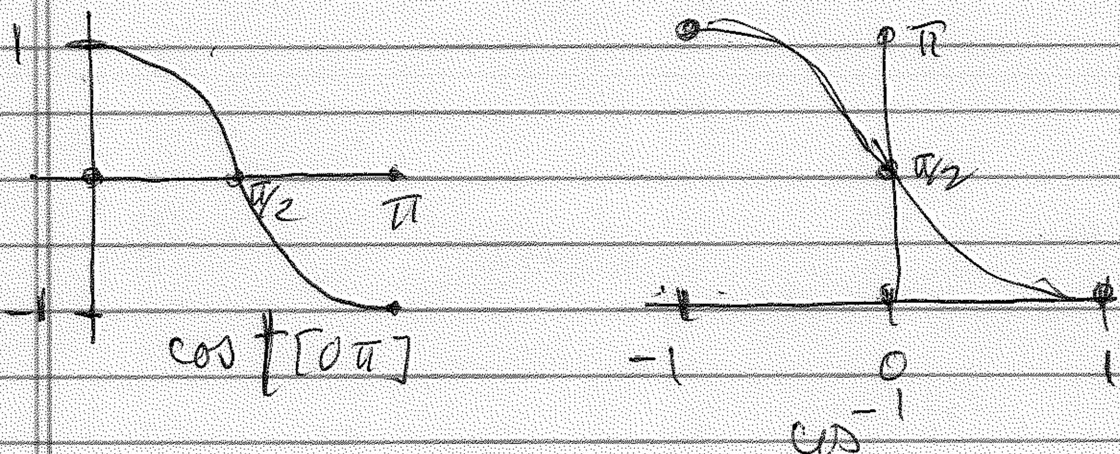
2. Spheres.

<u>Notation</u> $S^n = \{x \in \mathbb{E}^{n+1} : \|x\| = \}$ is called the <u>n-sphere</u>. For $r > 0$, $S^n_r = \{x \in \mathbb{E}^{n+1} : \|x\| = r\}$ is called the <u>n-sphere of radius r</u>. Always assume $n \geq 1$.

<u>Def</u> Let $\cos^{-1} : [-1,1] \to [0,\pi]$ be the inverse of the bijection

$$\cos|[0,\pi] : [0,\pi] \to [-1,1]$$



<u>Def</u> Define $\Theta : S^n \times S^n \to [0,\pi]$ by

$$\Theta(x,y) = \cos^{-1}(x \cdot y)$$

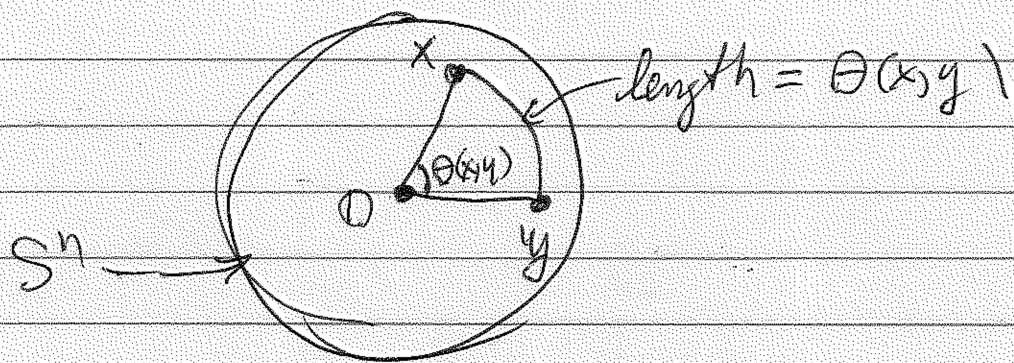
For $r > 0$, define $\Theta_r : S^n_r \times S^n_r \to [0, \pi r]$ by

$$\Theta_r(x,y) = r\cos^{-1}\left(\frac{x \cdot y}{r^2}\right) = r\,\Theta\left(\frac{x}{r}, \frac{y}{r}\right).$$

Intuitively, for $x, y \in S^n$:

$\Theta(x,y)$ = the measure of the angle $\measuredangle x0y$

= the length of the shortest circular arc in S^n joining x to y.



Similarly, for $x, y \in S^n_r$:

$\Theta_r(x,y)$ = the length of the shortest circular arc in S^n_r joining x to y.

Theorem 2.1. Θ is a metric on S^n. Also for $r > 0$, Θ_r is a metric on S^n_r.

Proof that Θ is symmetric.

For $x, y \in S^n$,

$$\Theta(x,y) = \cos^{-1}(x \cdot y) = \cos^{-1}(y \cdot x) = \Theta(y,x). \quad \square$$

Proof that Θ satisfies positivity.

Since $\Theta : S^n \times S^n \to [0, \pi]$, then $\Theta(x,y) \geq 0$ for all $x, y \in S^n$.

For $x \in S^n$, $\Theta(x,x) = \cos^{-1}(x \cdot x) = \cos^{-1}(1) = 0$.

Now suppose $x, y \in S^n$ and $\Theta(x,y) = 0$. Then $\cos^{-1}(x \cdot y) = 0$. Hence,

$$x \cdot y = \cos(0) = 1 = \|x\| \, \|y\|.$$

Therefore, Theorem 1.6.a (The Equality Case of the Cauchy Inequality) implies $\|y\| x = \|x\| y$. Thus $x = y$. $\square$

The proofs that for $r > 0$, Θ_r satisfies symmetry and positivity are similar.

Before proving that Θ satisfies the triangle inequality, we establish the Spherical Law of Cosines. (Radcliffe presents a different proof using properties of the cross product.) To state the spherical law of cosines we introduce angle measure in a sphere.

<u>Definition</u> Let $x, y, z \in S^n$ such that $y, z \neq \pm x$. Let

$$u = \frac{y - (y \cdot x)x}{\sqrt{1-(y \cdot x)^2}} \quad \text{and} \quad v = \frac{z - (z \cdot x)x}{\sqrt{1-(z \cdot x)^2}}$$

Then $u, v \in S^n$, $x \cdot u = 0 = x \cdot v$,

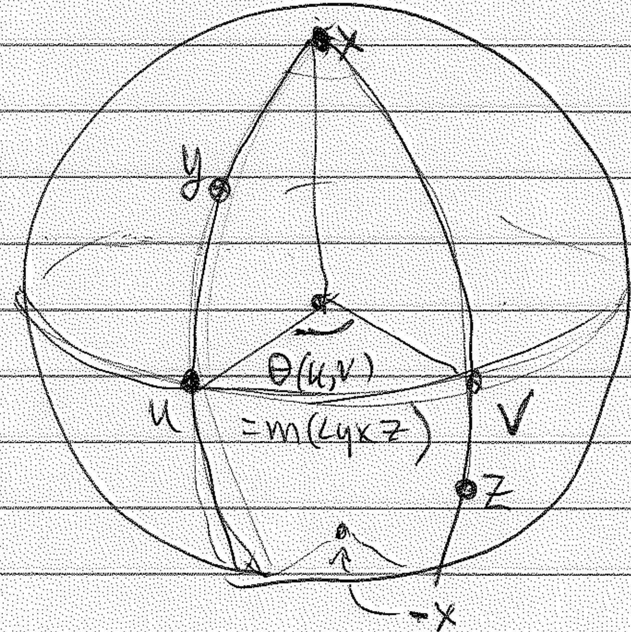
$y = (y \cdot x)x + \sqrt{1-(y \cdot x)^2}\, u$ and $z = (z \cdot x)x + \sqrt{1-(z \cdot x)^2}\, v$.

(<u>Exercise</u>: Verify these assertions.)

Define

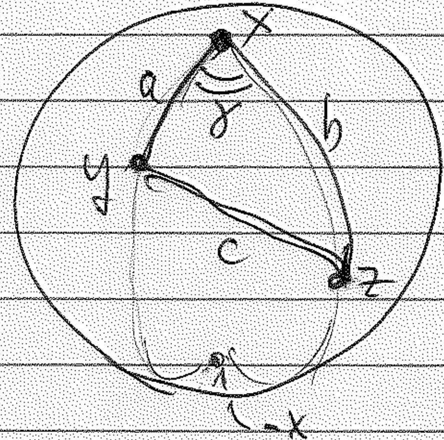
$$m(\angle yxz) = \theta(u, v) \quad \in [0, \pi]$$

and call $m(\angle yxz)$ the <u>measure of the angle</u> $\underline{\angle yxz}$.



The Spherical Law of Cosines 2.2. Let $x, y, z \in S^n$ such that $y, z \neq \pm x$.
Let $a = \theta(x,y)$, $b = \theta(x,z)$, $c = \theta(y,z)$ and $\gamma = m(\angle yxz)$. Then

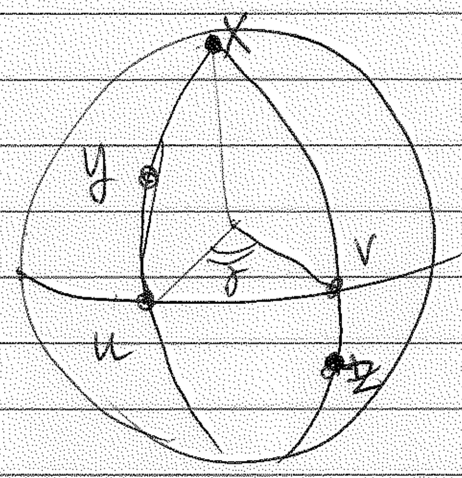
$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma$$



Proof The definitions of a, b and c imply $\cos a = x \cdot y$, $\cos b = x \cdot z$ and $\cos c = y \cdot z$.
Since $a, b \in [0, \pi]$, then $\sin a, \sin b \in [0, 1]$.
Therefore $\sin a = \sqrt{1 - \cos^2 a} = \sqrt{1 - (x \cdot y)^2}$ and $\sin b = \sqrt{1 - \cos^2 b} = \sqrt{1 - (x \cdot z)^2}$.

As in the definition of $m(\angle yxz)$, we introduce:

$$u = \frac{y - (y \cdot x)x}{\sqrt{1 - (y \cdot x)^2}} \quad \text{and} \quad v = \frac{z - (z \cdot x)x}{\sqrt{1 - (z \cdot x)^2}}$$



Then $u, v \in S^n$, $x \cdot u = 0 = x \cdot v$, $y = (y \cdot x)x + \sqrt{1-(y \cdot x)^2}\, u$,
$z = (z \cdot x)x + \sqrt{1-(z \cdot x)^2}\, v$ and $\gamma = m(\angle yxz) = \theta(u, v)$
Therefore, $\cos\gamma = u \cdot v$. Now:

$$\cos c = y \cdot z = (y \cdot x)(z \cdot x) + \sqrt{1-(y \cdot x)^2}\sqrt{1-(z \cdot x)^2}\,(u \cdot v)$$

$$= (\cos a)(\cos b) + (\sin a)(\sin b)(\cos\gamma). \quad \square$$

<u>Proof that θ satisfies the triangle inequality.</u> Let $x, y, z \in S^n$. We will prove $\theta(y,z) \le \theta(x,y) + \theta(x,z)$.

If $y = x$, then $\theta(y,z) = \theta(xz) \le \theta(xy) + \theta(xz)$
Observe that $\theta(x,-x) = \cos^{-1}(x \cdot (-x)) = \cos^{-1}(-1) = \pi$.
Hence if $y = -x$, then $\theta(y,z) \le \pi = \theta(x,y) \le \theta(xy) + \theta(xz)$.
Similarly, if $z = \pm x$, then $\theta(y,z) \le \theta(xy) + \theta(xz)$.

Now assume $y, z \neq \pm x$. Let
$a = \theta(x,y)$, $b = \theta(x,z)$ and $c = \theta(y,z)$.
We will prove $c \leq a+b$. Let $\gamma = m(\angle yxz)$.
Then the Spherical Law of Cosines implies

$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma.$$

Since $a, b \in [0,\pi]$, then $\sin a \geq 0$ and $\sin b \geq 0$.
Also $\cos \gamma \geq -1$. Thus $\sin a \sin b \cos \gamma \geq -\sin a \sin b$.
Hence,

$$\cos c \geq \cos a \cos b - \sin a \sin b = \cos(a+b).$$

Since $\cos | [0,\pi]$ is monotone decreasing,
then $c \leq a+b$. $\square$

Homework Problem 2.1. Let $r > 0$.

a) Formulate a definition of $m(\angle yxz)$ for
$x, y, z \in S^n_r$ such that $y, z \neq \pm x$.

b) Formulate and prove an appropriate version of the Spherical Law of Cosines for S^n_r.

c) Prove that θ_r satisfies the triangle inequality.

We now explore geodesic lines in S^n. We define great circles in S^n and show they are geodesic lines. We prove that any two points in S^n lie on a great circle, thereby establishing that S^n is a totally geodesic space. We prove that every geodesic line in S^n is a great circle and, more strongly that every unit speed local geodesic in S^n is the restriction of a great circle to some subinterval of $\mathbb{R}$.

Recall that Theorem 1.68 implies that a curve in a metric space is a unit speed local geodesic if and only if it is locally distance preserving. A unit speed local geodesic with domain $\mathbb{R}$ is called a geodesic line. A metric space is totally geodesic if every two points lie in a geodesic line.

<u>Definition</u> For $u, v \in S^n$ such that $u \cdot v = 0$, define $G_{uv} : \mathbb{R} \to S^n$ by

$$G_{uv}(t) = \cos(t)\,\mathbf{u} + \sin(t)\,\mathbf{v}.$$

($G_{uv}(\mathbb{R}) \subseteq S^n$ because $\|G_{uv}(t)\|^2 = \cos^2(t) + \sin^2(t) = 1$.) Call G_{uv} a <u>great</u> <u>circle</u> in S^n.

Lemma 2.3 If $u, v \in S^n$ and $u \cdot v = 0$, then for each $t \in \mathbb{R}$:

a) $G_{uv}(t+\frac{\pi}{2}) = G'_{uv}(t)$ and $G_{uv}(t) \cdot G_{uv}(t+\frac{\pi}{2}) = 0$,

b) $G_{uv}(t+\pi) = -G_{uv}(t) = G''_{uv}(t)$, and

c) $G_{u,-v}(t) = G_{uv}(-t)$.

Exercise Prove Lemma 2.3.

Remark. These equations can all be easily proved by direct computation. However, there is an alternation approach to proving $G_{uv}(t) \cdot G_{uv}(t+\frac{\pi}{2}) = 0$. First verify by direct computation that $G_{uv}(t+\pi/2) = G'_{uv}(t)$. Then differentiate the equation $\frac{1}{2} G_{uv}(t) \cdot G_{uv}(t) = \frac{1}{2}$ to obtain

$$0 = \frac{d}{dt}(1/2) = \frac{d}{dt}\left(\frac{1}{2} G_{uv}(t) \cdot G_{uv}(t)\right) = G_{uv}(t) \cdot G'_{uv}(t)$$

$$= G_{uv}(t) \cdot G_{uv}(t+\pi/2).$$

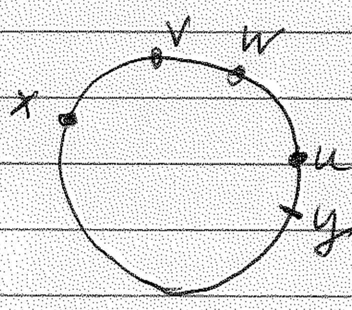
Theorem 2.4 Each great circle in S^n is a geodesic line.

Proof Let $u, v \in S^n$ such that $u \cdot v = 0$. We will prove that G_{uv} is locally distance preserving. Let $s, t \in \mathbb{R}$ such that $s \leq t \leq s + \pi$. Then

$$\theta(G_{uv}(s), G_{uv}(t)) = \cos^{-1}(G_{uv}(s) \cdot G_{uv}(t)) =$$
$$\cos^{-1}(\cos(s)\cos(t) + \sin(s)\sin(t)) =$$
$$\cos^{-1}(\cos(t-s)) = t - s$$

because $t - s \in [0, \pi]$ and $\cos^{-1}$ is the inverse of $\cos | [0, \pi]$. Therefore, $\theta(G_{uv}(s), G_{uv}(t)) = d(s,t)$. Thus, the restriction of G_{uv} to any interval of length π is distance preserving. □

Lemma 2.5. Suppose $u, v \in S^n$ such that $u \cdot v = 0$ and suppose $a \in \mathbb{R}$. Let $w = G_{uv}(a)$, $x = G_{uv}(a + \frac{\pi}{2})$ and $y = G_{uv}(a - \frac{\pi}{2})$. Then $w \cdot x = 0 = w \cdot y$ and for each $t \in \mathbb{R}$, $G_{wx}(t) = G_{uv}(t + a)$ and $G_{wy}(t) = G_{u,-v}(t - a) = G_{u,v}(-t + a)$.



<u>Proof</u> $w \cdot x = 0 = w \cdot y$ follows from Lemma 2.3.b.

For $t \in \mathbb{R}$: $G_{wx}(t) = \cos(t)w + \sin(t)x =$

$\cos(t)\, G_{uv}(a) + \sin(t)\, G_{uv}(a+\frac{\pi}{2}) =$

$\cos(t)\big(\cos(a)u + \sin(a)v\big) + \sin(t)\big(\cos(a+\frac{\pi}{2})u + \sin(a+\frac{\pi}{2})v\big)$

$= \cos(t)\big(\cos(a)u + \sin(a)v\big) + \sin(t)\big(-\sin(a)u + \cos(a)v\big) =$

$\big(\cos(t)\cos(a) - \sin(t)\sin(a)\big)u + \big(\sin(t)\cos(a) + \cos(t)\sin(a)\big)v$

$= \cos(t+a)u + \sin(t+a)v = G_{uv}(t+a).$

Also for $t \in \mathbb{R}$: $G_{wy}(t) = \cos(t)w + \sin(t)y =$

$\cos(t)\, G_{uv}(a) + \sin(t)\, G_{uv}(a-\frac{\pi}{2}) =$

$\cos(t)\big(\cos(a)u + \sin(a)v\big) + \sin(t)\big(\cos(a-\frac{\pi}{2})u + \sin(a-\frac{\pi}{2})v\big) =$

$\cos(t)\big(\cos(a)u + \sin(a)v\big) + \sin(t)\big(\sin(a)u - \cos(a)v\big) =$

$\big(\cos(t)\cos(a) + \sin(t)\sin(a)\big)u + \big(\cos(t)\sin(a) - \sin(t)\cos(a)\big)v =$

$\cos(t-a)u - \sin(t-a)v = G_{u,-v}(t-a) = G_{uv}(-t+a)$

by Lemma 2.3.c. $\square$

<u>Existence Theorem 2.6.</u> If $x, y \in S^n$ and $r, s \in \mathbb{R}$ such that ~~that~~ $r + \Theta(x,y) = s$, then there is a great circle $G_{u,v} : \mathbb{R} \to S^n$ such that $G_{uv}(r) = x$ and $G_{uv}(s) = y$.

<u>Proof</u> First assume $y \neq \pm x$.

Recall that Theorem 1.6 (The Equality Case of the Cauchy Inequality) tells us that if $x \cdot y = \pm 1 \overset{?}{=} \pm \|x\| \|y\|$, then $y = \pm x$.

Hence $x \cdot y \neq \pm 1$. Since $|x \cdot y| \leq \|x\| \|y\| = 1$ (by the Cauchy Inequality), then it follows that $|x \cdot y| < 1$. Therefore $1 - (x \cdot y)^2 > 0$.

So we can define

$$u = \frac{y - (x \cdot y)x}{\sqrt{1 - (x \cdot y)^2}}$$

Then $u \in S^n$, $x \cdot u = 0$ and $y = (x \cdot y)x + \sqrt{1 - (x \cdot y)^2}\, u$. (Verify these assertions.) Observe that

$G_{x,u}(0) = (\cos 0)x + \sin(0)u = x$ and

$G_{x,u}(\Theta(x,y)) = \cos(\Theta(x,y))x + \sin(\Theta(x,y))\, u$

Since $\theta(x,y) = \cos^{-1}(x \cdot y)$, then $\cos\theta(x,y) = x \cdot y$. Since $\theta(x,y) \in [0,\pi]$, then $\sin(\theta(x,y)) \geq 0$. Therefore,

$$\sin(\theta(x,y)) = \sqrt{1-\cos^2(\theta(x,y))} = \sqrt{1-(x \cdot y)^2}.$$

Hence, $G_{x,u}(\theta(x,y)) = (x \cdot y)x + \sqrt{1-(x \cdot y)^2}\,u = y$.

We have $r, s \in \mathbb{R}$ such that $r + \theta(x,y) = s$. Let $v = G_{x,u}(-r)$ and $w = G_{x,u}(-r + \pi/2)$. Then Lemma 2.5 implies $v \cdot w = 0$ and $G_{v,w}(t) = G_{xu}(t-r)$ for each $t \in \mathbb{R}$. Therefore,

$$G_{v,w}(r) = G_{xu}(0) = x \text{ and}$$

$$G_{v,w}(s) = G_{xu}(r-s) = G_{xu}(\theta(xy)) = y.$$

This completes the proof when $y \neq \pm x$.

Suppose $y = \pm x$. Since $n \geq 1$, there is a $u \in S^n$ such that $x \cdot u = 0$. Then $G_{x,u}(0) = \cos(0)x + \sin(0)u = x$ and $G_{x,u}(\pi) = -G_{xu}(0) = -x$ (by Lemma 2.3.b).

Again $r, s \in \mathbb{R}$ and $r + \theta(x,y) = s$.

