r, s)$ and we're done. So we can assume $p > 0$. Since $r = 0$ implies $p = 0$, then it follows that $r > 0$. Thus, we can assume $u_1 \circ u_1 = +1 = v_1 \circ v_1$. Corollary 3.11 implies there is an $f \in O(V, \circ)$ such that $f(u_1) = v_1$. Then $f(u_1), f(u_2), \ldots, f(u_n)$ is an orthonormal basis for $(V, \circ)$ of signature p, q. Let W be the vector subspace of V spanned by $f(u_1) = v_1$. Since $f(u_1), \ldots, f(u_n)$ and $v_1, \ldots, v_n$ are orthonormal sequences, it follows that $f(u_2), \ldots, f(u_n)$ and $v_2, \ldots, v_n$ are orthonormal sequences in $W^\perp$. Hence, by Lemma 3.2, $f(u_2), \ldots, f(u_n)$ and $v_2, \ldots, v_n$ are linearly independent sequences in $W^\perp$. Lemma 3.5 implies $V \overset{\perp}{=} W \oplus W^\perp$. Thus,

$$n = \dim V = \dim W + \dim W^\perp = 1 + \dim W^\perp.$$

So $\dim W^\perp = n - 1$. It follows that

$f(u_2), \ldots, f(u_n)$ and $v_2, \ldots, v_n$ are orthonormal bases for W^+.

Clearly, the signatures of $f(u_2), \ldots, f(u_n)$ and $v_2, \ldots, v_n$ are $(p-1, q)$ and $(r-1, s)$, respectively. Hence, the inductive hypothesis implies $(p-1, q) = (r-1, s)$. We conclude that $(p, q) = (r, s)$. $\square$

Theorem 3.8 makes the following definition possible.

Def Let $(V, \circ)$ be a finite dimensional s.b.f. The ordered pair (r, s) of non-negative integers is the signature of $(V, \circ)$ if (r, s) is the signature of some (and, hence, every) orthonormal basis for $(V, \circ)$.

Homework Problem 3.2. Let $(V, \circ)$ be a finite dimensional s.b.f. space. If $y, z \in V$ and $y \circ y = z \circ z$, then Theorem 3.10.d implies that $(y-z)\circ(y-z) \neq 0$ is a sufficient condition for the existence of a linear reflection Z_u such that $Z_u(y) = z$.

a) Prove that $(y-z)\circ(y-z) \neq 0$ is also a necessary condition for the existence of a linear reflection Z_u such that $Z_u(y) = z$.

There are instances of elements y and z in a finite dimensional s.b.f. space such that $y \circ y = z \circ z \neq 0$ and $(y-z)\circ(y-z) = 0$. Thus, in these instances, there is no linear reflection Z_u such that $Z_u(y) = z$.

b) Prove that if $(V, \circ)$ is a s.b.f. space of signature (r, s) such that either $r \geq 1$ and $s \geq 2$ or $r \geq 2$ and $s \geq 1$, then there are $y, z \in V$ such that $y \circ y = z \circ z \neq 0$ and $(y-z)\circ(y-z) = 0$.

Recall that Minkowski n-space M^n is a non-degenerate s.b.f. space of signature $(n-1, n)$. ($e_i \circ e_i = +1$ for $1 \le i \le n-1$ and $e_n \circ e_n = -1$.) The following result shows that under some (important) circumstances, if $y, z \in M^n$ and $y \circ y = z \circ z$, then $(y-z)\circ(y-z) \ne 0$ and hence, there is a linear reflection of M^n that moves y to z.

<u>Theorem 3.12.</u> If $y, z \in M^n$ such that $y \ne z$ and $y \circ y = z \circ z < 0$, then $(y-z)\circ(y-z) \ne 0$ and, hence, there is a linear reflection Z_u of M^n such that $Z_u(y) = z$.

<u>Remark.</u> Reconcile Theorem 3.12 with Homework Problem 3.1.6.

<u>Proof</u> Assume $y \circ y = z \circ z < 0$ and $(y-z)\circ(y-z) = 0$. We will prove $y = z$.

Let $y = (y_1, \dots, y_n)$ and $z = (z_1, \dots, z_n)$. Let $v = (y_1, \dots, y_{n-1})$ and $w = (z_1, \dots, z_{n-1})$, and let $a = y \circ y = z \circ z < 0$.