Let $v = G_{x,u}(-r)$ and $w = G_{x,u}(-r+\frac{\pi}{2})$.
Then $v \cdot w = 0$ and $G_{v,w}(t) = G_{x,u}(t-r)$
by Lemma 2.5. So $G_{v,w}(r) = G_{x,u}(0) = x$.

If $y = x$, then $\theta(x,y) = 0$ and

$G_{v,w}(s) = G_{vw}(r) = x = y$.

If $y = -x$, then $\theta(x,y) = \cos^{-1}(x\cdot(-x)) = \cos^{-1}(-1) = \pi$

and $G_{vw}(s) = G_{vw}(r+\pi) = -G_{vw}(r) = -x = y$
by Lemma 2.3.b. $\square$

<u>Corollary 2.7.</u> For every $n \geq 1$, S^n is totally geodesic.

<u>Proof</u> Theorem 2.6 implies that every pair of points in S^n lie on a great circle, and Theorem 2.4 tells us that every great circle is a geodesic line. $\square$

→ Uniqueness Theorem 2.8. Suppose $u, v, w, x \in S^n$ such that $u \cdot v = 0 = w \cdot x$ and suppose $r, s \in \mathbb{R}$ such that $r - s$ is not an integer multiple of π.
If $G_{u,v}(r) = G_{w,x}(r)$ and $G_{u,v}(s) = G_{w,x}(s)$, then $u = w$, $v = x$ and, hence, $G_{uv} = G_{wx}$.

Proof We have

$$\begin{cases} \cos(r)u + \sin(r)v = \cos(r)w + \sin(r)x \\ \cos(s)u + \sin(s)v = \cos(s)w + \sin(s)x \end{cases}$$

Multiplying the first equation by $\sin(s)$ and the second equation by $\sin(r)$ and subtracting yields:

$$(\sin(s)\cos(r) - \cos(s)\sin(r))u = (\sin(s)\cos(r) - \cos(s)\sin(r))w$$

Thus $\sin(s-r)u = \sin(s-r)w$.

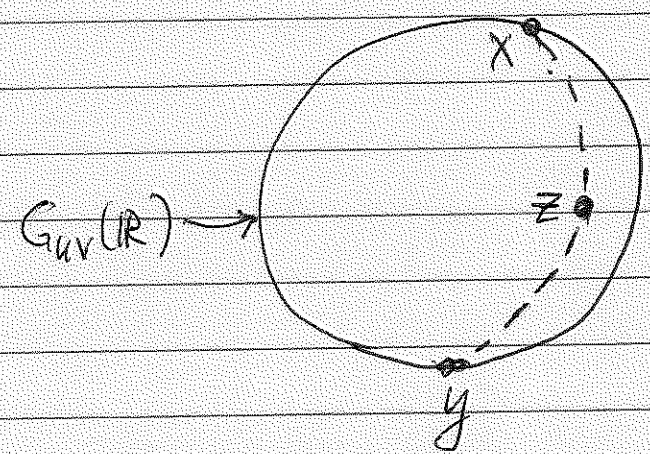
Since $s-r$ is not an integer multiple of π, then $\sin(s-r) \neq 0$. Hence, $u = w$.
Because $u = w$, the first two equations simplify to

$$\sin(r)v = \sin(r)x \text{ and } \sin(s)v = \sin(s)x.$$

Since $s-r$ is not an integer multiple of π, then at least one of $\sin(r)$, $\sin(s)$ is non-zero. Therefore, $v = x$. $\square$

Now for a key lemma.

<u>Lemma 2.9.</u> Suppose $u, v \in S^n$ such that $u \cdot v = 0$, r and $s \in \mathbb{R}$ such that $r < s < r + \pi$, $G_{uv}(r) = x$ and $G_{uv}(s) = z$. If $y \in S^n$ such that $\Theta(x,y) + \Theta(y,z) = \Theta(x,z)$, then $G_{u,v}(r + \Theta(x,y)) = y$.



<u>Proof</u> First, for simplicity, assume $r = 0$. Then $x = G_{uv}(0) = u$ and $0 < s < \pi$. Thus, $z = G_{uv}(s) = \cos(s)u + \sin(s)v$. Hence, $x \cdot z = u \cdot z = \cos(s)$. Since $0 < s < \pi$, then $\sin(s) > 0$. Therefore,

$$\sin(s) = \sqrt{1 - \cos^2(s)} = \sqrt{1 - (x \cdot z)^2}.$$

It follows that $z = (x \cdot z)x + \sqrt{1 - (x \cdot z)^2}\, v$. So

$$v = \frac{z - (x\cdot z)^2 x}{\sqrt{1-(x\cdot z)^2}} .$$

In the case that $y=x$, $\theta(x,y)=0$.
So $y = G_{uv}(0) = G_{uv}(r + \theta(x,y))$.

Assume $y \neq x$. Then
$0 < \theta(x,y) \leq \theta(x,y) + \theta(yz) = \theta(x,z) = \cos^{-1}(x\cdot z) = s < \pi$.
So $\cos^{-1}(x\cdot y) = \theta(x,y) \in (0,\pi)$. Thus,
$x\cdot y \in \cos(0,\pi) = (-1,1)$. Hence, $1-(x\cdot y)^2 > 0$
and we can define

$$w = \frac{y - (x\cdot y)x}{\sqrt{1-(x\cdot y)^2}} .$$

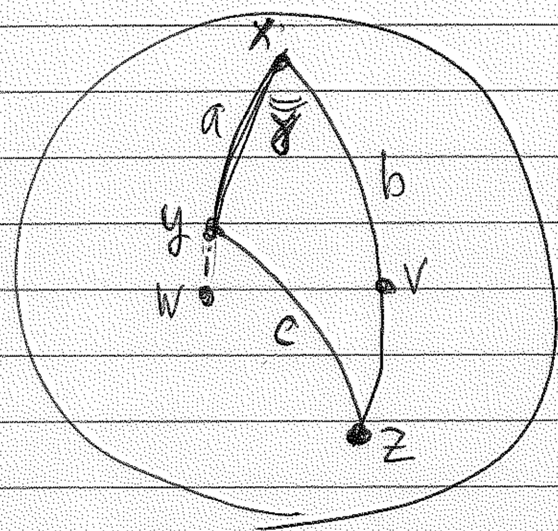
Then $w \in S^n$, $x\cdot w = 0$ and $y = (x\cdot y)x + \sqrt{1-(x\cdot y)^2}\, w$.
From the relationship between x, y, z,
v and w, it follows that

$$m(\angle yxz) = \theta(v,w).$$

Thus, $\cos(m(\angle yxz)) = \cos(\theta(v,w)) = v\cdot w$

Let $a = \theta(x,y)$, $b = \theta(x,z)$, $c = \theta(y,z)$
and $\gamma = m(\angle yxz)$. Then the Spherical
Law of Cosines implies

$$\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(\gamma).$$



On the other hand, since

$$b = \theta(x,z) = \theta(x,y) + \theta(y,z) = a + c,$$

then

$$\cos(c) = \cos(b-a) = \cos(a)\cos(b) + \sin(a)\sin(b).$$

Therefore,

$$\sin(a)\sin(b)\cos(\gamma) = \sin(a)\sin(b).$$

We saw above that $a = \theta(x,y) \in (0,\pi)$ and $b = \theta(xz) \in (0\pi)$. Hence, $\sin(a) > 0$ and $\sin(b) > 0$. Consequently, $\cos(\gamma) = 1$. Thus,

$$1 = \cos(m(\angle yxz)) = v \cdot w.$$

Therefore, Theorem 1.6.a implies $v = w$.
Hence, $y = (x \cdot y)x + \sqrt{1-(x \cdot y)^2}\, v$.
Recall that $x = u$, $x \cdot y = \cos(\theta(x,y))$ and $\sqrt{1-(x \cdot y)^2} = \sin(\theta(x,y))$ (because $\theta(xy) \in (0,\pi)$).

Therefore, $y = \cos(\theta(x,y))u + \sin(\theta(x,y))v$
$= G_{uv}(\theta(x,y)) = G_{uv}(r + \theta(x,y))$. This completes the proof under the simplifying assumption that $r=0$.

Now suppose r is any element of $\mathbb{R}$, $r < s < r+\pi$, $G_{uv}(r) = x$, $G_{uv}(s) = y$ and $z \in S^n$ such that $\theta(x,y) + \theta(y,z) = \theta(x,z)$. Let $w = G_{uv}(r + \pi/2)$. Then Lemma 2.5 implies $x \cdot w = 0$ and $G_{x,w}(t) = G_{uv}(t+r)$ for each $t \in \mathbb{R}$. Hence, $G_{x,w}(0) = G_{uv}(r) = x$ and $G_{x,w}(s-r) = G_{uv}((s-r)+r) = G_{uv}(s) = y$. Also $0 < s-r < \pi$. Thus, the previously proved $r=0$ case of this lemma implies $G_{xw}(\theta(x,y)) = y$. Hence $G_{uv}(r + \theta(x,y)) = y$. $\square$

Now we prove our main result about local geodesics in S^n: every unit speed local geodesic in S^n is the restriction of a great circle.

<u>Theorem 2.10.</u> If $f: J \to S^n$ is a unit speed local geodesic, then there is a great circle $G_{u,v}: \mathbb{R} \to S^n$ such that $f = G_{uv}|J$.

<u>Proof</u> Theorem 1.68 implies f is locally distance preserving. We first establish:

*) If $[a,b] \subset J$ such that $b < a + \pi$ and $f|[a,b]$ is distance preserving, then there is a great circle $G_{uv}: \mathbb{R} \to S^n$ such that $f|[a,b] = G_{uv}|[a,b]$.

Let $x = f(a)$ and $z = f(b)$. Since f is distance preserving then $\theta(x,z) = b - a$. Theorem 2.6 implies there is a great circle $G_{uv}: \mathbb{R} \to S^n$ such that $G_{uv}(a) = x$ and $G_{uv}(b) = z$.
We will prove $G_{uv}|[a,b] = f|[a,b]$.
Let $a \leq t \leq b$ and let $y = f(t)$. Since f is distance preserving then

$$\theta(x,y) + \theta(y,z) = (t-a) + (b-t) = b - a = \theta(x,z).$$

Hence, Lemma 2.9 implies

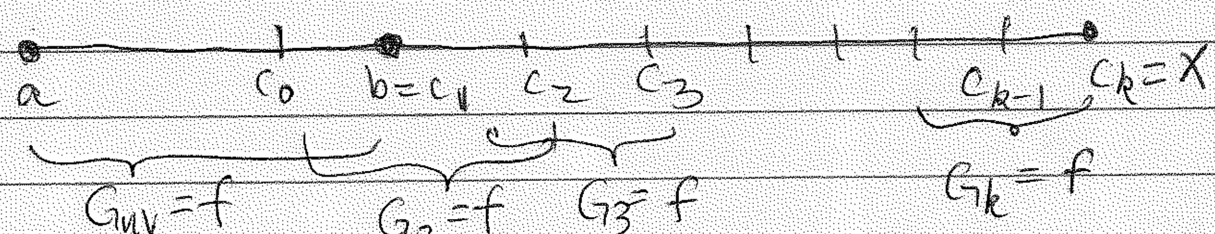
$$G_{uv}(t) = G_{uv}(a + \theta(x,y)) = y = f(t).$$

This proves $G_{uv}|[a,b] = f|[a,b]$.

Choose $[a,b] \subset J$ such that $f|[a,b]$ is distance preserving. Then *) provides a great circle $G_{uv} : \mathbb{R} \to S^n$ such that $f|[a,b] = G_{uv}|[a,b]$. We will prove $f = G_{uv}|J$.

Let $x \in J - [a,b]$. Either $x < a$ or $x > b$. We consider the case $x > b$. The proof in the case $x < a$ is similar.

Since f is locally distance preserving, there is a cover $\mathcal{C}$ of $[a,x]$ by relatively open subsets of $[a,x]$ such that $f|C$ is distance preserving for each $C \in \mathcal{C}$. Since $[a,x]$ is compact, then the cover $\mathcal{C}$ has a Lebesgue number $\delta > 0$. (Thus, if $S \subset [a,x]$ and $\operatorname{diam}(S) < \delta$, then $S \subset C$ for some $C \in \mathcal{C}$ and, hence, $f|S$ is distance preserving.) We may assume $\delta \leq \pi$. Let $(c_1, c_2, \ldots, c_k)$ be a partition of $[b,x]$ such that $0 < c_i - c_{i-1} < \delta/2$ for $2 \leq i \leq k$. Also choose $c_0 \in (a,b)$ so that $0 < c_1 - c_0 < \delta/2$. Then $0 < c_i - c_{i-2} \leq \pi$ and $f|[c_{i-2}, c_i]$ is distance preserving for $2 \leq i \leq k$.



For each i, $2 \le i \le k$, $*)$ provides a great circle $G_i : \mathbb{R} \to S^n$ such that $f \,|\, [c_{i-2}, c_i] = G_i \,|\, [c_{i-2}, c_i]$.

Observe that $G_{uv}(c_0) = f(c_0) = G_2(c_0)$ and $G_{uv}(c_1) = f(c_1) = G_2(c_1)$, and $0 < c_1 - c_0 < \pi$. Hence, the Uniqueness Theorem 2.8 implies $G_{uv} = G_2$. Observe that for $3 \le i \le k$,

$G_{i-1}(c_{i-2}) = f(c_{i-2}) = G_i(c_{i-2})$,

$G_{i-1}(c_{i-1}) = f(c_{i-1}) = G_i(c_{i-1})$, and $0 < c_i - c_{i-1} < \pi$. Hence, the Uniqueness Theorem 2.8 implies $G_{i-1} = G_i$. Thus

$$G_{uv} = G_2 = G_3 = \cdots = G_k.$$

Thus $G_{uv}(x) = G_k(x) = f(x)$. $\square$

Homework Problem 2.2. Let $r > 0$.

a) Formulate an appropriate notion of great circle in S^n_r.
b) Prove the analogue of Theorem 2.4 for S^n_r.
c) Prove the analogue of Theorem 2.6 for S^n_r.
d) Prove the analogue of Theorem 2.10 for S^n_r.

Next we identify the totally geodesic subspaces of S^n.

Theorem 2.7k. If V is a vector subspace of $\mathbb{E}^{n+1}$ of dimension ≥ 2, then $V \cap S^n$ is a totally geodesic subspace of S^n.

Proof. Let $x, y \in V \cap S^n$.

First consider the case $y = \pm x$. Since $\dim(V) \geq 2$, then we can enlarge x to a 2-element orthonormal sequence x, v in V. Then $v \in V \cap S^n$ and $x \cdot v = 0$. Then $G_{x,v}(\mathbb{R}) \subset S^n$. Also, since $G_{x,v}(t) = \cos(t)x + \sin(t)v$ is a linear combination of x and v and $x, v \in V$, then $G_{x,v}(t) \in V$. Thus $G_{x,v}(\mathbb{R}) \subset V \cap S^n$. Since $G_{xv}(0) = x$ and $G_{xv}(\pi) = -x$, and $y = \pm x$, then $x, y \in G_{uv}(\mathbb{R})$.

Second assume $y \neq \pm x$. Then $x \cdot y \neq \pm 1$ by Theorem 1.6r. So we can define

$$v = \frac{y - (x \cdot y)x}{\sqrt{1 - (x \cdot y)^2}}.$$

Then, as we've observed previously, $v \in S^n$, $x \cdot v = 0$, $G_{x,v}: \mathbb{R} \to S^n$, $G_{x,v}(0) = x$ and $G_{x,v}(\theta(x,y)) = y$. Since $x, y \in V$ and v is a linear combination of x and y, then $v \in V$. For each $t \in \mathbb{R}$, $G_{x,v}(t) = \cos(t)x + \sin(t)v$ is a linear combination of x and v. Thus, $G_{x,v}(t) \in V$. Thus, $G_{x,v}: \mathbb{R} \to V \cap S^n$.

This proves $V \cap S^n$ is a totally geodesic subspace of S^n. $\square$

→ Our next goal is to prove the converse of Theorem 2.11. However, it is convenient to prove the following lemma first.

<u>Lemma 2.12.</u> Suppose $x, y \in S^n$ and $G_{u,v}: \mathbb{R} \to S^n$ is a great circle such that $x, y \in G_{u,v}(\mathbb{R})$. If $z \in S^n$ and z is a linear combination of x and y, then $z \in G_{u,v}(\mathbb{R})$.

<u>Proof</u> Let $r, s \in \mathbb{R}$ such that $G_{u,v}(r) = x$ and $G_{u,v}(s) = y$. Let $w = G_{u,v}(r + \frac{\pi}{2})$. Then Lemma 2.5 implies $w \cdot x = 0$ and

$G_{x,w}(t) = G_{uv}(t+r)$ for all $t \in \mathbb{R}$.
Hence, $G_{x,w}(0) = G_{uv}(r) = x$ and
$G_{x,w}(s-r) = G_{uv}(s) = y$. Therefore,
$y = \cos(s-r)x + \sin(s-r)w$.

Assume $z \in S^n$ and z is a linear combination of x and y. Then $z = ax + by$ for some $a, b \in \mathbb{R}$. Therefore,

$$\begin{aligned} z &= ax + b(\cos(s-r)x + \sin(s-r)w) \\ &= (a + b\cos(s-r))x + (b\sin(s-r))w. \end{aligned}$$

Thus z is a linear combination of x and w. Since $x, w \in S^n$ and $x \cdot w = 0$, then

$$z = (z \cdot x)x + (z \cdot w)w.$$

Hence, $1 = \|z\|^2 = (z \cdot x)^2 + (z \cdot w)^2$.
The definition of $\theta(x,z)$ implies
$z \cdot x = \cos(\theta(x,z))$ and $\theta(x,z) \in [0, \pi]$.
Therefore, $\sin(\theta(x,z)) \geq 0$ and, hence,

$$\sin(\theta(x,z)) = \sqrt{1 - \cos^2(\theta(x,z))} = \sqrt{1 - (z \cdot x)^2} = \pm z \cdot w$$

Thus, $z \cdot w = \pm \sin(\theta(x,z))$.

In the case that $z \cdot w = + \sin(\theta(x,z))$:
$z = \cos(\theta(x,z))x + \sin(\theta(x,z))w =$
$G_{x,w}(\theta(x,z)) = G_{u,v}(\theta(x,z)+r)$.

In the case that $z \cdot w = - \sin(\theta(x,z))$:

$z = \cos(\theta(x,z))x - \sin(\theta(x,z))w =$

$\cos(-\theta(x,z))x + \sin(-\theta(x,z))w =$

$G_{x,w}(-\theta(x,z)) = G_{u,v}(-\theta(x,z)+r)$.

We conclude that $z \in G_{u,v}(\mathbb{R})$ in either case. $\square$

<u>Theorem 2.13.</u> If T is a totally geodesic subspace of S^n, then there is a vector subspace V of $\mathbb{E}^{n+1}$ of dimension ≥ 2 such that $T = V \cap S^n$.

<u>Proof</u> We begin by proving:

(*) If $u_1, u_2, \cdots, u_k$ is an orthonormal sequence in T and V is the vector subspace of $\mathbb{E}^{n+1}$ generated by $u_1, u_2 \cdots u_k$ (i.e., V is the set of all linear combinations of $u_1, u_2, \cdots, u_k$), then $V \cap S^n \subset T$.

We prove (*) by induction. To begin, let $u_1 \in T$, and let $V = \{au_1 : a \in \mathbb{R}\}$. Clearly, $V \cap S^n = \{u_1, -u_1\}$. Since T is totally geodesic and $u_1 \in T$, then there is a great circle $G_{uv} : \mathbb{R} \to T$ such that $u_1 \in G_{uv}(\mathbb{R})$. Therefore, there is an $r \in \mathbb{R}$ such that $G_{uv}(r) = u_1$. Then $G_{u,v}(r+\pi) = -u_1$ by Lemma 2.3.c. Hence, $-u_1 \in G_{uv}(\mathbb{R}) \subset T$. Thus $V \cap S^n = \{u_1, -u_1\} \subset T$.

Next let $k \geq 1$ and assume that if $u_1, u_2, \ldots, u_k$ is any k-element orthonormal sequence in T and V is the vector subspace of $\mathbb{E}^{n+1}$ generated by $u_1, u_2, \ldots, u_k$, then $V \cap S^n \subset T$.

Suppose $u_1, u_2, \ldots, u_k, u_{k+1}$ is a $(k+1)$-element orthonormal sequence in T. Let V be the vector subspace of $\mathbb{E}^{n+1}$ generated by $u_1, u_2, \ldots, u_k, u_{k+1}$. We must prove $V \cap S^n \subset T$.

Let W be the vector subspace of $\mathbb{E}^{n+1}$ generated by $u_1, u_2, \dots, u_k$. Then our inductive hypothesis implies $W \cap S^n \subset T$.

Let $x \in V \cap S^n$. We must prove $x \in T$. We can write $x = \sum_{i=1}^{k+1} (x \cdot u_i) u_i$. Let $y = \sum_{i=1}^{k} (x \cdot u_i) u_i$. Then $y \in W$.

First consider the case $y = 0$. Then $x = (x \cdot u_{k+1}) u_{k+1}$. Since $\|x\| = 1$, then $x = \pm u_{k+1}$. Since $u_{k+1} \in T$, then $-u_{k+1} \in T$. (See the proof of the $k=1$ case.) Hence, $x \in T$.

Second consider the case $y \neq 0$. Then $y/\|y\| \in W \cap S^n \subset T$. Also $u_{k+1} \in T$. Since T is totally geodesic, then there is a great circle $G_{u,v} : \mathbb{R} \to T$ such that u_{k+1} and $y/\|y\| \in G_{uv}(\mathbb{R})$. Note that $x \in S^n$ and $x = (x \cdot u_{k+1}) u_{k+1} + \|y\| \left(y/\|y\| \right)$.

Hence, Lemma 2.12 implies $x \in G_{uv}(\mathbb{R})$. Therefore $x \in T$. This proves $V \cap S^n \subset T$.

We have now completed the inductive proof of the assertion (*).

Now let $u_1, u_2, \ldots, u_k$ be a maximal orthonormal sequence in T. Let V be the vector subspace of E^{n+1} generated by $u_1, u_2, \ldots, u_k$. Then (*) implies $V \cap S^n \subset T$.

We will prove $V \cap S^n = T$. Assume not. Then there is an $x \in T - V$. Let $y = \sum_{i=1}^{k} (x \cdot u_i) u_i$. Then $y \in V$.

We will prove $y \neq 0$. Assume $y = 0$. Then $0 = x \cdot y = \sum_{i=1}^{k} (x \cdot u_i)^2$. Hence, $x \cdot u_i = 0$ for $1 \leq i \leq k$. Therefore, $u_1, u_2, \ldots, u_k, x$ is a $(k+1)$-element orthonormal sequence in T. This contradicts the maximality of $u_1, u_2, \ldots, u_k$. We conclude that $y \neq 0$.

Since $y \in V$ and $x \notin V$, then $x-y \neq 0$.

Therefore, we can define $u_{k+1} = \dfrac{x-y}{\|x-y\|}$.

Then $u_{k+1} \in S^n$. Observe that for $1 \leq j \leq k$, $y \cdot u_j = \sum_{i=1}^{k} (x \cdot u_i)(u_i \cdot u_j) = x \cdot u_j$.

Hence, for $1 \leq j \leq k$,

$$u_{k+1} \cdot u_j = \frac{x \cdot u_j - y \cdot u_j}{\|x-y\|} = 0.$$

Thus $u_1, u_2, \ldots, u_k, u_{k+1}$ is a $(k+1)$-element orthonormal sequence in S^n.

$$x \in T \text{ and } y/\|y\| \in V \cap S^n \subset T.$$

Hence, there is a great circle $G_{uv} : \mathbb{R} \to T$ such that x and $y/\|y\| \in G_{uv}(\mathbb{R})$.

Observe that

$$u_{k+1} = \left(\frac{1}{\|x-y\|}\right) x + \left(\frac{-\|y\|}{\|x-y\|}\right)\left(y/\|y\|\right).$$

Thus, $u_{k+1} \in S^n$ and u_{k+1} is a linear combination of x and $y/\|y\|$. Therefore, Lemma 2.12

implies $u_{k+1} \in G_{u,v}(\mathbb{R})$. Hence, $u_{k+1} \in T$. So $u_1, u_2, \ldots, u_k, u_{k+1}$ is a $(k+1)$-element orthonormal sequence in T. This contradicts the maximality of $u_1, u_2, \ldots, u_k$. We conclude that $V \cap S^n = T$.

Finally we show why $\dim(V) \geq 2$. $T = V \cap S^n$ contains a point and, hence, a great circle $G_{uv} : \mathbb{R} \to T$. Then $G_{uv}(0), G_{uv}(\pi/2)$ is a 2-element orthonormal sequence in V. Consequently, $\dim(V) \geq 2$. $\square$

<u>Corollary 2.14.</u> A subset T of S^n is a totally geodesic subspace if and only if $T = V \cap S^n$ where V is a vector subspace of $\mathbb{E}^{n+1}$ of dimension ≥ 2. $\square$

Homework Problem 2.3 Let $r > 0$. Prove the analogue of Corollary 2.14 for S^n_r.

Definition A subset T of S^n is a k-dimensional metric sphere if T is isometric to S^k_r for some $r > 0$.

Theorem 2.15. If V is a $(k+1)$-dimensional vector subspace of $\mathbb{E}^{n+1}$, then $V \cap S^n$ is isometric to S^k.

Homework Problem 2.4 Prove Theorem 2.15.

Theorem 2.16. A subset T of S^n is a metric sphere if and only if $T = V \cap S^n$ for some vector subspace V of $\mathbb{E}^{n+1}$.

Homework Problem 2.5 a) Prove that if X and Y are metric spaces, X is totally geodesic, $r > 0$ and $f: X \to Y$ is a bijection such that $d(f(x), f(x')) = r\,d(x, x')$ for all $x, x' \in X$, then Y is totally geodesic.

b) Prove that if u is a unit vector in $\mathbb{E}^{n+1}$ and $0 < |a| < 1$, then $P(u,a) \cap S^n$ is not a metric sphere.

c) Prove Theorem 2.16

Corollary 2.17. Every metric sphere in S^n is isometric to S^k for some $k \le n$.

<u>Theorem 2.18.</u> The function

$$f \mapsto f|S^n : O(\mathbb{E}^{n+1}) \to \mathcal{I}(S^n)$$

is an isomorphism.

<u>Proof</u> Let $f \in O(\mathbb{E}^{n+1})$. Then f is distance preserving and $f(0) = 0$. Hence, f preserves dot products by Corollary 1.10. Therefore, for $x \in S^n$:

$$\|f(x)\| = \sqrt{f(x) \cdot f(x)} = \sqrt{x \cdot x} = \|x\| = 1.$$

Thus, $f(x) \in S^n$. So $f|S^n$ maps S^n to itelf.

Since $f^{-1} : \mathbb{E}^{n+1} \to \mathbb{E}^{n+1}$ is also an isometry and $f^{-1}(0) = f^{-1}(f(0)) = 0$, then $f^{-1} \in O(\mathbb{E}^{n+1})$. So the argument in the previous paragraph shows $f^{-1}|S^n$ maps S^n to itself. Furthermore, $(f^{-1}|S^n) \circ (f|S^n) = f^{-1} \circ f|S^n = id_{S^n}$ and $(f|S^n) \circ (f^{-1}|S^n) = f \circ f^{-1}|S^n = id_{S^n}$. Thus, $f|S^n : S^n \to S^n$ is a bijection.

For $x, y \in S^n$:

$\cos(\theta(f(x), f(y))) = f(x) \cdot f(y) = x \cdot y = \cos(\theta(x, y))$

Since $\cos|[0, \pi]$ is one-to-one, then $\theta(f(x), f(y)) = \theta(x, y)$. Thus, $f|S^n : S^n \to S^n$ is distance preserving. Consequently, $f|S^n \in \mathcal{I}(S^n)$.

For $f, g \in O(\mathbb{E}^n)$, since $(g|S^n)\circ(f|S^n) = g\circ f|S^n$, then $f \mapsto f|S^n : O(\mathbb{E}^{n+1}) \to \mathcal{I}(S^n)$ is a group homomorphism.

To prove $f \mapsto f|S^n : O(\mathbb{E}^{n+1}) \to \mathcal{I}(S^n)$ is injective, let $f \in O(\mathbb{E}^n)$ such that $f|S^n = id_{S^n}$. We will prove $f = id_{\mathbb{E}^{n+1}}$. Let $x \in \mathbb{E}^{n+1}$. If $x=0$, then $f(x) = 0 = x$ because $f \in O(\mathbb{E}^n)$. Assume $x \neq 0$. Since f is distance preserving and $f(0)=0$, the Corollary 1.1.4 implies f is linear. Therefore,

$$f(x) = \|x\| f\left(\frac{x}{\|x\|}\right) = \|x\| (f|S^n)\left(\frac{x}{\|x\|}\right) = \|x\|\left(\frac{x}{\|x\|}\right) = x.$$

Thus $f = id_{\mathbb{E}^{n+1}}$. Hence $f \mapsto f|S^n$ is injective.

To prove $f \mapsto f|S^n : O(\mathbb{E}^{n+1}) \to \mathcal{I}(S^n)$ is surjective, let $g \in \mathcal{I}(S^n)$. Define $\bar{g} : \mathbb{E}^{n+1} \to \mathbb{E}^{n+1}$ by

$$\bar{g}(x) = \begin{cases} 0 & \text{if } x = 0 \\ \|x\| g\left(\frac{x}{\|x\|}\right) & \text{if } x \neq 0 \end{cases},$$

We will prove $\bar{g}$ preserves dot products.

Let $x, y \in \mathbb{E}^{n+1}$. Clearly, if either $x=0$ or $y=0$, then clearly

$$\bar{g}(x)\cdot\bar{g}(y) = 0 = x\cdot y.$$

Assume $x\neq 0$ and $y\neq 0$. Since g is an isometry of S^n, then $\theta(g(\frac{x}{\|x\|}), g(\frac{y}{\|y\|})) = \theta(\frac{x}{\|x\|}, \frac{y}{\|y\|})$.

Therefore,

$$\bar{g}(x)\cdot\bar{g}(y) = \left(\|x\| g\left(\tfrac{x}{\|x\|}\right)\right)\cdot\left(\|y\| g\left(\tfrac{y}{\|y\|}\right)\right) =$$
$$\|x\|\,\|y\|\left(g\left(\tfrac{x}{\|x\|}\right)\cdot g\left(\tfrac{y}{\|y\|}\right)\right) = \|x\|\,\|y\| \cos\theta\left(g\left(\tfrac{x}{\|x\|}\right), g\left(\tfrac{y}{\|y\|}\right)\right) =$$
$$\|x\|\,\|y\| \cos\theta\left(\tfrac{x}{\|x\|}, \tfrac{y}{\|y\|}\right) = \|x\|\,\|y\|\left(\tfrac{x}{\|x\|}\right)\cdot\left(\tfrac{y}{\|y\|}\right) =$$
$$x\cdot y.$$

Hence, $\bar{g}$ preserves dots products. Thus, $\bar{g}$ is distance preserving and $\bar{g}(0)=0$ by Corollary 1.10. Consequently, $\bar{g}\in O(\mathbb{E}^{n+1})$.

Observe that for $x\in S^n$, $\bar{g}(x) = \|x\| g\left(\frac{x}{\|x\|}\right) = 1 g\left(\frac{x}{1}\right) = g(x)$. Thus $\bar{g}|S^n = g$. This proves $f\mapsto f|S^n$: $O(\mathbb{E}^{n+1}) \to I(S^n)$ is surjective.

We conclude that $f\mapsto f|S^n : O(\mathbb{E}^{n+1}) \to I(S^n)$ is an isomorphism. $\square$

Theorem 2.19 For any curve $\gamma : [a,b] \to S^n$, the spherical length of γ $L_S(\gamma)$ and the Euclidean length of γ $L_E(\gamma)$ are equal. Thus, γ is spherically rectifiable if and only if γ is Euclideanly rectifiable.

Proof Define $\rho : \mathbb{R} \to [0, \infty)$ by

$$\rho(\theta) = \sqrt{2}\sqrt{1-\cos\theta}$$

First we prove:

a) $\|x-y\| = \rho(\theta(x,y))$ for all $x, y \in S^n$.

$\|x-y\|^2 = \|x\|^2 - 2x\cdot y + \|y\|^2 = 2 - 2\cos\theta(x,y)$.

Hence, $\|x-y\| = \sqrt{2}\sqrt{1-\cos\theta(x,y)} = \rho(\theta(x,y))$.

Observe that $\rho'(\theta) = \sqrt{2}\,\dfrac{\sin\theta}{2\sqrt{1-\cos\theta}} =$

$$\frac{1}{\sqrt{2}}\sqrt{\frac{\sin^2\theta}{1-\cos\theta}} = \frac{1}{\sqrt{2}}\sqrt{\frac{1-\cos^2\theta}{1-\cos\theta}} = \frac{1}{\sqrt{2}}\sqrt{1+\cos\theta}.$$

Hence:

b) $\rho'(\theta) \le 1$ and $\rho'(0) = 1$.

Next we prove:

c) $\rho(\theta) \le \theta$ for $\theta > 0$.

Let $\theta > 0$. Since $\rho(0) = 0$, the Mean Value Theorem implies there is a $\bar{\theta} \in (0, \theta)$ such that

$$\rho(\theta) = \rho(\theta) - \rho(0) = \rho'(\bar{\theta})\theta \le \theta.$$

Hence:

d) $\|x - y\| \le \theta(x, y)$ for all $x, y \in S^n$.

Now if $P = (c_0 c_1 \ldots c_k)$ is any partition of $[a, b]$, then

$$L_E(\gamma, P) = \sum_{i=1}^{k} \|\gamma(c_i) - \gamma(c_{i-1})\| \le \sum_{i=1}^{k} \theta(\gamma(c_{i-1}), \gamma(c_i))$$

$$= L_S(\gamma, P) \le L_S(\gamma).$$

Thus $L_E(\gamma) \le L_S(\gamma)$.

Next we prove:

e) $\lim_{\theta \to 0} \frac{\rho(\theta)}{\theta} = 1$.

Since $\rho(0) = 0$, $\lim_{\theta \to 0} \frac{\rho(\theta)}{\theta} = \lim_{\theta \to 0} \frac{\rho(\theta) - \rho(0)}{\theta - 0} = \rho'(0) = 1$.

Let $\varepsilon > 0$. Then $\exists\, \xi > 0$ such that $0 < \theta < \xi$ implies $1 - \varepsilon < \frac{\rho(\theta)}{\theta}$. Hence, for $x, y \in S^n$, if $\theta(x,y) < \xi$, then $(1-\varepsilon)\,\theta(x,y) < \|x - y\|$. Since $\gamma : [a,b] \to S^n$ is uniformly continuous, $\exists\, \delta > 0$ such that if $s, t \in [a,b]$ and $|s - t| < \delta$, then $\theta(\gamma(s), \gamma(t)) < \xi$. Let P be any partition of $[a,b]$. Then P is refined by a partition $Q = (c_0, c_1, \ldots, c_k)$ of $[a,b]$ such that $c_i - c_{i-1} < \delta$ for $1 \le i \le k$. Therefore, $L_S(\gamma, P) \le L_S(\gamma, Q)$, $\theta(\gamma(c_{i-1}), \gamma(c_i)) < \xi$ and, hence,
$$(1-\varepsilon)\,\theta(\gamma(c_{i-1}), \gamma(c_i)) \le \|\gamma(c_{i-1}) - \gamma(c_i)\|.$$
Hence,
$$(1-\varepsilon) L_S(\gamma, P) \le (1-\varepsilon) L_S(\gamma, Q) = \sum_{i=1}^{k} (1-\varepsilon)\,\theta(\gamma(c_{i-1}), \gamma(c_i))$$
$$< \sum_{i=1}^{k} \|\gamma(c_{i-1}) - \gamma(c_i)\| = L_E(\gamma, Q) \le L_E(\gamma).$$

Thus, $(1-\varepsilon) L_S(\gamma) \le L_E(\gamma)$. Since $\varepsilon > 0$ is arbitrary, then $L_S(\gamma) \le L_E(\gamma)$. This completes the proof that $L_S(\gamma) = L_E(\gamma)$. $\square$

Determinants and Volume

Def Let $\mathbb{R}_m^n$ denote the set of all $m \times n$ entries with real entries. The determinant is a function

$$\det : \mathbb{R}_n^n \to \mathbb{R}$$

that satisfies the following three axioms.

a) Multilinearity : For $A \in \mathbb{R}_n^n$, $\det(A)$ is a linear function of each column (and each row) of A.

b) The Alternating Property: If $A, B \in \mathbb{R}_n^n$ and B is obtained from A by interchanging two columns (or rows), then $\det(B) = -\det(A)$.

c) Normalization: If $I_n \in \mathbb{R}_n^n$ is the identity matrix, then $\det(I_n) = 1$.

The existence and uniqueness of the determinant function is a standard theorem of linear algebra.

We mention several other basic useful theorems about determinants.

The Alternating Property (b) has the following consequence.

d) If $A \in \mathbb{R}^n_n$ and two distinct columns (or rows) of A are equal, then $\det(A) = 0$.

e) <u>The Product Formula.</u> If $A, B \in \mathbb{R}^n_n$, then $\det(AB) = \det(A)\det(B)$.

f) <u>The Transpose Formula.</u> If $A \in \mathbb{R}^n_n$, then $\det(A^T) = \det(A)$.

<u>Lemma 2.20.</u> If $f \in O(\mathbb{E}^n)$ is represented by the matrix $A = (f(e_1) \cdots f(e_n))$, then $\det(A) = \pm 1$.

<u>Proof</u> Since $f \in O(\mathbb{E}^n)$, then $A^T = A^{-1}$ by Corollary 1.81. Hence,

$$1 = \det(I_n) = \det(AA^{-1}) = \det(A)\det(A^{-1}) = \det(A)\det(A^T) = (\det(A))^2.$$

Therefore, $\det(A) = \pm 1$. $\square$

<u>Def</u> For $x_1, \ldots, x_k \in \mathbb{E}^n$, the <u>$k$-parallelopiped in $\mathbb{E}^n$ with edges $x_1, \ldots, x_k$</u> is the set

$$\Pi(x_1, \ldots, x_k) = \{\textstyle\sum_{i=1}^k a_i x_i : a_i \in [0,1] \text{ for } 1 \le i \le k\}.$$

Let $\mathcal{P}^n_k$ denote the set of all k-parallelopipeds in $\mathbb{E}^n$.

Def A function $V^n_k : \mathcal{P}^n_k \to [0,\infty)$ is called a k-dimensional volume function if it satisfies the following three axioms.

α) $V^n_k(\Pi(x_1 \cdots x_{i-1}, ax_i, x_{i+1} \cdots x_k)) = |a| V^n_k(\Pi(x_1 \cdots x_k))$

β) If y is a linear combination of $x_1 \cdots x_{i-1}\ x_{i+1} \cdots x_k$, then $V^n_k(\Pi(x_1 \cdots x_{i-1}, x_i + y, x_{i+1} \cdots x_n) = V^n_k(\Pi(x_1 \cdots x_k))$

γ) If $u_1 \cdots u_k$ is an orthonormal sequence in $\mathbb{E}^n$, then $V^n_k(\Pi(u_1 \cdots u_k)) = 1$.

For $x_1 \cdots x_k \in \mathbb{E}^n$, let $(x_1 \cdots x_k) \in \mathbb{R}^k_n$ denote the $n \times k$ matrix with columns $x_1 \cdots x_k$.

Theorem 2.21 A k-dimensional volume function $V^n_k : P^n_k \to [0, \infty)$ is defined as follows. For $x_1, \ldots, x_k \in \mathbb{E}^n$, let $u_{k+1} \cdots u_n$ be an $n-k$ element orthonormal sequence in $\mathbb{E}^n$ such that $x_i \cdot u_j = 0$ for $1 \leq i \leq k$, $k+1 \leq j \leq n$ and define

$$V^n_k(\Pi(x_1 \cdots x_k)) = |\det(x_1 \cdots x_k\, u_{k+1} \cdots u_n)|.$$

Furthermore this volume function is unique.

<u>Proof</u> First we show that the definition of $V_k^n(\Pi(x_1 \cdots x_k))$ is independent of the choice of $u_{k+1} \cdots u_n$. First, if $x_1 \cdots x_k$ are linearly dependent, then $V_k^n(\Pi(x_1 \cdots x_k)) = 0$ regardless of the choice of $u_{k+1} \cdots u_n$. So assume $x_1 \cdots x_k$ are linearly independent and span the vector subspace V of $\mathbb{E}^n$. Let $V^\perp = \{y \in \mathbb{E}^n : x \cdot y = 0 \text{ for every } x \in V\}$. Then $u_{k+1} \cdots u_n$ is an orthonormal basis for $V^\perp$. Suppose $v_{k+1} \cdots v_n$ is another orthonormal basis for $V^\perp$. Then there is an $f \in O(\mathbb{E}^n)$ such that $f(x) = x$ for each $x \in V$ and $f(u_i) = v_i$ for $k+1 \le i \le n$. Let $A \in \mathbb{R}_n^n$ be the matrix representative of f. Thus, $Ax_i = f(x_i) = x_i$ for $1 \le i \le k$ and $Au_i = f(u_i) = v_i$ for $k+1 \le i \le n$. Thus

$$A(x_1 \cdots x_k\, u_{k+1} \cdots u_n) = (x_1 \cdots x_k\, v_{k+1} \cdots v_n).$$

Also $\det(A) = \pm 1$ by Lemma 2.20. Hence,

$$|\det(x_1 \cdots x_k\, v_{k+1} \cdots v_n)| = |\det(A(x_1 \cdots x_k\, u_{k+1} \cdots u_n))| =$$

$$|\det(A)|\,|\det(x_1 \cdots x_k\, u_{k+1} \cdots u_n)| = |\det(x_1 \cdots x_k\, u_{k+1} \cdots u_n)|$$

It follows that the definition of $V_k^n(\Pi(x_1 \cdots x_k))$ is independent of the choice of $u_{k+1} \cdots u_n$.

Next we verify that V^n_k satisfies axioms α), β) and γ).

Clearly axiom α) follows directly from the multilinearity property a) of the determinant.

Axiom β) follows from multilinearity and property d). <u>Exercise</u>: Verify this assertion.

If $u_1, \ldots, u_n$ is an orthonormal basis for $\mathbb{E}^n$, then $(u_1 \cdots u_n)$ is the matrix representative of the element of $\mathcal{O}(\mathbb{E}^n)$ that sends e_i to u_i for $1 \le i \le n$. Thus $|\det(u_1 \cdots u_k u_{k+1} \cdots u_n)| = 1$ by property d). Axiom γ) follows from this observation.

Finally we prove that any k-dimensional volume function that satisfies axioms α), β) and γ) is unique.