$0 = (y-z)\circ(y-z) = 2a - 2y\circ z$. Hence,

$y \circ z = a$. Hence:

$$a = y \circ y = \|v\|^2 - y_n^2$$

$$a = z \circ z = \|w\|^2 - z_n^2$$

$$a = y \circ z = v \circ w - y_n z_n$$

Thus, $v \circ w = a + y_n z_n$, $\|v\|^2 = a + y_n^2$, $\|w\|^2 = a + z_n^2$.
The Cauchy inequality implies
$(v \circ w)^2 \leq \|v\|^2 \|w\|^2$. Hence,

$$(a + y_n z_n)^2 \leq (a + y_n^2)(a + z_n^2)$$

Therefore,

$$a^2 + 2a y_n z_n + y_n^2 z_n^2 \leq a^2 + a(y_n^2 + z_n^2) + y_n^2 z_n^2.$$

Thus, $2a y_n z_n \leq a(y_n^2 + z_n^2)$.
Since $a < 0$, then $2 y_n z_n \geq y_n^2 + z_n^2$.
Thus, $0 \geq y_n^2 - 2 y_n z_n + z_n^2 = (y_n - z_n)^2$.
Consequently, $y_n = z_n$. Hence,

$y - z = (y_1 - z_1, \ldots, y_{n-1} - z_{n-1}, 0)$.

So $0 = (y - z) \circ (y - z) = \sum_{i=1}^{n-1} (y_i - z_i)^2$.

Thus, $y_i = z_i$ for $1 \leq i \leq n-1$. Therefore, $y = z$. $\square$

Observe that the Euclidean norm on $\mathbb{E}^n$ satisfies the following two equations:

$$\|ax\|^2 = a^2\|x\|^2$$

$$\frac{1}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right) = x \cdot y.$$

for all $x, y \in \mathbb{E}^n$ and $a \in \mathbb{R}$. This observation motivates the following definition:

Def. Let V be a vector space. A function $Q: V \to \mathbb{R}$ is a quadratic form on V if:

a) $Q(ax) = a^2 Q(x)$ for all $x \in V$ and $a \in \mathbb{R}$,

b) A bilinear form $\circ: V \times V \to \mathbb{R}$ is defined by

$$x \circ y = \frac{1}{2}\left(Q(x+y) - Q(x) - Q(y)\right).$$

Theorem 3.13: Sylvester's Law of Inertia. If V is a finite dimensional vector space and $Q: V \to \mathbb{R}$ is a quadratic form, then there is a basis $u_1, \dots, u_n$ for V and integers $r, s \geq 0$ depending only on Q such that $r+s \leq n$ and such that for each $x \in V$, if $x = \sum_{i=1}^n a_i u_i$ where $a_1, \dots, a_n \in \mathbb{R}$, then

$$Q(x) = \sum_{i=1}^r a_i^2 - \sum_{i=r+1}^{r+s} a_i^2.$$

Proof Define the bilinear form $\circ : V \times V \to \mathbb{R}$ by $x \circ y = \frac{1}{2}(Q(x+y) - Q(x) - Q(y))$. $\circ$ is clearly symmetric. Let (r,s) be the signature of $(V, \circ)$ and let $u_1, \dots u_n$ be an orthonormal basis for $(V, \circ)$ ordered so that $u_i \circ u_i = +1$ for $1 \le i \le r$, $u_i \circ u_i = -1$ for $r+1 \le i \le r+s$, and u_i is degenerate for $r+s+1 \le i \le n$.

Let $x \in V$ and let $a_1, \dots, a_n \in \mathbb{R}$ such that $x = \sum_{i=1}^n a_i x_i$. First observe that

$$x \circ x = \frac{1}{2}(Q(x+x) - Q(x) - Q(x)) = \frac{1}{2}(Q(2x) - 2Q(x))$$
$$= \frac{1}{2}(4Q(x) - 2Q(x)) = \frac{1}{2}(2Q(x)) = Q(x).$$

Hence,

$$Q(x) = x \circ x = \left(\sum_{i=1}^n a_i u_i\right) \circ \left(\sum_{i=1}^n a_i u_i\right) =$$
$$\sum_{i=1}^n a_i^2 (u_i \circ u_i) = \sum_{i=1}^r a_i^2 - \sum_{i=r+1}^{r+s} a_i^2. \quad \square$$

Recall that if $f : \mathbb{E}^m \to \mathbb{E}^n$ is a function that preserves dot products, then f must be linear by Corollary 1.18. We now explore the extent to which this result extends to finite dimensional s.b.f. spaces.

Theorem 3.14. Suppose $(V, \circ)$ and $(W, \circ)$ are finite dimensional s.b.f. spaces with signatures (p, q) and (r, s), respectively; and suppose $f : V \to W$ is a bilinear form preserving function. (In other words, $f(x) \circ f(y) = x \circ y$ for all $x, y \in V$.) If $r + s = \dim W$ and either $p = r$ or $q = s$, then f is linear.

Homework Problem 3.3. Suppose $(V, \circ)$ and $(W, \circ)$ are finite dimensional s.b.f. spaces with signatures (p, q) and (r, s), respectively; and suppose $f : V \to W$ is a bilinear form preserving function.

a) Prove that $p \le r$ and $q \le s$.

b) Prove that if $r + s < \dim W$, then f need not be linear

c) Prove that if $p < r$ and $q < s$, then f need not be linear.

d) Prove Theorem 3.14.