Let $x_1 \cdots x_k \in \mathbb{E}^n$. Perform the Gram-Schmidt process on $x_1 \cdots x_k$. This process yields a sequence $y_1, u_1, y_2, u_2, \ldots \in \mathbb{E}^n$ such that $y_1 = x_1$, $u_1 = y_1/\|y_1\|, \ldots, y_i = x_i - \sum_{j=1}^{i-1}(x_i \cdot u_j)u_j$, $u_i = y_i/\|y_i\|$. The process terminates either if some $y_i = 0$ or with y_k, u_k. ~~Also~~ the sequence $u_1, u_2, u_3, \ldots$ is orthonormal.

Axiom β) implies

$$V_k^n(\Pi(u_1 \cdots u_{i-1} x_i x_{i+1} \cdots x_k)) = V_k^n(\Pi(u_1 \cdots u_{i-1} y_i x_{i+1} \cdots x_k)).$$

Axiom α) implies

$$V_k^n(\Pi(u_1 \cdots u_{i-1} y_i x_{i+1} \cdots x_k)) = \begin{cases} |y_i| V_k^n(\Pi(u_1 \cdots u_{i-1} u_i x_{i+1} \cdots x_k)) & \text{if } y_i \neq 0 \\ 0 & \text{if } y_i = 0 \end{cases}$$

Consequently, either

$V_k^n(\Pi(x_1 \cdots x_k)) = 0$ if some $y_i = 0$ or

$V_k^n(\Pi(x_1 \cdots x_k)) = |y_1| \cdots |y_k| \, V_k^n(\Pi(u_1 \cdots u_k))$.

Axiom δ) implies $V_k^n(\Pi(u_1 \cdots u_k)) = 1$.

Thus $V_k^n(\Pi(x_1 \cdots x_k)) = |y_1| \cdots |y_k|$ if no $y_i = 0$.

This proves V_k^n is unique. $\Box$.

For $x_1, \dots, x_n \in \mathbb{E}^n$, not only does the absolute value of $\det(x_1 \cdots x_n)$ determine the volume of $\Pi(x_1 \cdots x_n)$. Also the sign of $\det(x_1 \cdots x_n)$ indicates whether the sequence $x_1 \cdots x_n$ is positively oriented or negatively oriented. We assert this because the sign of $\det(x_1 \cdots x_n)$ has two characteristics we would want from an orientation indicator: $\det(e_1 \cdots e_n) = \det(I_n) = 1$, and $\det(x_1 \cdots x_n)$ changes sign if we interchange x_i and x_j.

Here is an alternative formula for determining the volume of a k-parallelopiped in $\mathbb{E}^n$.

<u>Theorem 2.22.</u> <u>If $x_1, \dots, x_k \in \mathbb{E}^n$ and $A = (x_1 \cdots x_k) \in \mathbb{R}_n^k$, then</u>

$$V_k(\Pi(x_1, \dots, x_k)) = \sqrt{\det(A^T A)}.$$

<u>Proof</u> Let $u_{k+1}, \dots, u_n$ be an orthonormal sequence in $\mathbb{E}^n$ such that $x_i \cdot u_{k+j} = 0$ for $1 \le i \le k$, $1 \le j \le n-k$. Let $A = (x_1 \cdots x_k)$, $B = (u_{k+1} \cdots u_n)$ and $C = (A|B) = (x_1 \cdots x_k u_{k+1} \cdots u_n)$.

Then $V_k^n(\Pi(x_1 \cdots x_n)) = |\det(C)| =$

$\sqrt{(\det(C))^2} = \sqrt{\det(C^T)\det(C)} = \sqrt{\det(C^TC)}$.

Since $C = (A|B)$, $C^T = \left(\frac{A^T}{B^T}\right)$ and

$$C^TC = \left(\frac{A^T}{B^T}\right)(A|B) = \left(\begin{array}{c|c} A^TA & A^TB \\ \hline B^TA & B^TB \end{array}\right)$$

For $1 \le i \le k$, $1 \le j \le n-k$, the $(i,j)^{th}$ entry of A^TB is $x_i^T u_{j+k} = x_i \cdot u_{j+k} = 0$. So $A^TB = \mathbb{O}$

For $1 \le i \le n-k$, $1 \le j \le k$, the $(i,j)^{th}$ entry of B^TA is $u_{i+k}^T x_j = u_{i+k} \cdot x_j = 0$. So $B^TA = 0$.

For $1 \le i \le n-k$, $1 \le j \le n-k$, the $(i,j)^{th}$ entry of B^TB is $u_{i+k}^T u_{j+k} = u_{i+k} \cdot u_{j+k} = \delta_{ij}$. So $B^TB = I_{n-k}$.

Thus $C^TC = \left(\begin{array}{c|c} A^TA & \mathbb{O} \\ \hline 0 & I_{n-k} \end{array}\right)$. Therefore,

$\det(C^TC) = \det(A^TA)$. Hence,

$V_k^n(\Pi(x_1 \cdots x_n)) = \sqrt{\det(A^TA)}$. $\square$

Recall: for $x, y \in \mathbb{E}^3$, $x \times y = \det\begin{pmatrix} x & y & \begin{matrix} e_1 \\ e_2 \\ e_3 \end{matrix} \end{pmatrix}$

Hence, for $z \in \mathbb{E}^3$, $(x \times y) \cdot z = \det(x\, y\, z)$.

Definition For $x_1, \ldots, x_{n-1} \in \mathbb{E}^n$,

define $x_1 \times x_2 \times \cdots \times x_{n-1} = \det\begin{pmatrix} x_1 & x_2 & \cdots & x_{n-1} & \begin{matrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{matrix} \end{pmatrix} \in \mathbb{E}^n$

Lemma 2.23 Let $x_1, \ldots, x_{n-1} \in \mathbb{E}^n$.

a) For $y \in \mathbb{E}^n$, $(x_1 \times \cdots \times x_{n-1}) \cdot y = \det(x_1 \cdots x_{n-1}\, y)$.

b) $(x_1 \times \cdots \times x_{n-1}) \cdot x_i = 0$ for $1 \le i \le n-1$.

c) If $u \in \mathbb{E}^n$, $\|u\| = 1$ and $x_i \cdot u = 0$ for $1 \le i \le n-1$, then $x_1 \times \cdots \times x_{n-1} = \pm \|x_1 \times \cdots \times x_n\| u$. $\square$

Theorem 2.24 For $x_1, \ldots, x_{n-1} \in \mathbb{E}^n$,

$$V_{n-1}^n(\Pi(x_1, \ldots, x_{n-1})) = \|x_1 \times \cdots \times x_{n-1}\|.$$

Proof Let $u \in \mathbb{E}^n$ be a unit vector such that $x_i \cdot u = 0$ for $1 \le i \le n-1$. Then

$V_{n-1}^n(\Pi(x_1 \cdots x_{n-1})) = |\det(x_1 \cdots x_{n-1}\, u)| = |(x_1 \times \cdots \times x_{n-1}) \cdot u| =$

$|(\pm \|x_1 \times \cdots \times x_{n-1}\| u) \cdot u\| = \|x_1 \times \cdots \times x_{n-1}\|$. $\square$

Lemma 2.25 For $x_1 \dots x_k \in \mathbb{E}^m$, if $f: \mathbb{E}^m \to \mathbb{E}^n$ is a linear map, then $f(\Pi(x_1 \dots x_k)) = \Pi(f(x_1), \dots, f(x_k))$.

Proof The following statements are equivalent:

$y \in f(\Pi(x_1 \dots x_k))$,

$y = f(\sum_{i=1}^k a_i x_i)$ where $a_i \in [0\,1]$ for $1 \le i \le k$.

$y = \sum_{i=1}^k a_i f(x_i)$ where $a_i \in [0\,1]$ for $1 \le i \le k$.

$y \in \Pi(f(x_1), \dots, f(x_k))$. $\square$

We recall some notation and results of multivariate calculus.

Def Let U be an open subset of $\mathbb{E}^k$. A function $f: U \to \mathbb{E}^n$ is differentiable if for each $x \in U$, there is a linear function $df_x: \mathbb{E}^k \to \mathbb{E}^n$ called the differential of f at x such that

$$\lim_{h \to 0} \frac{\| f(x+h) - f(x) - df_x(h) \|}{\|h\|} = 0$$

If $f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$, then the matrix representative of df_x is the derivative matrix

$$f'(x) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_k}\right) = \left(\frac{\partial f_i}{\partial x_j}\right) \in \mathbb{R}^k_n.$$

Thus, $df_x(e_j) = \frac{\partial f}{\partial x_j}$ for $1 \le j \le k$.

Def Let U be an open subset of $\mathbb{E}^k$. A function $f: U \to \mathbb{E}^n$ is of class C^0 if it is continuous. For $k \geq 1$, we inductively define $f: U \to \mathbb{E}^n$ to be of class C^k if each $\frac{\partial f}{\partial x_j}: U \to \mathbb{E}^n$ is of class C^{k-1} for $1 \le j \le k$.

$f: U \to \mathbb{E}^n$ is of class C^∞ if it is of class C^r for each $r \geq 0$.

Def Let U, V be open subsets of $\mathbb{E}^n$. A function $f: U \to V$ is a diffeomorphism of class C^r $(r \geq 0)$ if $f: U \to V$ is a bijection such that both $f: U \to V$ and $f^{-1}: V \to U$ are of class C^r.

Def Let $A \subset \mathbb{E}^k$. A function $f: A \to \mathbb{E}^n$ is differentiable (of class C^r) if there is an open subset U of $\mathbb{E}^k$ such that $A \subset U$ and there is a differentiable function $g: U \to \mathbb{E}^n$ (of class C^r) such that $g|A = f$.

<u>Def</u> Let $A \subset \mathbb{E}^k$ and $B \subset \mathbb{E}^n$. A function $f: A \to B$ is a <u>diffeomorphism</u> (of class C^r) if $f: A \to B$ is a bijection and $f: A \to \mathbb{E}^n$ and $f^{-1}: B \to \mathbb{E}^k$ are differentiable (of class C^r) as in the previous definition.

We recall the change of variables formula from one-variable calculus. Suppose $f: [a,b] \to [c,d]$ is a diffeomorphism of class C^1 such that $f'(x) > 0$ for all $x \in [a,b]$. If $g: [c,d] \to \mathbb{R}$ is a continuous function, then

$$\int_c^d g(y)\,dy = \int_a^b g(f(x))f'(x)\,dx$$

The factor $f'(x)$ in the right-hand integral is essential in this equation because it accounts for how much f "blows up" or "shrinks" length near x in transforming dx to dy.

We observe that $f'(x) = V_1(df_x(\Pi(1)))$; in other words, the factor by which f blows up (or shrinks) length near x is the 1-dimensional volume of the 1-parallelopiped

$df_x(\Pi(1)) = df_x([0\,1])$. Here's the proof:

$$V_1'(df_x(\Pi(1)) = V_1'(\Pi(df_x(1))) =$$
$$|\det(df_x(1))| = df_x(1) = f'(x).$$

So we can rewrite the one-variable change of variables formula as

$$\int_c^d g(y)\,dy = \int_a^b g(f(x))\,V_1'(df_x(\Pi(1)))\,dx.$$

This version of the change of variables formula generalizes to higher dimensions.

One more insight is needed to state a high dimensional version of the change of variables formula. If U is an open subset of $\mathbb{E}^k$, $f: U \to \mathbb{E}^n$ is a differentiable function, and $x \in U$, then the factor by which f blows up or shrinks k-dimensional volume near x is

$$V_k^n(df_x(\Pi(e_1,\dots,e_k))).$$

Now we state:

Change of Variables Formula 2.26

Let U be an open subset of $\mathbb{E}^k$, let V be an open subset of $\mathbb{E}^n$, let $f: U \to V$ be a function such that $f: U \to f(U)$ is a diffeomorphism of class C^1. Let $g: V \to \mathbb{R}$ be a continuous function. Let $R \subset U$ be a k-dimensional measurable set.* Then

$$\int_{f(R)} g(y)\,dy = \int_R g(f(x))\, V_k^n(df_x(\Pi(e_1 \cdots e_k)))\,dx$$

Def If $S \subset \mathbb{E}^n$ is a k-dimensional measurable set*, then its k-dimensional volume is

$$V_k^n(S) = \int_S 1\,dy$$

Corollary 2.27. If $R \subset U \subset \mathbb{E}^k$, $V \subset \mathbb{E}^n$ and $f: U \to V$ are as in the statement of Theorem 2-26, then

$$V_k^n(f(R)) = \int_R V_k^n(df_x(\Pi(e_1 \cdots e_k)))\,dx.$$

* We will not explore the details of the the theory of k-dimensional measurable sets in $\mathbb{E}^n$. We will leave this topic at the level of intuition. Note, however, that any set which is a finite union of translated rectangular k-parallelopipeds is a k-dimensional measurable set.

We can now rewrite the Change of Variables Formula 2.26 and Corollary 2.27 using our observations about the function V_k^n.

<u>Corollary 2.28</u> If $R \subset U \subset E^k$, $V \subset E^n$, $f: U \to V$ and $g: V \to \mathbb{R}$ are as in the statement of Theorem 2.26, then

$$\int_{f(R)} g(y)\, dy = \int_R g(f(x)) \sqrt{\det\left((f'(x))^T (f'(x)) \right)}\; dx$$

and

$$V_k^n(f(R)) = \int_R \sqrt{\det\left((f'(x))^T (f'(x)) \right)}\; dx$$

<u>Proof</u> It suffices to prove

$$V_k^n(df_x(\Pi(e_1 \cdots e_k))) = \sqrt{\det\left((f'(x))^T (f'(x)) \right)}\,.$$

Using Lemma 2.25 and Theorem 2.22 we have:

$$V_k^n(df_x(\Pi(e_1 \cdots e_k))) = V_k^n(\Pi(df_x(e_1), \ldots, df_x(e_k))) =$$

$$V_n^k(\Pi(f'(x))) = \sqrt{\det\left((f'(x))^T (f'(x)) \right)}\,. \quad \square$$

Observe that if U and V are open subsets of $\mathbb{E}^n$ and $f: U \to V$ is a differentiable function, then $f'(x)$ is an $n \times n$ matrix. Hence,

$$\sqrt{\det\left((f'(x))^T (f'(x))\right)} = \sqrt{\det\left((f'(x))^T\right)\det(f'(x))} =$$

$$\sqrt{(\det(f'(x)))^2} = |\det(f'(x))|.$$

Recall:

<u>Definition</u> If U and V are open subsets of $\mathbb{E}^n$ and $f: U \to V$ is a differentiable function, then for each $x \in U$, $\det(f'(x))$ is called the <u>Jacobian</u> of f at x.

The preceding observation yields:

<u>Corollary 2.29.</u> If $R \subset U \subset \mathbb{E}^k$, $V \subset \mathbb{E}^n$, $f: U \to V$ and $g: V \to \mathbb{R}$ are as in the statement of Theorem 2.26, and if $k = n$, then

$$\int_{f(R)} g(y)\,dy = \int_R g(f(x))\,|\det(f'(x))|\,dx \quad \text{and}$$

$$V_n^n(f(R)) = \int_R |\det(f'(x))|\,dx,$$

This Corollary is the traditional form of the high dimensional change of variables formula.

Now we state a special case of the change of variables formula for $k = n-1$.

Observe that if U is an open subset of $\mathbb{E}^{n-1}$, and $f : U \to \mathbb{E}^n$ is a differentiable function, then for $x \in U$, Theorem 2.24 yields:

$$V^n_{n-1}\left(df_x\left(\Pi(e_1 \cdots e_{n-1})\right)\right) =$$

$$V^n_{n-1}\left(\Pi\left(df_x(e_1), \ldots, df_x(e_{n-1})\right)\right) =$$

$$\| df_x(e_1) \times \cdots \times df_x(e_{n-1}) \| =$$

$$\left\| \frac{\partial f}{\partial x_1}(x) \times \cdots \times \frac{\partial f}{\partial x_{n-1}}(x) \right\|.$$

In this situation, Theorems 2.26 and 2.27 yield:

<u>Corollary 2.30.</u> If $R \subset U \subset \mathbb{E}^k$, $V \subset \mathbb{E}^n$, $f : U \to V$ and $g : V \to \mathbb{R}$ are as in the statement of Theorem 2.26, and if $k = n-1$, then

$$\int_{f(R)} g(y)\, dy = \int_R g(f(x)) \left\| \frac{\partial f}{\partial x_1}(x) \times \cdots \times \frac{\partial f}{\partial x_{n-1}}(x) \right\| dx$$

and

$$V^n_{n-1}(f(R)) = \int_R \left\| \frac{\partial f}{\partial x_1}(x) \times \cdots \times \frac{\partial f}{\partial x_{n-1}}(x) \right\| dx.$$

<u>Corollary 2.31.</u> Let U be an open subset of $\mathbb{E}^k$, let $f: U \to \mathbb{E}^n$ be a function such that $f: U \to f(U)$ is a diffeomorphism of class C^1, and let R be a k-dimensional measurable subset of U. Let $r>0$ and define $g: \mathbb{E}^n \to \mathbb{E}^n$ by $g(x) = rx$. Then

$$V_k^n(g \circ f(R)) = r^k V_k^n(f(R)).$$

<u>Homework Problem 2.6</u> Prove Corollary 2.31.

Def A coordinate chart for S^n is a differentiable function $f: R \to S^n$ where R is an n-manifold possibly with boundary (or corners) in $\mathbb{E}^n$.

Def The Archimedian coordinate chart (or Archimedian projection)

$$A: [0, 2\pi] \times B^{n-1} \to S^n$$

is defined by the equation

$$A(\theta, x) = (\sqrt{1-\|x\|^2}\cos\theta, \sqrt{1-\|x\|^2}\sin\theta, x_1, \dots, x_{n-1})$$

where $\theta \in [0, 2\pi]$ and $x = (x_1, \dots, x_{n-1}) \in B^{n-1}$

Theorem 2.32 a) $A: [0, 2\pi] \times B^{n-1} \to S^n$ is onto.

b) $A|(0, 2\pi) \times \mathrm{int}(B^{n-1})$ is a diffeomorphism onto its image

c) $A((0, 2\pi) \times \mathrm{int}(B^{n-1})) \cap A(\partial([0, 2\pi] \times B^{n-1})) = \emptyset$

Proof of a) Let $y = (y_1, y_2, \dots, y_{n+1}) \in S^n$.

Let $x = (y_3, \dots, y_{n+1}) \in B^{n-1}$. If $(y_1, y_2) = (0,0)$, then $\|x\| = 1$ and $A(\theta, x) = y$ for every $\theta \in [0, 2\pi]$. If $(y_1, y_2) \neq (0,0)$, then $y_1^2 + y_2^2 = 1 - \|x\|^2$. So there is a $\theta \in [0, 2\pi]$ such that $(\cos\theta, \sin\theta) = \dfrac{(y_1, y_2)}{\sqrt{1 - \|x\|^2}}$. Hence, $A(\theta, x) = (y_1, y_2, y_3, \dots, y_{n+1}) = y$.

<u>Proof of b)</u> If (θ, x) and $(\theta', x') \in (0, 2\pi) \times \operatorname{int}(B^{n-1})$ and $A(\theta, x) = A(\theta', x')$, then $x = x'$ and $\|x\| < 1$. Hence, $(\cos\theta, \sin\theta) = (\cos\theta', \sin\theta')$. Thus $\theta = \theta'$. This proves $A \mid (0, 2\pi) \times \operatorname{int}(B^{n-1})$ is injective.

It remains to prove that $dA_{(\theta,x)} : E^n \to E^{n+1}$ is injective for each $(x, \theta) \in (0, 2\pi) \times \operatorname{int}(B^{n-1})$.

Let $\rho = \sqrt{1 - \|x\|^2}$. Then $\dfrac{\partial \rho}{\partial x_i} = -\dfrac{x_i}{\rho}$ for $1 \le i \le n-1$.

Recall that the linear map $dA_{(\theta,x)}$ is represented by the derivative matrix $A'(\theta, x) \in \mathbb{R}^n_{n+1}$ which has the form

$$\begin{pmatrix} -\rho\sin\theta & -(x_1/\rho)\cos\theta & \cdots & -(x_{n-1}/\rho)\cos\theta \\ \rho\cos\theta & -(x_1/\rho)\sin\theta & \cdots & -(x_{n-1}/\rho)\sin\theta \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Let $\tau_1, \tau_2 : \mathbb{E}^{n+1} \to \mathbb{E}^n$ be the projections

$$\tau_1(y_1 y_2 \cdots y_{n+1}) = (y_2, \cdots, y_{n+1}) \text{ and}$$

$$\tau_2(y_1 y_2 \cdots y_{n+1}) = (y_1\ y_3 \cdots y_{n+1}).$$

Then τ_1 and τ_2 are represented by the matrices

$$T_1 = (0\ e_1 \cdots e_n) \text{ and } T_2 = (e_1\ 0\ e_2 \cdots e_n) \in \mathbb{R}^n_{n+1}$$

Thus, $\tau_i \circ dA_{(\theta,x)} : \mathbb{E}^n \to \mathbb{E}^n$ is represented by the matrix $T_i \cdot A'(\theta, x) \in \mathbb{R}^n_n$ for $i = 1, 2$.

Note that if $A = \begin{pmatrix} a_1 \\ \vdots \\ a_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1}_n$, then

$T_1 \cdot A = \begin{pmatrix} a_2 \\ \vdots \\ a_{n+1} \end{pmatrix}$ and $T_2 \cdot A = \begin{pmatrix} a_1 \\ a_3 \\ \vdots \\ a_{n+1} \end{pmatrix}$. Hence,

$$T_1 \cdot A'(\theta, x) = \begin{pmatrix} \rho\cos\theta & -(x_1/\rho)\sin\theta & \cdots & -(x_{n-1}/\rho)\sin\theta \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

and

$$T_2 \cdot A'(\theta, x) = \begin{pmatrix} -\rho\sin\theta & -(x_1/\rho)\cos\theta & \cdots & -(x_{n-1}/\rho)\cos\theta \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Therefore, $\det(T_1 A'(\theta,x)) = \rho\cos\theta$ and $\det(T_2 A'(\theta x)) = -\rho\sin\theta$.

For $(\theta,x) \in (0,2\pi) \times (\mathrm{int}(B^{n-1}))$, $\rho \neq 0$ and either $\cos\theta \neq 0$ or $\sin\theta \neq 0$!

Hence, either $\det(T_1 A'(\theta,x)) \neq 0$ or $\det(T_2 A'(\theta x)) \neq 0$. Thus, either

$\tau_1 \circ dA_{(\theta,x)} : \mathbb{E}^n \to \mathbb{E}^n$ is an isomorphism or

$\tau_2 \circ dA_{(\theta,x)} : \mathbb{E}^n \to \mathbb{E}^n$ is an isomorphism.

Hence, $dA_{(\theta,x)} : \mathbb{E}^n \to \mathbb{E}^{n+1}$ is always injective.

It follows by a variation of the Inverse Function Theorem that

$$A|(0,2\pi)\times \mathrm{int}(B^{n-1}) : (0,2\pi)\times\mathrm{int}(B^{n-1}) \to A((0,2\pi)\times\mathrm{int}(B^{n-1}))$$

has a differentiable inverse. Thus, $A|(0,2\pi)\times\mathrm{int}(B^{n-1})$ is a diffeomorphism onto its image.

c) $\partial([0,2\pi]\times B^{n-1}) = (\{0,2\pi\}\times B^{n-1}) \cup ([0,2\pi]\times S^{n-2})$.

Let $(\theta,x) \in (0,2\pi)\times\mathrm{int}(B^{n-1})$ and let $(\theta',x') \in \partial([0,2\pi]\times B^{n-1})$.

First suppose $(\theta',x') \in [0,2\pi]\times S^{n-2}$.

Since $\|x\| < 1$, then $A(\theta, x) = (\sqrt{1-\|x\|^2}\cos\theta, \sqrt{1-\|x\|^2}\sin\theta, \ldots)$
where $(\sqrt{1-\|x\|^2}\cos\theta, \sqrt{1-\|x\|^2}\sin\theta) \neq (0,0)$.

Since $\|x'\| = 1$, then $A(\theta', x') = (0, 0, \ldots)$
Hence, $A(\theta, x) \neq A(\theta', x')$.

Second suppose $(\theta', x') \in \{0, 2\pi\} \times B^{n-1}$.
Assume $A(\theta, x) = A(\theta', x')$. Then $x = x'$.
Since $\|x\| < 1$, it follows that $(\cos\theta, \sin\theta) =$
$(\cos\theta', \sin\theta')$. But this is impossible
if $\theta \in (0, 2\pi)$ and $\theta' \in \{0, 2\pi\}$.
We conclude $A(\theta, x) \neq A(\theta', x')$.

This proves $A((0, 2\pi) \times \operatorname{int}(B^{n-1}))$
and $A(\partial([0, 2\pi] \times B^{n-1}))$ are disjoint. $\square$

<u>Theorem 2.33</u> For each $(\theta, x) \in [0,2\pi] \times B^{n-1}$,

$$\left\| \frac{\partial A}{\partial \theta}(\theta, x) \times \frac{\partial A}{\partial x_1}(\theta, x) \times \cdots \times \frac{\partial A}{\partial x_{n-1}}(\theta, x) \right\| = 1.$$

Hence, for each n-dimensional measurable subset R of $[0,2\pi] \times B^{n-1}$,

$$V_n^{n+1}(A(R)) = V_n^n(R).$$

In particular,

$$V_n^{n+1}(S^n) = V_n^n([0,2\pi] \times B^{n-1}) = 2\pi V_{n-1}^{n-1}(B^{n-1}).$$

<u>Proof</u> Again let $\rho = \sqrt{1 - \|x\|^2}$. Then

$\frac{\partial \rho}{\partial x_i} = -\frac{x_i}{\rho}$ for $1 \le i \le n-1$. Since

$A(\theta, x) = (\rho\cos\theta, \rho\sin\theta, x_1 \ldots x_{n-1})$, then

$$\begin{pmatrix} \frac{\partial A}{\partial \theta}(\theta, x) & \frac{\partial A}{\partial x_1}(\theta, x) & \cdots & \frac{\partial A}{\partial x_{n-1}}(\theta, x) & \begin{matrix} e_1 \\ \vdots \\ e_{n+1} \end{matrix} \end{pmatrix} =$$

$$\begin{pmatrix} -\rho\sin\theta & -\frac{x_1}{\rho}\cos\theta & -\frac{x_2}{\rho}\cos\theta & \cdots & -\frac{x_{n-1}}{\rho}\cos\theta & e_1 \\ \rho\cos\theta & -\frac{x_1}{\rho}\sin\theta & -\frac{x_2}{\rho}\sin\theta & \cdots & -\frac{x_{n-1}}{\rho}\sin\theta & e_2 \\ 0 & 1 & 0 & \cdots & 0 & e_3 \\ 0 & 0 & 1 & \cdots & 0 & e_4 \\ & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & e_{n+1} \end{pmatrix}.$$

Hence, in $\frac{\partial A}{\partial \theta} \times \frac{\partial A}{\partial x_1} \times \cdots \times \frac{\partial A}{\partial x_{n-1}} = \det\begin{pmatrix} \frac{\partial A}{\partial \theta} & \frac{\partial A}{\partial x_1} & \cdots & \frac{\partial A}{\partial x_{n-1}} & \begin{matrix} e_1 \\ \vdots \\ e_{n+1} \end{matrix} \end{pmatrix}$:

the coefficient of e_1 is $\pm \rho \cos\theta$,
the coefficient of e_2 is $\pm \rho \sin\theta$,
the coefficient of e_3 is $\pm x_1(\sin^2\theta + \cos^2\theta) = \pm x_1$
the coefficient of e_4 is $\pm x_2(\sin^2\theta + \cos^2\theta) = \pm x_2$

$\vdots$

the coefficient of e_{n+1} is $\pm x_{n-1}(\sin^2\theta + \cos^2\theta) = \pm x_{n-1}$.

Therefore, $\left\| \frac{\partial A}{\partial \theta} \times \frac{\partial A}{\partial x_1} \times \cdots \times \frac{\partial A}{\partial x_{n-1}} \right\| =$

$\rho^2 \cos^2\theta + \rho^2 \sin^2\theta + x_1^2 + \cdots + x_{n-1}^2 =$

$\rho^2 + \|x\|^2 = (1 - \|x\|^2) + \|x\|^2 = 1$.

This proves $\left\| \frac{\partial A}{\partial \theta} \times \frac{\partial A}{\partial x_1} \times \cdots \times \frac{\partial A}{\partial x_{n-1}} \right\| = 1$.

Let $D = [0, 2\pi] \times B^{n-1}$. Then
$\operatorname{int} D = (0, 2\pi) \times \operatorname{int}(B^{n-1})$ and $\partial D =$
$(\{0, 2\pi\} \times B^{n-1}) \cup ([0, 2\pi] \times S^{n-2})$. Then $V_n^n(\partial D) = 0$.
Theorem 2.31 implies that
$A|\operatorname{int} D : \operatorname{int} D \to A(\operatorname{int}(D))$ is a diffeomorphism
and $A(\operatorname{int} D) \cap A(\partial D) = \emptyset$.

Let R be an n-dimensional measurable subset of D. Then $V_n^n(R \cap (\partial D)) = 0$,

$A| R \cap int(D) : R \cap int(D) \to A(R \cap int(D))$
is a diffeomorphism, and
$A(R \cap int(D)) \cap A(R \cap (\partial D)) = \emptyset$.
Since differentiable functions preserve volume 0, then $V_n^{n+1}(A(R \cap \partial D)) = 0$
Therefore:
$V_n^{n+1}(A(R)) = V_n^{n+1}(A(R \cap int(D))) + V_n^{n+1}(A(R \cap (\partial D))) =$
$V_n^{n+1}(A(R \cap (int D)) =$

$$\int_{R \cap (int D)} \left\| \frac{\partial A}{\partial \theta} \times \frac{\partial A}{\partial x_1} \times \cdots \times \frac{\partial A}{\partial x_{n-1}} \right\| dx \quad \text{(by Corollary 2.30)} =$$

$$\int_{R \cap (int D)} 1 \, dx = V_n^n(R \cap (int D)) =$$

$$V_n^n(R \cap (int D)) + V_n^n(R \cap (\partial D)) = V_n^n(R).$$

Finally, since $A(D) = S^n$, then
$V_n^{n+1}(S^n) = V_n^{n+1}(A(D)) = V_n^n(D) =$
$V_n^n([0, 2\pi] \times B^{n-1}) = 2\pi V_{n-1}^{n-1}(B^{n-1})$. $\square$

<u>Theorem 2.34</u> $V^{n+1}_{n+1}(B^{n+1}) = \frac{1}{n+1} V^{n+1}_{n}(S^n)$.

<u>Proof</u> Let R be an n-dimensional measurable subset of E^n, let Z be a measure 0 subset of R, and let $f: R \to S^n$ be an onto differentiable map such that $f|R-Z$ is a diffeomorphism onto its image. (For example, let f be the Archimedian projection.) Then

$$V^{n+1}_n(S^{n+1}) = V^{n+1}_n f(R-Z) =$$

$$\int_{R-Z} \left\| \frac{\partial f}{\partial x_1} \times \cdots \times \frac{\partial f}{\partial x_n} \right\| dx = \int_R \left\| \frac{\partial f}{\partial x_1} \times \cdots \times \frac{\partial f}{\partial x_n} \right\| dx.$$

Define $g: R \times [0,1] \to B^{n+1}$ by $g(x,r) = rf(x)$. $(R \times \{0,1\}) \cup (Z \times [0,1])$ has measure 0, g is differentiable and onto and $g|(R-Z)\times(0,1)$ is a diffeomorphism onto its image. To verify this last assertion, observe that $g|(R-Z)\times(0,1)$ is injective. It remains to prove that $dg_{(x,r)}: E^{n+1} \to E^{n+1}$ is an isomorphism for each $(x,r) \in (R-Z)\times(0,1)$. For $1 \le i \le n$,

$$dg_{(x,r)}(e_i) = \frac{\partial g}{\partial x_i}(x,r) = r\frac{\partial f}{\partial x_i}(x) \text{ and}$$

$dg_{(x,r)}(e_{n+1}) = \frac{\partial g}{\partial r}(x,r) = f(x)$. Since $f|R-Z$ is a diffeomorphism and $x \in R-Z$, then ~~the~~ $df_x(e_i) = \frac{\partial f}{\partial x_i}(x)$, $1 \le i \le n$, are linearly independent. Therefore, the $dg_{(x,r)}(e_i) = r\frac{\partial f}{\partial x_i}(x)$, $1 \le i \le n$, are linearly independent. Since $f(y) \cdot f(y) = 1$ for all $y \in R$, then

$$0 = \frac{\partial}{\partial x_i}\big(f(x) \cdot f(x)\big) = 2 f(x) \cdot \frac{\partial f}{\partial x_i}(x) \text{ for } 1 \le i \le n.$$

Thus, $dg_{(x,r)}(e_{n+1}) \cdot dg_{(x,r)}(e_i) = f(x) \cdot r\frac{\partial f}{\partial x_i}(x) = 0$ for $1 \le i \le n$. Also $\|dg_{(x,r)}(e_{n+1})\| = \|f(x)\| = 1$. It follows that $dg_{(x,r)}(e_i)$, $1 \le i \le n+1$, are linearly independent. Hence $dg_{(x,r)} : \mathbb{E}^{n+1} \to \mathbb{E}^{n+1}$ is an isomorphism. Consequently, $g|(R-Z)\times(0,1)$ is a diffeomorphism onto its image.

It follows that $V^{n+1}_{n+1}(B^{n+1}) = V^{n+1}_{n+1}\big(g((R-Z)\times(0,1))\big) =$

$$\int_{(R-Z)\times(0,1)} |\det(g'(y))|\, dy = \int_{R\times[0,1]} |\det(g'(y))|\, dy$$

where $y = (x,r)$.

Thus, $V_{n+1}^{n+1}(B^{n+1}) = \int_{R\times[0,1]} \left| \det\left(\frac{\partial g}{\partial x_1} \cdots \frac{\partial g}{\partial x_n} \frac{\partial g}{\partial r}\right)\right| dy$

$= \int_{R\times[0,1]} \left| \det\left(r\frac{\partial f}{\partial x_1} \cdots r\frac{\partial f}{\partial x_n}\ f\right)\right| dy =$

$\int_{R\times[0,1]} r^n \left| \det\left(\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n}\ f\right)\right| dy =$

$\int_R \left| \det\left(\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n}\ f\right)\right| dx \int_0^1 r^n dr$

by Fubini's Theorem. Thus,

$V_{n+1}^{n+1}(B^{n+1}) = \frac{1}{n+1} \int_R \left| \det\left(\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n}\ f\right)\right| dx$

$= \frac{1}{n+1} \int_R \left| \left(\frac{\partial f}{\partial x_1} \times \cdots \times \frac{\partial f}{\partial x_n}\right) \cdot f \right| dx$

We showed previously that $\frac{\partial f}{\partial x_i}(x) \cdot f(x) = 0$ for $1 \le i \le n$. Hence, Lemma 2.23c implies

$\frac{\partial f}{\partial x_1} \times \cdots \times \frac{\partial f}{\partial x_n} = \pm \left\| \frac{\partial f}{\partial x_1} \times \cdots \times \frac{\partial f}{\partial x_n} \right\| f(x)$.

Therefore $\left(\frac{\partial f}{\partial x_1} \times \cdots \times \frac{\partial f}{\partial x_n}\right) \cdot f = \pm \left\| \frac{\partial f}{\partial x_1} \times \cdots \times \frac{\partial f}{\partial x_n} \right\|$.

So $\int_R \left| \left(\frac{\partial f}{\partial x_1} \times \cdots \times \frac{\partial f}{\partial x_n} \right) \cdot f \right| dx =$

$\int_R \left\| \frac{\partial f}{\partial x_1} \times \cdots \times \frac{\partial f}{\partial x_n} \right\| dx = V_n^{n+1}(S^n)$.

Thus, $V_{n+1}^{n+1}(B^{n+1}) = \frac{1}{n+1} V_n^{n+1}(S^n)$. $\square$

Theorems 2.33 and 2.34 imply $V_{n+1}^{n+1}(B^{n+1}) = \frac{1}{n+1} V_n^{n+1}(S^n)$ and $V_n^{n+1}(S^n) = 2\pi\, V_{n-1}^{n-1}(B^{n-1})$. These equations allow us to construct the following table.

space	volume	space	volume
B^0	1	B^1	2
S^1	2π	S^2	4π
B^2	π	B^3	$\frac{4}{3}\pi$
S^3	$2\pi^2$	S^4	$8/3\ \pi^2$
B^4	$1/2\ \pi^2$	B^5	$8/15\ \pi^2$
S^5	π^3	S^6	$16/15\ \pi^3$
B^6	$1/6\ \pi^3$	B^7	$16/105\ \pi^3$
S^7	$1/3\ \pi^4$	S^8	$32/105\ \pi^4$
B^8	$1/24\ \pi^4$	B^9	$32/945\ \pi^4$
S^9	$1/12\ \pi^5$	S^{10}	$64/945\ \pi^5$

We can express these results in closed form as follows.

$$\begin{cases} V_{2n}^{2n}(B^{2n}) = \dfrac{\pi^n}{n!} & V_{2n+1}^{2n+1}(B^{2n+1}) = \dfrac{2(2\pi)^n}{(2n+1)!!} \\ V_{2n+1}^{2n+2}(S^{2n+1}) = \dfrac{2\pi^{n+1}}{n!} & V_{2n}^{2n+1}(S^{2n}) = \dfrac{2(2\pi)^n}{(2n-1)!!} \end{cases}$$

where $(2n+1)!! = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1) \cdot (2n+1)$.

<u>Homework Problem 2.7</u>. For $x \in S^2$ and $0 < r < \pi$, let
$C(x,r) = \{y \in S^2 : \Theta(x,y) = r\}$ and $D(x,r) = \{y \in S^2 : \Theta(x,y) \leq r\}$.
Then $C(x,r)$ is the <u>circle in S^2 of radius r centered at x</u> and $D(x,r)$ is the <u>disk in S^2 of radius r centered at x</u>. Express the circumference of $C(x,r)$ (measured in S^2) and the area of $D(x,r)$ (measured in S^2) as functions of r.

Recall: $\cos^{-1} : [-1,1] \to [0,\pi]$ is the inverse of $\cos|[0,\pi] : [0,\pi] \to [-1,1]$. Also recall that for $u, v \in S^n$, $\Theta(u,v) = \cos^{-1}(u \cdot v)$. We now enlarge the domain of Θ.

<u>Def</u> For $x, y \in \mathbb{E}^n - \{0\}$, define
$$\Theta(x,y) = \cos^{-1}\left(\frac{x}{\|x\|} \cdot \frac{y}{\|y\|}\right).$$
Thus, $x \cdot y = \|x\| \, \|y\| \cos(\Theta(x,y))$.

<u>Definition</u> Let $f: \mathbb{E}^k \to \mathbb{E}^n$ be a function.

- f is <u>angle preserving</u> if for all $x, y \in \mathbb{E}^k - \{0\}$, $f(x), f(y) \in \mathbb{E}^n - \{0\}$ and $\theta(f(x), f(y)) = \theta(x, y)$.
- f <u>preserves orthogonality</u> if for all $x, y \in \mathbb{E}^k$, $x \cdot y = 0$ implies $f(x) \cdot f(y) = 0$.
- f is a <u>similarity</u> with <u>scale factor</u> $r > 0$ if $\|f(x) - f(y)\| = r\|x - y\|$ for all $x, y \in \mathbb{E}^n$.

<u>Lemma 2.35.</u> If $f: \mathbb{E}^k \to \mathbb{E}^n$ is a linear function, then the following are equivalent.

a) f is angle preserving.

b) f preserves orthogonality.

c) f is a similarity.

d) There is an $s > 0$ such that $f(x) \cdot f(y) = s(x \cdot y)$ for all $x, y \in \mathbb{E}^k$.

<u>Homework Problem 2.8.</u> Prove Lemma 2.35.

<u>Homework Problem 2.9.</u> Prove that if a function $f: \mathbb{E}^n \to \mathbb{E}^n$ is a similarity with scale factor $r \neq 1$, then f has a fixed point.

Def Let U be an open subset of $\mathbb{E}^k$. A differentiable function $f: U \to \mathbb{E}^n$ is conformal if $df_x: \mathbb{E}^k \to \mathbb{E}^n$ is angle preserving for each $x \in U$.

The following lemma provides examples of conformal functions

Lemma 2.36 If $U \subset \mathbb{E}^k$ and $V \subset \mathbb{E}^m$ are open subsets and $f: U \to V$ and $g: V \to \mathbb{E}^n$ are conformal functions, then $g \circ f: U \to \mathbb{E}^n$ is conformal.

b) If $U, V \subset \mathbb{E}^n$ are open subsets and $f: U \to V$ is a conformal diffeomorphism, then $f^{-1}: V \to U$ is conformal

c) If $f: \mathbb{E}^k \to \mathbb{E}^n$ is a distance preserving, then f is conformal.

d) Let $r > 0$ and define the dilation $D_r: \mathbb{E}^n \to \mathbb{E}^n$ by $D_r(x) = rx$. Then D_r is conformal.

e) If U is an open subset of $\mathbb{C}$ $(= \mathbb{E}^2)$ and $f: U \to \mathbb{C}$ is a holomorphic function* such that $f'(z) \neq 0$ for each $z \in U$, then f is conformal.

* $f: U \to \mathbb{C}$ is holomorphic if its complex derivative $f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$ exists for all $z \in U$.

Proof of a) First observe that if $\varphi : \mathbb{E}^k \to \mathbb{E}^m$ and $\psi : \mathbb{E}^m \to \mathbb{E}^n$ preserve orthogonality, then so does $\psi \circ \varphi : \mathbb{E}^k \to \mathbb{E}^m$. Indeed, if $x, y \in \mathbb{E}^k$ and $x \cdot y = 0$, then $\varphi(x) \cdot \varphi(y) = 0$. Hence, $\psi \circ \varphi(x) \cdot \psi \circ \varphi(y) = \psi(\varphi(x)) \cdot \psi(\varphi(y)) = 0$.

For $x \in U$, the Chain Rule implies $d(g \circ f)_x = dg_{f(x)} \circ df_x$. Since f and g are conformal, then df_x and $dg_{f(x)}$ preserve orthogonality. Hence, the observation in the preceding paragraph implies $dg_{f(x)} \circ df_x$ preserves orthogonality. Thus $d(g \circ f)_x$ preserves orthogonality. This proves $g \circ f$ is conformal. $\square$

Proof of b) First observe that if $\varphi : \mathbb{E}^n \to \mathbb{E}^n$ is an angle preserving linear isomorphism, then so is $\varphi^{-1} : \mathbb{E}^n \to \mathbb{E}^n$. Indeed, for $x, y \in \mathbb{E}^n - \{0\}$:
$$\theta(\varphi^{-1}(x), \varphi^{-1}(y)) = \theta(\varphi(\varphi^{-1}(x)), \varphi(\varphi^{-1}(y))) = \theta(x, y).$$

If $f : U \to V$ is a conformal diffeomorphism, then $f^{-1} : V \to U$ is a diffeomorphism. For $y \in V$, if $x = f^{-1}(y)$, then $d(f^{-1})_y = (df_x)^{-1}$. Since f is conformal, then df_x is angle preserving. Hence, $d(f^{-1})_y$ is angle preserving by the preceding observation. This proves f^{-1} is conformal. $\square$

c) Let $f: \mathbb{E}^k \to \mathbb{E}^n$ be a distance preserving function. Define $f_0: \mathbb{E}^k \to \mathbb{E}^n$ by $f_0(x) = f(x) - f(0)$. Then f_0 is distance preserving: $\|f_0(x) - f_0(y)\| =$
$\|(f(x)-f(0)) - (f(y)-f(0))\| = \|f(x)-f(y)\| = \|x-y\|$.
Also $f_0(0) = f(0) - f(0) = 0$. Hence, $f_0: \mathbb{E}^k \to \mathbb{E}^n$ is linear by Corollary 1.17.

We will show that $f: \mathbb{E}^k \to \mathbb{E}^n$ is differentiable and $df_x = f_0$ for each $x \in \mathbb{E}^k$. Since f is distance preserving, then Theorem 1.13 implies f is strongly affine. Hence, for $x, x+h \in \mathbb{E}^k$,

$$f(x+h) = f(x+h-0) = f(x) + f(h) - f(0) = f(x) + f_0(h).$$

Thus, $f(x+h) - f(x) - f_0(h) = 0$. Hence

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - f_0(h)\|}{\|h\|} = 0.$$

This proves f is differentiable and $df_x = f_0$.

Since f_0 is linear and distance preserving, it is a linear similarity. Since $df_x = f_0$ for each $x \in \mathbb{E}^k$, it follows that f is conformal. $\square$

d) Let $r>0$. Clearly D_r is linear. Since $D_r(x+h) - D_r(x) - D_r(h) = 0$, then

$$\lim_{h \to 0} \frac{\| D_r(x+h) - D_r(x) - D_r(h) \|}{\|h\|} = 0.$$

Thus D_r is differentiable and $d(D_r)_x = D_r$ for each $x \in \mathbb{E}^n$. Since $\|D_r(x) - D_r(y)\| = \|D_r(x-y)\| = r\|x-y\|$ for all $x, y \in \mathbb{E}^n$, then $d(D_r)_x = D_r$ is a similarity for each $x \in \mathbb{E}^n$. It follows that D_r is conformal. $\square$

e) Let U be an open subset of $\mathbb{C}$ and let $f: U \to \mathbb{C}$ be a holomorphic function such that $f'(z) \neq 0$ for each $z \in U$. We can write

$$f(z) = u(z) + i v(z)$$

where $u, v : U \to \mathbb{R}$. Then u and v satisfy the Cauchy Riemann equations:

$$\left\{ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \right\}.$$

[Proof: $\frac{\partial u}{\partial x}(x+iy) + i \frac{\partial v}{\partial x}(x+iy) = \frac{\partial f}{\partial x}(x+iy) =$

$$\lim_{h \to 0} \frac{f(x+h+iy) - f(x+iy)}{h} = \lim_{h \to 0} \frac{f(x+iy+h) - f(x+iy)}{h} = f'(x+iy).$$

Also $\frac{\partial u}{\partial y}(x+iy) + i\frac{\partial v}{\partial y}(x+iy) = \frac{\partial f}{\partial y}(x+iy) =$

$\lim_{h\to 0} \frac{f(x+i(y+h)) - f(x+iy)}{h} = i \lim_{h\to 0} \frac{f(x+iy+ih) - f(x+iy)}{ih} =$

$i f'(x+iy)$. Thus

$$\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = i f' = i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = -\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}.$$

Therefore, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$. $\square$]

Now regard U as a subset of $\mathbb{E}^2$ and f as a function from U to $\mathbb{E}^2$. Then for $x \in U$, $f(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$. Thus, the derivative matrix of f is

$$f'(x) = \begin{pmatrix} \frac{\partial u}{\partial x}(x) & \frac{\partial u}{\partial y}(x) \\ \frac{\partial v}{\partial x}(x) & \frac{\partial v}{\partial y}(x) \end{pmatrix}.$$

Hence, the Cauchy Riemann equations imply

$$f'(x) = \begin{pmatrix} \frac{\partial u}{\partial x}(x) & -\frac{\partial v}{\partial x}(x) \\ \frac{\partial v}{\partial x}(x) & \frac{\partial u}{\partial x}(x) \end{pmatrix}.$$

Since $\frac{\partial u}{\partial x}(x) + i\frac{\partial v}{\partial x}(x) = f'(x) \neq 0$,
then $\left(\frac{\partial u}{\partial x}(x), \frac{\partial v}{\partial x}(x)\right) \neq 0$. Fix $x \in U$ and
let $a = \frac{\partial u}{\partial x}(x)$, $b = \frac{\partial v}{\partial x}(x)$. Then $(a,b) \neq (0,0)$
and $f'(x) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Let $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in E^2$.
Then $df_x(y) = f'(x) \cdot y = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} =$
$\begin{pmatrix} ay_1 - by_2 \\ by_1 + ay_2 \end{pmatrix}$. Hence,
$\|df_x(y)\|^2 = (ay_1 - by_2)^2 + (by_1 + ay_2)^2 =$
$a^2y_1^2 - 2aby_1y_2 + b^2y_2^2 + b^2y_1^2 + 2aby_1y_2 + a^2y_2^2 =$
$(a^2+b^2)(y_1^2+y_2^2) = (a^2+b^2)\|y\|^2$.
Thus, $\|df_x(y) - df_x(z)\| = \|df_x(y-z)\| =$
$\sqrt{a^2+b^2}\,\|y-z\|$. Since $(a,b) \neq (0,0)$, then
$\sqrt{a^2+b^2} > 0$. Thus, df_x is a similarity.
This proves f is conformal. $\square$

We define one more type of conformal function.

<u>Def</u> For $c \in \mathbb{E}^n$ and $r > 0$, let

$$S(c, r) = \{x \in \mathbb{E}^n : \|x - c\| = r\},$$

Call $S(c,r)$ the <u>hypersphere in $\mathbb{E}^n$ of radius r centered at c</u>.

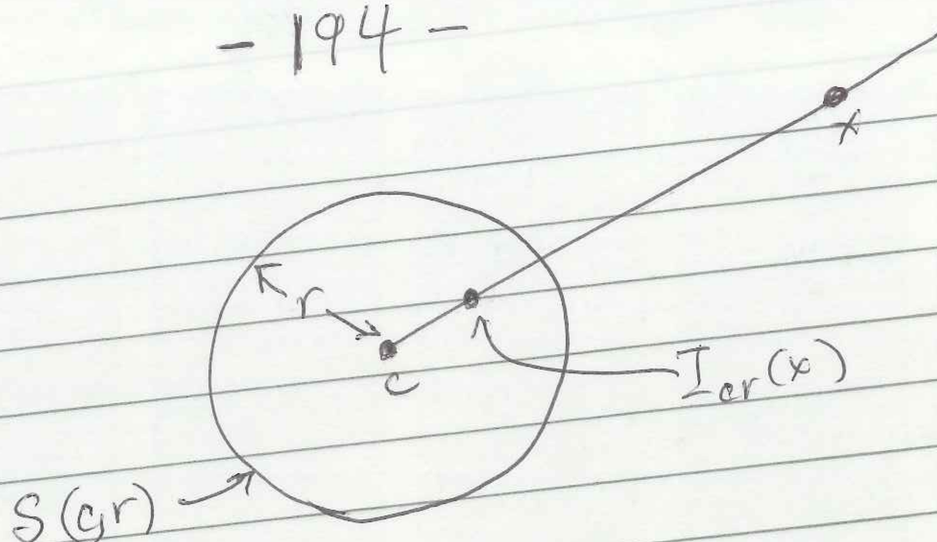
<u>Def</u> For $c \in \mathbb{E}^n$ and $r > 0$, the <u>inversion in $S(c,r)$</u> is the function

$$I_{c,r} : \mathbb{E}^n - \{c\} \to \mathbb{E}^n - \{c\}$$

with the properties that for each $x \in \mathbb{E}^n - \{c\}$, $I_{c,r}(x)$ lies on the ray $\{c + t(x-c) : t > 0\}$ and $\|I_{c,r}(x) - c\| \|x - c\| = r^2$. Hence,

$$I_{c,r}(x) = c + \frac{r^2}{\|x-c\|^2}(x-c)$$

for each $x \in \mathbb{E}^n - \{c\}$. (<u>Proof</u>. $I_{c,r}(x) = c + t(x-c)$ for some $t > 0$. Hence, $r^2 = \|I_{c,r}(x) - c\| \|x - c\| = \|t(x-c)\| \|x-c\| = t\|x-c\|^2$. Thus, $t = r^2/\|x-c\|^2$. Therefore, $I_{c,r}(x) = c + \frac{r^2}{\|x-c\|^2}(x-c)$. $\square$)

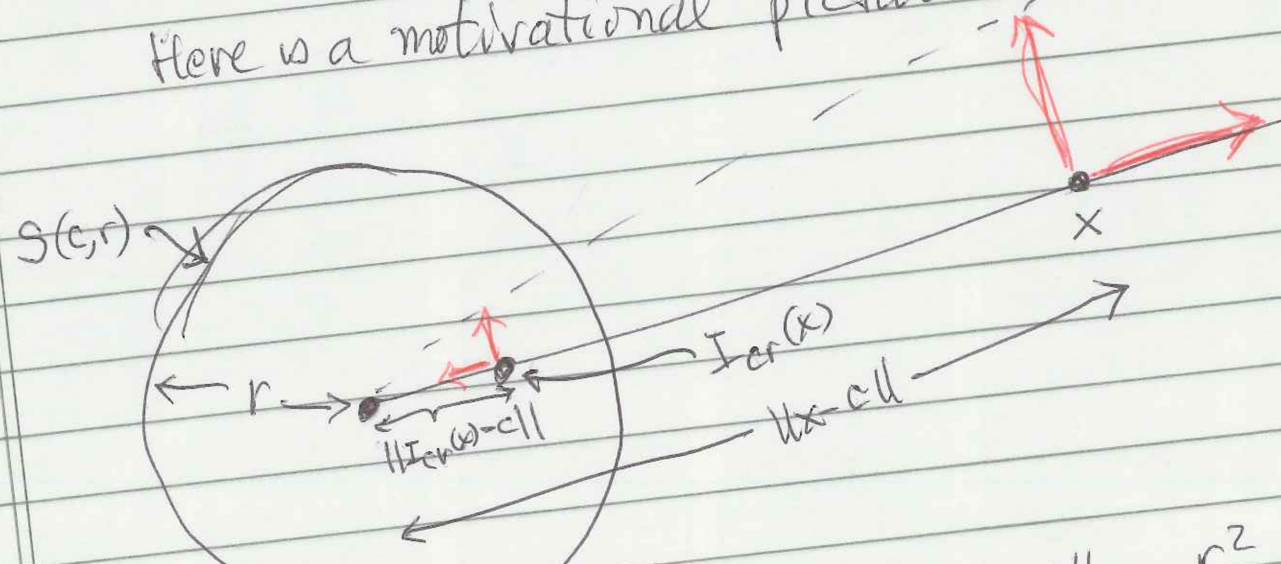


<u>Lemma 2.37</u> Every inversion is conformal.

<u>Proof.</u> Let $c \in \mathbb{E}^n$ and $r > 0$. We will prove that for $x \in \mathbb{E}^n - \{c\}$,

$$d(I_{cr})_x = \frac{r^2}{\|x-c\|^2} Z_{\frac{x-c}{\|x-c\|}, 0}$$

Here is a motivational picture.



Since $\|I_{cr}(x)-c\| \; \|x-c\| = r^2$, then $\frac{\|I_{cr}(x)-c\|}{\|x-c\|} = \frac{r^2}{\|x-c\|^2}$

First we observe that if U is an open subset of $\mathbb{E}^k$ and $f: U \to \mathbb{E}^m$ is a differentiable function, then for each $x \in U$ and each unit vector v in $\mathbb{E}^k$,

$$df_x(v) = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$

<u>Proof:</u> $0 = \lim_{t \to 0} \dfrac{\| f(x+tv) - f(x) - df_x(tv) \|}{\|tv\|} =$

$$\lim_{t \to 0} \left\| \frac{f(x+tv) - f(x) - df_x(tv)}{t\|v\|} \right\| =$$

$$\lim_{t \to 0} \left\| \frac{f(x+tv) - f(x)}{t} - df_x(v) \right\| .$$

Now let $x \in \mathbb{E}^n - \{c\}$ and let v be a unit vector in $\mathbb{E}^n$. We will evaluate $d(I_{c,r})_x(v)$.

$$d(I_{c,r})_x(v) = \lim_{t \to 0} \frac{I_{c,r}(x+tv) - I_{c,r}(x)}{t} =$$

$$\lim_{t \to 0} \frac{1}{t} \left(\left(c + \frac{r^2}{\|x+tv-c\|^2}(x+tv-c) \right) - \left(c + \frac{r^2}{\|x-c\|^2}(x-c) \right) \right) =$$

$$\lim_{t \to 0} \frac{r^2}{t}\left(\frac{\|x-c\|^2(x-c+tv) - \|x-c+tv\|^2(x-c)}{\|x-c\|^2\,\|x-c+tv\|^2}\right) =$$

$$\lim_{t \to 0} \frac{r^2}{t}\left(\frac{\|x-c\|^2(x-c+tv) - (\|x-c\|^2 - 2t(x-c)\cdot v + t^2\|v\|^2)(x-c)}{\|x-c\|^2\,\|x-c+tv\|^2}\right) =$$

$$\lim_{t \to 0} \frac{r^2}{t}\left(\frac{\|x-c\|^2(x-c) + t\|x-c\|^2 v - \|x-c\|^2(x-c) - 2t((x-c)\cdot v)(x-c) + t^2(x-c)}{\|x-c\|^2\,\|x-c+tv\|^2}\right) =$$

$$\lim_{t \to 0} \frac{r^2}{\|x-c\|^2}\left(\frac{\|x-c\|^2 v - 2(v\cdot(x-c))(x-c) + t(x-c)}{\|x-c+tv\|^2}\right) =$$

$$\frac{r^2}{\|x-c\|^2}\left(v - 2\left(v\cdot\left(\frac{x-c}{\|x-c\|}\right)\right)\frac{x-c}{\|x-c\|}\right) =$$

$$\frac{r^2}{\|x-c\|^2} Z_{\frac{x-c}{\|x-c\|},0}(v) .$$

Thus, $d(I_{c,r})_x(v) = \frac{r^2}{\|x-c\|^2} Z_{\frac{x-c}{\|x-c\|},0}(v)$ for every unit vector v in E^n. Since $d(I_{c,r})_x$ and $\frac{r^2}{\|x-c\|^2} Z_{\frac{x-c}{\|x-c\|},0}$ are linear functions, it follows that $d(I_{c,r})_x = \frac{r^2}{\|x-c\|^2} Z_{\frac{x-c}{\|x-c\|},0}$.

Finally for $x \in \mathbb{E}^n - \{c\}$ and $y, z \in \mathbb{E}^n$,

since $Z_{\frac{x-c}{\|x-c\|},0}$ is an isometry:

$\| d(I_{c,r})_x(y) - d(I_{c,r})_x(z) \| =$

$\left\| \frac{r^2}{\|x-c\|^2} Z_{\frac{x-c}{\|x-c\|},0}(y) - \frac{r^2}{\|x-c\|^2} Z_{\frac{x-c}{\|x-c\|},0}(z) \right\| =$

$\frac{r^2}{\|x-c\|^2} \left\| Z_{\frac{x-c}{\|x-c\|},0}(y) - Z_{\frac{x-c}{\|x-c\|},0}(z) \right\| =$

$\frac{r^2}{\|x-c\|^2} \|y-z\|.$

Hence, $d(I_{c,r})_x$ is a similarity for each $x \in \mathbb{E}^n - \{c\}$. It follows that $I_{c,r}$ is conformal. $\square$

Theorem 2.38. Let $c \in \mathbb{E}^n$ and $r > 0$. Then the inversion I_{cr} acts on the collection of all hyperplanes and hyperspheres as follows.

a) If P is a hyperplane in $\mathbb{E}^n$ such that $c \in P$, then $I_{cr}(P - \{c\}) = P - \{c\}$.

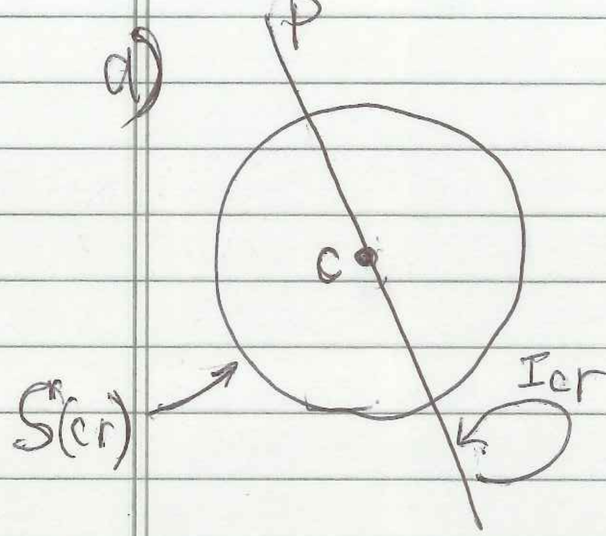
b) If P is a hyperplane in $\mathbb{E}^n$ such that $c \notin P$, then $I_{cr}(P) \cup \{c\}$ is a hypersphere in $\mathbb{E}^n$

c) If S is a hypersphere in $\mathbb{E}^n$ such that $c \in S$, then $I_{cr}(S - \{c\})$ is a hyperplane in $\mathbb{E}^n$ such that $c \notin I_{cr}(S - \{c\})$.

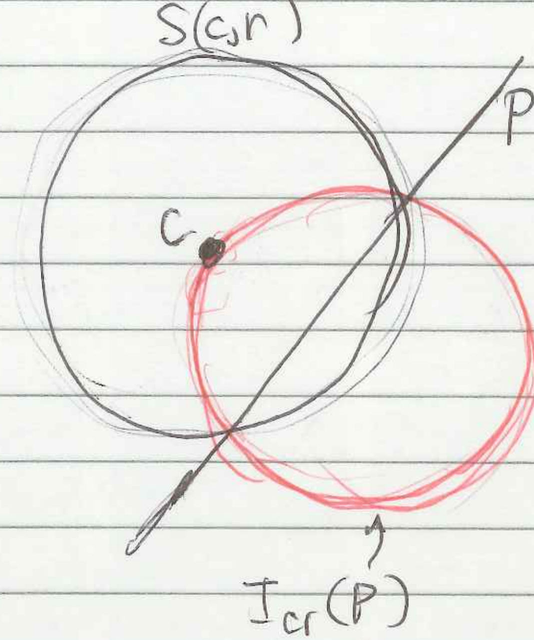
d) If S is a hypersphere in $\mathbb{E}^n$ such that $c \notin S$, then $I_{cr}(S)$ is a hypersphere in $\mathbb{E}^n$ such that $c \notin I_{cr}(S)$.

Homework Problem 2.10. Prove Theorem 2.38.

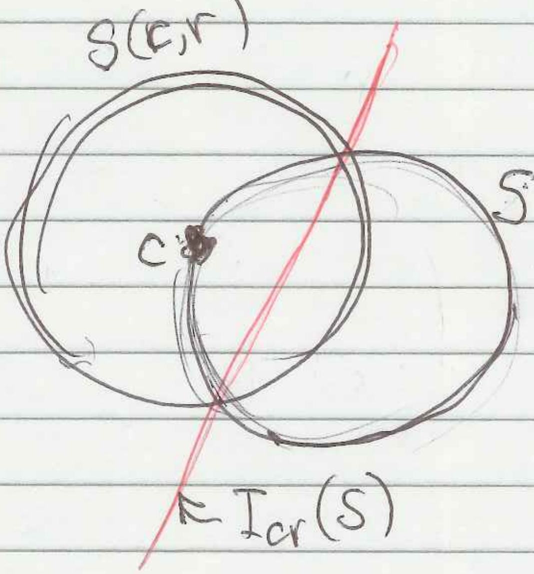
a)



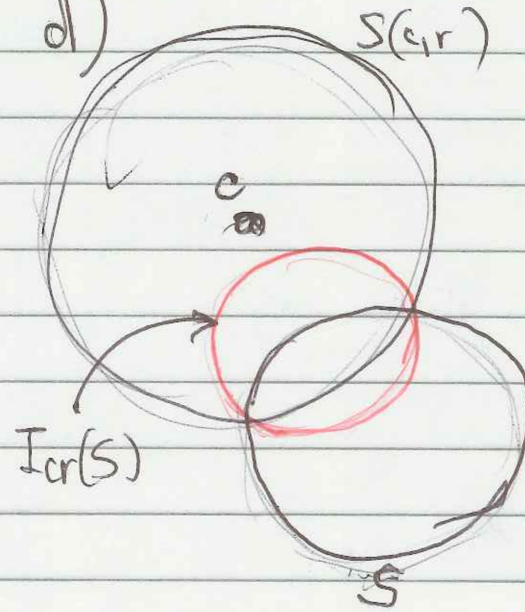
b)



c)



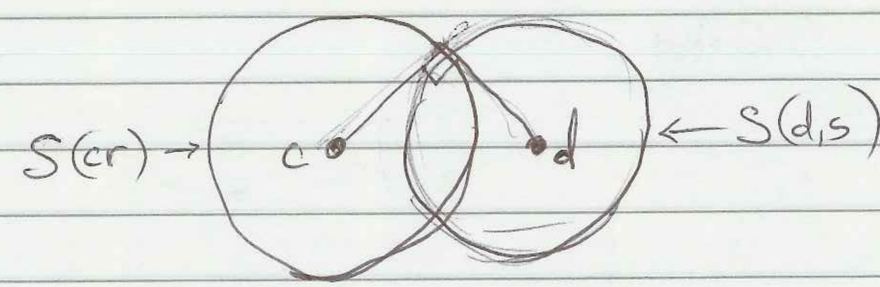
d)



Theorem 2.39. If $c \in \mathbb{E}^n$ and $r, s > 0$, then for each $x \in \mathbb{E}^n - \{c\}$, $I_{cs} \circ I_{cr}(x) = c + \_\_\_\_ (x-c)$.

Homework Problem 2.11. Fill in the blank in Theorem 2.39 and prove it.

Def Two hyperspheres $S(c,r)$ and $S(d,s)$ in $\mathbb{E}^n$ are orthogonal if $S(cr) \cap S(ds) \neq \emptyset$ and $(x-c)\cdot(x-d) = 0$ for every $x \in S(cr) \cap S(ds)$



Theorem 2.40 Suppose $S(cr)$ and $S(ds)$ are hyperspheres in $\mathbb{E}^n$. Then the following are equivalent.

a) $S(cr)$ and $S(ds)$ are orthogonal

b) There is an $x \in S(cr) \cap S(ds)$ such that $(x-c)\cdot(x-d) = 0$.

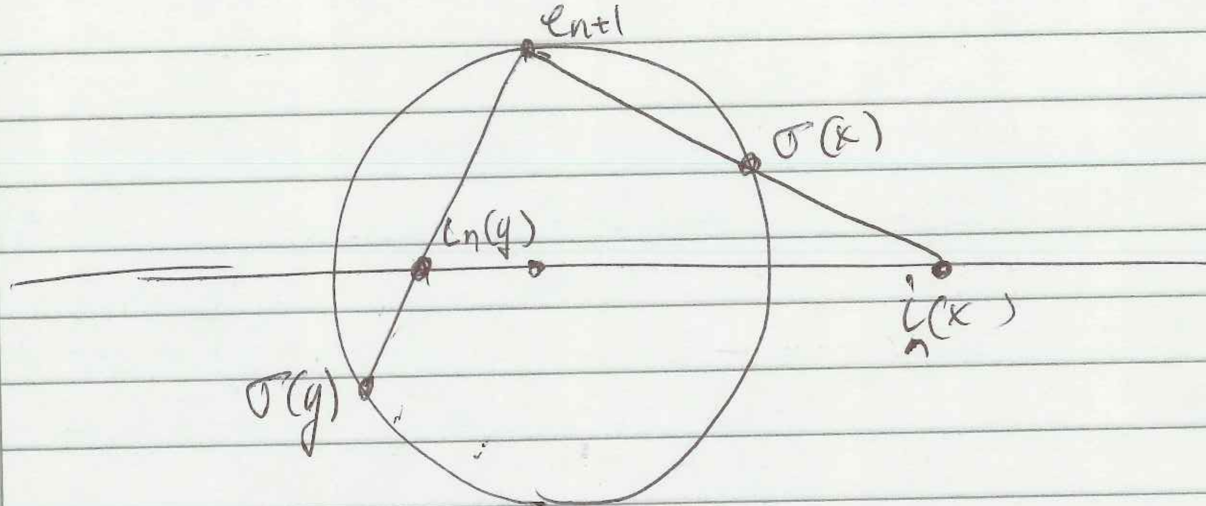
c) $r^2 + s^2 = \|c-d\|^2$

d) $I_{cr}(S(d,s)) = S(ds)$.

Homework Problem 2.12 Prove Theorem 2.40.

Note that we can replace the statement in d) by $I_{d,s}(S(cr)) = S(cr)$ without affecting the truth of the theorem.

Def Define $i_n : \mathbb{E}^n \to \mathbb{E}^{n+1}$ by $i_n(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0)$. Define <u>stereographic projection</u> $\sigma : \mathbb{E}^n \to S^n - \{e_{n+1}\}$ by
$$\sigma(x) = \left(\frac{2}{\|x\|^2+1}\right) i_n(x) + \left(\frac{\|x\|^2-1}{\|x\|^2+1}\right) e_{n+1}$$



Observations

a) Since $\frac{2}{\|x\|^2+1} + \frac{\|x\|^2-1}{\|x\|^2+1} = 1$, then $\sigma(x)$ lies on the line determined by $i_n(x)$ and e_{n+1}

b) Since $i_n(x) \cdot e_{n+1} = 0$, then
$$\|\sigma(x)\|^2 = \frac{4\|x\|^2}{(\|x\|^2+1)^2} + \frac{(\|x\|^2-1)^2}{(\|x\|^2+1)^2} = 1.$$

Hence, $\sigma(\mathbb{E}^n) \subset S^n$.

c) Since $\frac{\|x\|^2-1}{\|x\|^2+1} \neq 1$ for all $x \in \mathbb{E}^n$, then $\sigma(\mathbb{E}^n) \subset S^n - \{e_{n+1}\}$.

Lemma 2.41. Stereographic projection $\sigma: \mathbb{E}^n \to S^n - \{e_{n+1}\}$ is a diffeomorphism.

Proof Define $\pi_n: \mathbb{E}^{n+1} \to \mathbb{E}^n$ by $\pi_n(y_1 \cdots y_{n+1}) = (y_1 \cdots y_n)$ and define $\tau: S^n - \{e_{n+1}\} \to \mathbb{E}^n$ by

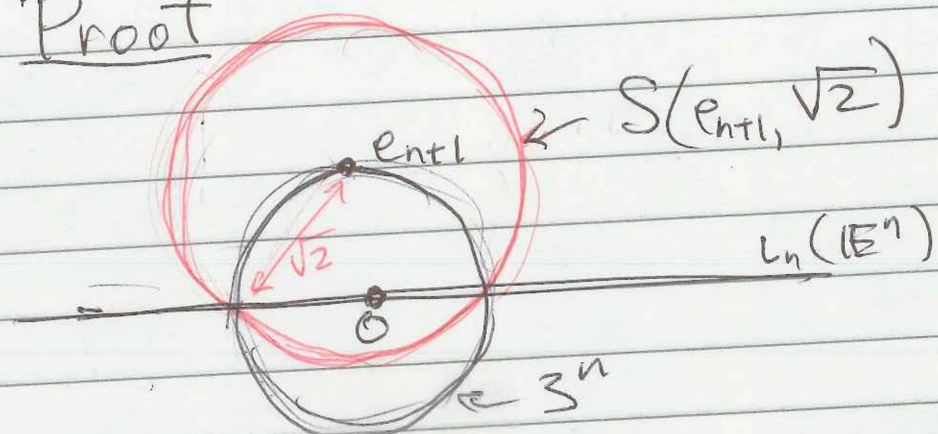
$$\tau(y) = \frac{1}{1-y_{n+1}} \pi_n(y).$$

Then $\sigma: \mathbb{E}^n \to S^n - \{e_{n+1}\}$ and $\tau: S^n - \{e_{n+1}\} \to \mathbb{E}^n$ are differentiable and $\tau \circ \sigma = id_{\mathbb{E}^n}$ and $\sigma \circ \tau = id_{S^n - \{e_{n+1}\}}$. Thus $\sigma: \mathbb{E}^n \to S^n - \{e_{n+1}\}$ is a diffeomorphism

Exercise Verify that $\tau \circ \sigma = id_{\mathbb{E}^n}$ and $\sigma \circ \tau = id_{S^n - \{e_{n+1}\}}$.

Lemma 2.42. Stereographic projection $\sigma : \mathbb{E}^n \to S^n - \{e_{n+1}\}$ is conformal.

Proof



We assert that $\sigma = I_{e_{n+1}, \sqrt{2}} \circ i_n$.

Exercise: Verify this assertion

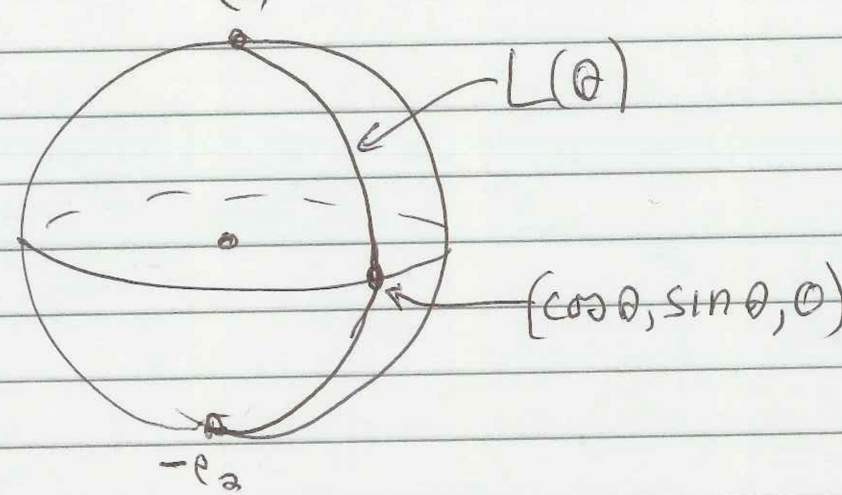
Since $i_n : \mathbb{E}^n \to \mathbb{E}^{n+1}$ is distance preserving, it is conformal by Lemma 2.36.c. $I_{e_{n+1}, \sqrt{2}}$ is conformal by Lemma 2.37. Hence, $\sigma = I_{e_{n+1}, \sqrt{2}} \circ i_n$ is conformal by Lemma 2.36.a. $\square$

The Mercator Projection.

For $\theta \in \mathbb{R}$, let

$$L(\theta) = \{ (\sqrt{1-z^2}\cos\theta, \sqrt{1-z^2}\sin\theta, z) : -1<z<1 \}$$

$L(\theta)$ is a meridian of longitude (minus $\pm e_3$) in S^2



The Mercator Projection is a conformal covering map $M: \mathbb{E}^2 \to S^2 - \{e_3, -e_3\}$ such that $M(\{\theta\} \times \mathbb{R}) = L(\theta)$ for each $\theta \in \mathbb{R}$. More precisely, M has the form

$$M(\theta, y) = (\cos(\theta)\cos(\varphi(y)), \sin(\theta)\cos(\varphi(y)), \sin(\varphi(y)))$$

where $\varphi : \mathbb{R} \to (-\pi/2, \pi/2)$ is an appropriately chosen increasing diffeomorphism. (Since

$\sin|(-\frac{\pi}{2},\frac{\pi}{2}) : (-\frac{\pi}{2},\frac{\pi}{2}) \to (-1,1)$ is a diffeomorphism, then $\sin\circ\varphi : \mathbb{R} \to (-1,1)$ is a diffeomorphism.) The trick is to choose $\varphi : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ so that M is conformal.

Observe that

$$\frac{\partial M}{\partial \theta}(\theta,y) = (-\sin\theta\cos(\varphi(y)), \cos(\theta)\cos(\varphi(y)), 0)$$

and

$$\frac{\partial M}{\partial y}(\theta,y) = (-\cos(\theta)\sin(\varphi(y))\varphi'(y), -\sin\theta\cos(\varphi(y))\varphi'(y), \cos(\varphi(y))\varphi'(y))$$

Hence, $\frac{\partial M}{\partial \theta}(\theta,y) \cdot \frac{\partial M}{\partial y}(\theta,y) = 0.$

For M to be conformal, it suffices that $\|\frac{\partial M}{\partial \theta}(\theta,y)\| = \|\frac{\partial M}{\partial y}(\theta,y)\|$. Indeed, assume $\|\frac{\partial M}{\partial \theta}(\theta,y)\| = \|\frac{\partial M}{\partial y}(\theta,y)\| = \mu(\theta,y)$ for all $(\theta,y) \in \mathbb{E}^2$. Then for $x = (a,b) \in \mathbb{E}^2$,

$$dM_{(\theta,y)}(x) = M'(\theta,y)\begin{pmatrix} a \\ b \end{pmatrix} = \left(\frac{\partial M}{\partial \theta}(\theta,y) \; \frac{\partial M}{\partial y}(\theta,y)\right)\begin{pmatrix} a \\ b \end{pmatrix}$$

$$= a\frac{\partial M}{\partial \theta}(\theta,y) + b\frac{\partial M}{\partial y}(\theta,y),$$ Hence,

$$\|dM_{(\theta,y)}(x)\|^2 = a^2\left\|\frac{\partial M}{\partial \theta}(\theta,y)\right\|^2 + b^2\left\|\frac{\partial M}{\partial y}(\theta,y)\right\|^2$$

$$= (a^2+b^2)(\mu(\theta,y))^2 = \cancel{\|x\|}(\mu(\theta,y))^2\|x\|^2.$$

So $\|dM_{(\theta,y)}(x)\| = \mu(\theta,y)\|x\|$.

Thus, $dM_{(\theta,y)}$ is a similarity.
Consequenly M is conformal.

$\left\|\frac{\partial M}{\partial \theta}(\theta,y)\right\| = \cos(\varphi(y))$ and

$\left\|\frac{\partial M}{\partial y}(\theta,y)\right\| = \varphi'(y)$.

Thus, $\varphi : \mathbb{R} \to (-\frac{\pi}{2}, \pi/2)$ must be a solution of the differential equation

$$\varphi'(y) = \cos(\varphi(y)).$$

We now solve for φ

$$\frac{d\varphi}{dy} = \cos(\varphi)$$

$$\therefore dy = \frac{d\varphi}{\cos(\varphi)} = \sec(\varphi)\, d\varphi$$

$$\therefore y = \int \sec(\varphi)\, d\varphi$$

(The usual representation of $\int \sec(\varphi)\, d\varphi$ gives $y = \ln|\sec\varphi + \tan\varphi| + C$.
It is not clear how to solve this equation for φ as a function of y. So we use a different representation of $\int \sec(\varphi)\, d\varphi$.)

$$y = \ln\left|\tan\left(\frac{\varphi}{2} + \frac{\pi}{4}\right)\right| + C$$

(Verify that $\frac{dy}{d\varphi} = \sec(\varphi)$.)

We might as well require $\varphi(0) = 0$. Since $\ln\left|\tan\left(\frac{0}{2} + \frac{\pi}{4}\right)\right| = \ln 1 = 0$, we choose $C = 0$. Solving for φ in terms of y yields

$$e^y = \left|\tan\left(\frac{\varphi}{2} + \frac{\pi}{4}\right)\right|$$

Since $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \left(\frac{\varphi}{2} + \frac{\pi}{4}\right) \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow \tan\left(\frac{\varphi}{2} + \frac{\pi}{4}\right) > 0$, then

$$e^{y} = \tan\left(\frac{\varphi}{2} + \frac{\pi}{4}\right)$$

$\tan \big| \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ is a diffeomorphism.

Let $\tan^{-1}$ denote the inverse of $\tan \big| \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

Thus $\tan^{-1}(e^{y}) = \frac{\varphi}{2} + \frac{\pi}{4}$.

So $\varphi(y) = 2\tan^{-1}(e^{y}) - \frac{\pi}{2}$.

~~With~~ (Verify that $\varphi'(y) = \cos(\varphi(y))$,)

With this choice of $\varphi : \mathbb{R} \to (-\pi/2, \pi/2)$, the Mercator projection $M : E^2 \to S^2 - \{e_3, -e_3\}$ is a conformal covering map. Thus,

$\left(M \big| [0, 2\pi] \times \mathbb{R}\right)^{-1}$ can be used to create a conformal map of S^2 in $[0, 2\pi] \times \mathbb{R}$.

The Mercator projection of the Earth was used by marine navigators from its ~~introdu~~ introduction in 1569 to the early 20^{th} century.

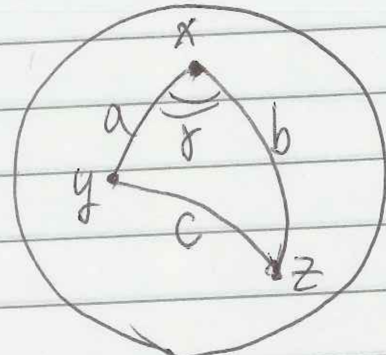
More Spherical Trigonometry

Recall:

The Spherical Law of Cosines

Let $x, y, z \in S^n$ such that $y, z \neq -x$. Let $a = \theta(x,y)$, $b = \theta(x,z)$, $c = \theta(y,z)$ and $\gamma = m(\angle yxz)$. Then

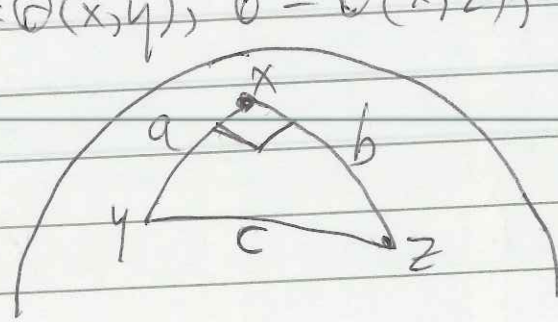
$$\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(\gamma)$$



If $\gamma = \pi/2$, then the Spherical Law of Cosines implies:

The Spherical Pythagorean Theorem 2.43

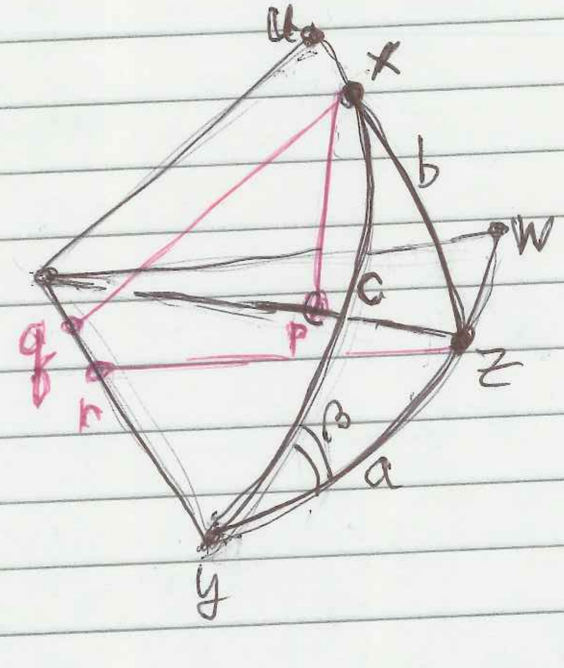
If $x, y, z \in S^n$ such that $y, z \neq -x$ and $m(\angle yxz) = \pi/2$, then $\cos(c) = \cos(a)\cos(b)$ where $a = \theta(x,y)$, $b = \theta(x,z)$, $c = \theta(y,z)$.



Our next goal is to prove the Spherical Law of Cosines. First we prove this result for right triangles.

The Spherical Law of Sines for Right Triangles 2.44. Let $x, y, z \in S^n$ so that $\pm x, \pm y, \pm z$ are distinct points and $m(\angle xzy) = \pi/2$. Let $b = \Theta(x,z)$, $c = \Theta(x,y)$ and $\beta = m(\angle xyz)$. Then

$$\sin(\beta) = \frac{\sin(b)}{\sin(c)}.$$



Proof Let $a = \Theta(y,z)$. Then the Spherical Pythagorean Theorem implies $\cos(a)\cos(b) = \cos(c)$. Since $\cos(a) = y \cdot z$, $\cos(b) = x \cdot z$ and $\cos(c) = x \cdot y$, then we have $x \cdot y = (x \cdot z)(y \cdot z)$.

Let $u = \dfrac{x-(x\cdot y)y}{\|x-(x\cdot y)y\|}$ and $w = \dfrac{z-(z\cdot y)y}{\|z-(z\cdot y)y\|}$.

Then $\beta \equiv \Theta(u,w)$. So $\cos(\beta) = u\cdot w$.

Let $p = (x\cdot z)z$, $q = (x\cdot y)y$ and $r = (z\cdot y)y$.

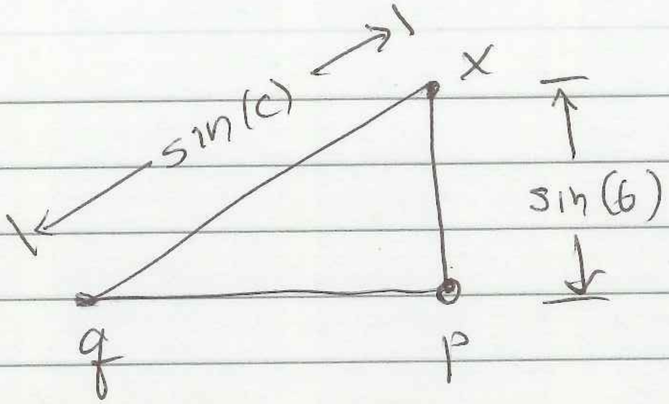
Then $u = \dfrac{x-q}{\|x-q\|}$ and $w = \dfrac{z-r}{\|z-r\|}$

We focus on the planar triangle Δxpq. First we observe that thes triangle has a right angle at p.

$(x-p)\cdot(p-q) = (x-(x\cdot z)z)\cdot((x\cdot z)z-(x\cdot y)y) =$
$(x\cdot z)^2 - (x\cdot y)^2 - (x\cdot z)^2\|z\|^2 + (x\cdot z)(x\cdot y)(y\cdot z) =$
$(x\cdot y)(x\cdot z)(y\cdot z) - (x\cdot y)^2 = (x\cdot y)^2 - (x\cdot y)^2 = 0$
by the Spherical Pythagorean Theorem.

$\|x-p\| = \|x-(x\cdot z)z\| = \sqrt{\|x\|^2 - 2(x\cdot z)^2 + (x\cdot z)^2\|z\|^2} =$
$\sqrt{1-(x\cdot z)^2} = \sqrt{1-\cos^2(b)} = \sin(b)$.

$\|x-q\| = \|x-(x\cdot y)y\| = \sqrt{\|x\|^2 - 2(x\cdot y)^2 + (x\cdot y)^2\|y\|^2} =$
$\sqrt{1-(x\cdot y)^2} = \sqrt{1-\cos^2(c)} = \sin(c)$.



Thus, $\sin(m(\angle xqp)) = \dfrac{\sin(b)}{\sin(c)}$

Finally ~~we~~ argue that $m(\angle xqp)$ is either β or $\pi-\beta$.

$$p-q = (x\cdot z)z - (x\cdot y)y = (x\cdot z)z - (x\cdot z)(y\cdot z)y$$

(by the Spherical Pythagorean Theorem)

$$= (x\cdot z)(z-(y\cdot z)y) = (x\cdot z)(z-r).$$

Thus $\dfrac{p-q}{\|p-q\|} = \dfrac{x\cdot z}{|x\cdot z|}\,\dfrac{z-r}{\|z-r\|} = \pm w.$

Therefore, $\dfrac{p-q}{\|p-q\|}\cdot\dfrac{x-q}{\|x-q\|} = \pm w\cdot u = \pm\cos(\beta)$

So $\cos(m(\angle xqp)) = \pm\cos(\beta) = \cos(\beta)$ or $\cos(\pi-\beta)$.

Hence, $m(\angle xqp) = \beta$ or $\pi-\beta$.

Since $\sin(\beta) = \sin(\pi - \beta)$, then $\sin(m(\angle xqp)) = \sin\beta$. We conclude that

$$\sin(\beta) = \frac{\sin(b)}{\sin(c)} . \square$$

<u>Homework Problem 2.14</u> There is a derivation of the formula

$$\sin(\beta) = \frac{\sin(b)}{\sin(c)}$$

from the formulas

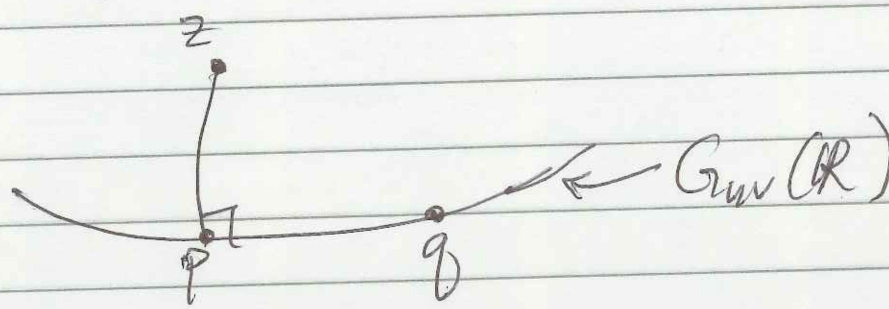
$$\cos(b) = \cos(a)\cos(c) + \sin(a)\sin(c)\cos(\beta)$$

$$\cos(c) = \cos(a)\cos(b)$$

using only trigonometric identities and algebra. Find such a derivation.

Before we can prove the general form of the Spherical Law of Sines, we must establish that we can "drop a perpendicular" from a point of S^n to a great circle not containing the point.

Lemma 2.45 Let $u, v \in S^n$ such that $u \cdot v = 0$ and let $z \in S^n - G_{uv}(\mathbb{R})$. Then there is a point $p \in G_{uv}(\mathbb{R})$ such that $m(\angle zpq) = \pi/2$ for every $q \in G_{uv}(\mathbb{R})$ such that $q \neq \pm p$.



Outline of proof.

Case 1: $z \cdot u = 0 = z \cdot v$. In this case, we assert:

a) $m(\angle zpq) = \pi/2$ for all $p, q \in G_{uv}(\mathbb{R})$, $q \neq \pm p$.

Case 2: $(z \cdot u, z \cdot v) \neq (0,0)$. Let

$$p = \frac{(z \cdot u)u + (z \cdot v)v}{\sqrt{(z \cdot u)^2 + (z \cdot v)^2}}$$

In this case, we assert:

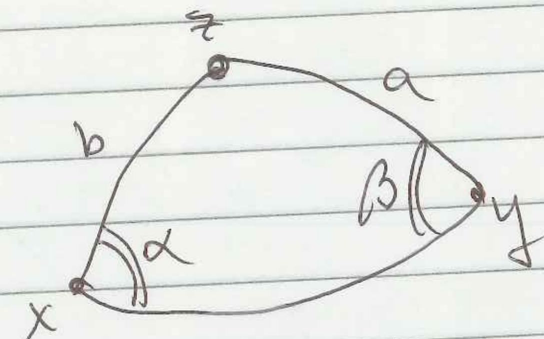
b) $p \in G_{uv}(\mathbb{R})$ and $m(\angle zpq) = \pi/2$ for all $q \in G_{uv}(\mathbb{R})$, $q \neq \pm p$

Homework Problem 2.15 Prove the preceding two assertions.

The Spherical Law of Sines 2.46

Let $x, y, z \in S^n$ such that $\pm x, \pm y$ and $\pm z$ are distinct points. Let $a = \theta(y,z)$, $b = \theta(x,z)$, $\alpha = m(\angle zxy)$ and $\beta = m(\angle zyx)$.

Then $$\frac{\sin(\alpha)}{\sin(a)} = \frac{\sin(\beta)}{\sin(b)}$$



Proof Let $u, v \in S^n$ such that $u \cdot v = 0$ and $x, y \in G_{uv}(\mathbb{R})$. If $z \in G_{uv}(\mathbb{R})$, then $\alpha, \beta \in \{0, \pi\}$. So $\sin(\alpha) = \sin(\beta) = 0$, and the theorem is true. Assume $z \notin G_{uv}(\mathbb{R})$.

Drop a perpendicular from z to a point $p \in G_{uv}(\mathbb{R})$. Then $m(\angle zpq)$ for any $q \in G_{uv}(\mathbb{R})$ such that $q \neq \pm p$.

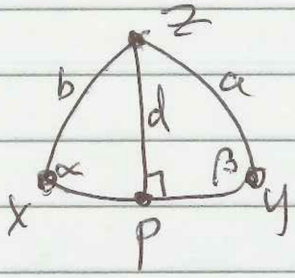
If $p = x$, then $\alpha = \pi/2$ and we can apply Theorem 2.44. If $p = -x$, then $\alpha = m(\angle z(-p)y) = m(\angle zpy) = \pi/2$, and we can again apply Theorem 2.44.

<u>Exercise</u> Verify that $m(\angle z(-p)y) = m(\angle zpy)$.

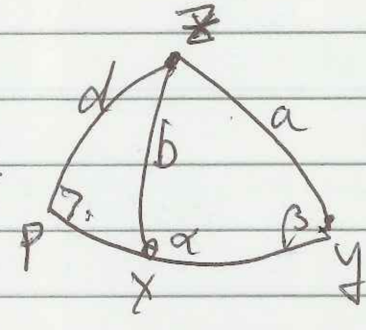
Thus, we can assume $p \neq \pm x$. Similarly, we can assume $p \neq \pm y$.

Let $d = \theta(z, p)$. We must now consider four cases pictured here.

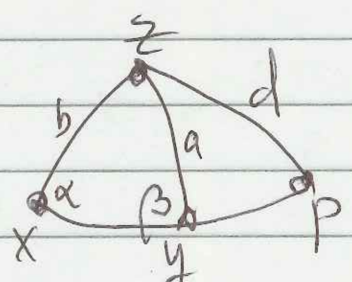
<u>Case 1:</u>



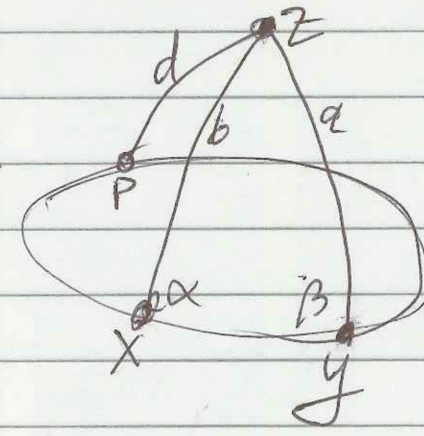
<u>Case 2</u>



<u>Case 3</u>



<u>Case 4</u>



<u>Case 1: $m(\angle zxp) = \alpha$ and $m(\angle zyp) = \beta$</u>

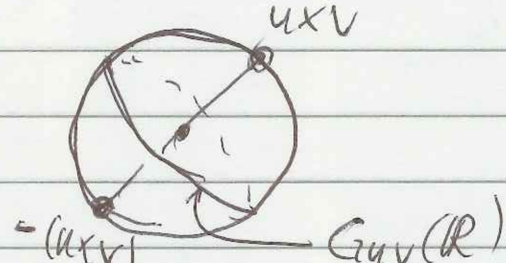
Then Theorem 2.44 implies $\sin(d) = \sin(\alpha)\sin(b)$ and $\sin(d) = \sin(\beta)\sin(a)$. Equating and dividing by $\sin(a)\sin(b)$ yields the result.

Case 2 $m(\angle zxp) = \pi - \alpha$ and $m(\angle zyp) = \beta$

Then Theorem 2.44 implies $\sin(d) = \sin(\pi - \alpha)\sin(b)$ $= \sin(\alpha)\sin(b)$ and $\sin(d) = \sin(\beta)\sin(a)$. Again equate and divide by $\sin(a)\sin(b)$.

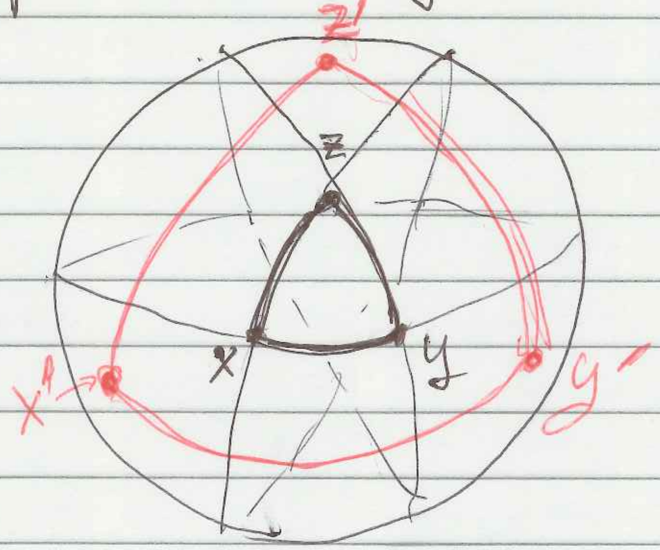
Cases 3 and 4 are similar. $\square$

Def If $G_{uv} : \mathbb{R} \to S^2$ is a great circle in S^2, then the two points $u \times v$ and $-(u \times v)$ are called the poles of G_{uv}. If V is the 2-dimensional vector subspace of $\mathbb{E}^3$ such that $V \cap S^2 = G_{uv}(\mathbb{R})$ and L is the 1-dimensional vector subspace of $\mathbb{E}^3$ that is orthogonal to V, then $L \cap S^2 = \{u \times v, -(u \times v)\}$.



Def Let x, y, z be non-collinear points in S^2. Let $V_{x,y}$, $V_{y,z}$, V_{zx} be the 2-dimensional vector subspaces of $\mathbb{E}^3$ that are spanned by $\{x, y\}$, $\{y, z\}$ and $\{z, x\}$, respectively. Let $C_{xy} = V_{xy} \cap S^2$, $C_{yz} = V_{yz} \cap S^2$ and $C_{zx} = V_{zx} \cap S^2$; these are (images of) great circles in S^2. The poles of C_{xy} are $\pm \frac{x \times y}{\|x \times y\|}$, the poles of C_{yz} are $\pm \frac{y \times z}{\|y \times z\|}$, and the poles of C_{zx} are $\pm \frac{z \times x}{\|z \times x\|}$. Let z' be the pole of C_{xy} that lies on the same side of V_{xy} as z. Thus, $(x \times y) \cdot z$ and $(x \times y) \cdot z'$ have the same sign.

Let x' be the pole of C_{yz} that lies on the same side of V_{yz} as x. Thus $(y \times z) \cdot x$ and $(y \times z) \cdot x'$ have the same sign. Let y' be the pole of C_{zx} that lies on the same side of V_{zx} as y. Thus, $(z \times x) \cdot y$ and $(z \times x) \cdot y'$ have the same sign. The spherical triangle $\Delta x'y'z'$ is called the <u>polar triangle</u> of the spherical triangle Δxyz.



In the preceding definition, $z' = \pm \frac{x \times y}{\|x \times y\|}$.

Therefore, if $(x \times y) \cdot z > 0$, then we must choose $z' = \frac{x \times y}{\|x \times y\|}$ to insure that $(x \times y) \cdot z' > 0$.

Similarly: if $(y \times z) \cdot x > 0$, then $x' = \frac{y \times z}{\|y \times z\|}$, and if $(z \times x) \cdot y > 0$, then $y' = \frac{z \times x}{\|z \times x\|}$.

Observe, that $(x \times y) \cdot z = (y \times z) \cdot x = (z \times x) \cdot y$ because these three numbers are equal to the determinants

$\det(x\, y\, z)$, $\det(y\, z\, x)$ and $\det(z\, x\, y)$

which are equal. Also observe that since $(y \times x) \cdot z = -(x \times y) \cdot z$, then by reordering x, y, z if necessary, we can guarantee that $(x \times y) \cdot z$, $(y \times z) \cdot x$ and $(z \times x) \cdot y$ are positive, in which case

$$x' = \frac{y \times z}{\|y \times z\|}, \quad y' = \frac{z \times x}{\|z \times x\|}, \quad z' = \frac{x \times y}{\|x \times y\|}.$$

Theorem 2.47 Let x, y, z be non-collinear points in S^2. If the spherical triangle $\Delta x'y'z'$ is the polar triangle of the spherical triangle Δxyz, then x', y', z' are non-collinear points in S^2 and Δxyz is the polar triangle of $\Delta x'y'z'$.

To prove Theorem 2.47, we need:

Lemma 2.48. For $x, y, z \in E^3$,

$$(x \times y) \times z = (x \cdot z)y - (y \cdot z)x.$$

<u>Remark</u>. This lemma says that $(x \times y) \times z$ is a linear combination of x and y. This should not surprise us because $(x \times y) \times z$ must be orthogonal to $x \times y$. Hence, it must lie in the 2-dimensional vector subspace of $\mathbb{E}^3$ spanned by x and y.

<u>Proof of Lemma 2.48</u>. To prove the lemma, one can simply calculate the coordinates of $(x \times y) \times z$ and of $(x \cdot z)y - (y \cdot z)x$ and observe that they are equal. We take a different approach.

Define $f, g : \mathbb{E}^3 \times \mathbb{E}^3 \times \mathbb{E}^3 \to \mathbb{E}^3$ by $f(x,y,z) = (x \times y) \times z$ and $g(x,y,z) = (x \cdot z)y - (y \cdot z)x$. Observe that f and g are multilinear functions; in other words ~~f~~ f and g are linear in each of the variables x, y, z. Hence, to prove $f = g$, it suffices to prove f and g agree on all 27 triples e_i, e_j, e_k of standard orthonormal basis elements. To this end, observe that

$$f(e_i, e_j, e_k) = g(e_i, e_j, e_k) = \begin{cases} 0 \text{ if } i \neq j \neq k \neq i \\ 0 \text{ if } i = j \\ e_j \text{ if } i = k \neq j \\ -e_i \text{ if } i \neq j = k \end{cases} . \quad \square$$

Proof of Theorem 2.47. Reorder x, y, z if necessary so that we can assume $(x \times y) \cdot z > 0$. Thus, $x' = \frac{y \times z}{\|y \times z\|}$, $y' = \frac{z \times x}{\|z \times x\|}$ and $z' = \frac{x \times y}{\|x \times y\|}$.

First we prove $(x' \times y') \cdot z' > 0$ ①

$$x' \times y' = \frac{y \times z}{\|y \times z\|} \times \frac{z \times x}{\|z \times x\|} \underset{\uparrow}{=} \frac{(y \cdot (z \times x))z - (z \cdot (z \times x))y}{\|y \times z\| \, \|z \times x\|} =$$

by Lemma 2.48

$$\frac{(y \cdot (z \times x))z}{\|y \times z\| \, \|z \times x\|}.$$ Hence, $(x' \times y') \cdot z' =$

$$\frac{(y \cdot (z \times x))z}{\|y \times z\| \, \|z \times x\|} \cdot \frac{x \times y}{\|x \times y\|} = \frac{((z \times x) \cdot y)((x \times y) \cdot z)}{\|y \times z\| \, \|z \times x\| \, \|x \times y\|}.$$

Since $(z \times x) \cdot y > 0$ and $(x \times y) \cdot z > 0$, then $(x' \times y') \cdot z' > 0$. Therefore z' is not orthogonal to $x' \times y'$. Hence, z' does not lie in the 2-dimensional vector subspace spanned by x' and y'. So x', y' and z' are non-collinear in S^2.

Next we prove $\frac{x' \times y'}{\|x' \times y'\|} = z$.

We just showed $x' \times y' = \frac{(z \times x) \cdot y}{\|y \times z\| \, \|z \times x\|} z$.

Since $\|z\|=1$ and $(z\times x)\circ y > 0$, then

$$\|x'\times y'\| = \frac{(z\times x)\circ y}{\|y\times z\|\,\|z\times x\|}.$$ Consequently,

$$\frac{x'\times y'}{\|x'\times y'\|} = z.$$

We can prove $\frac{y'\times z'}{\|y'\times z'\|} = x$ and $\frac{z'\times x'}{\|z'\times x'\|} = y$

in a similar fashion.

<u>Exercise</u> Verify this assertion.

It follows that the spherical triangle Δxyz is the polar triangle of the spherical triangle $\Delta x'y'z'$. □

<u>Theorem 2.49</u> Let x, y, z be non-collinear points on S^2, and let the spherical triangle $\Delta x'y'z'$ be the polar triangle of the spherical triangle Δxyz. Then:

$$\Theta(x',y') = \pi - m(\angle xzy),$$

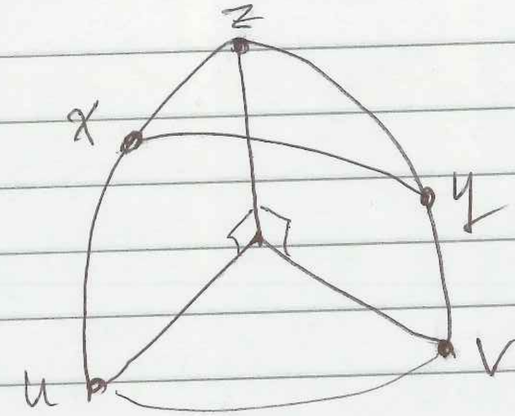
$$\Theta(y',z') = \pi - m(\angle yxz) \text{ and}$$

$$\Theta(z',x') = \pi - m(\angle zyx).$$

Proof Reorder x, y, z if necessary so that $(x \times y) \cdot z > 0$. Let

$$u = \frac{x-(x\cdot z)z}{\|x-(x\cdot z)z\|} \text{ and } v = \frac{y-(y\cdot z)z}{\|y-(y\cdot z)z\|}.$$

Then $u, v \in S^2$, $u \cdot z = 0 = v \cdot z$ and $m(\angle xzy) = \theta(u,v)$.



Note that $x' \cdot z = \frac{y \times z}{\|y\times z\|} \cdot z = 0$ and $y' \cdot z = \frac{z \times x}{\|z \times x\|} \cdot z = 0$. Since u, v, x' and y' are orthogonal to z, they all lie in the 2-dimensional vector subspace V of $\mathbb{E}^3$ orthogonal to z.

We make four observations.

1) $u \cdot y' = \frac{x-(x\cdot z)z}{\|x-(x\cdot z)z\|} \cdot \frac{z\times x}{\|z \times x\|} =$

$$\frac{x\cdot(z\times x) - (x\cdot z)(z\cdot(z\times x))}{\|x-(x\cdot z)z\|\,\|z\times x\|} = 0.$$

2) $v \cdot x' = \dfrac{y-(y\cdot z)z}{\|y-(y\cdot z)z\|} \cdot \dfrac{y \times z}{\|y \times z\|} =$

$$\frac{y\cdot(y\times z) - (y\cdot z)(z\cdot(y\times z))}{\|y-(y\cdot z)\|\ \|y\times z\|} = 0.$$

3) $u \cdot x' = \dfrac{x-(x\cdot z)z}{\|x-(x\cdot z)z\|} \cdot \dfrac{y\times z}{\|y\times z\|} =$

$$\frac{(y\times z)\cdot x - (x\cdot z)(z\cdot(y\times z))}{\|x-(x\cdot z)z\|\ \|y\times z\|} = \frac{(y\times z)\cdot x}{\|x-(x\cdot z)z\|\ \|y\times z\|} > 0$$

because $(y\times z)\cdot x > 0$.

4) $v \cdot y' = \dfrac{y-(y\cdot z)z}{\|y-(y\cdot z)z\|} \cdot \dfrac{z\times x}{\|z\times x\|} =$

$$\frac{(z\times x)\cdot y - (y\cdot z)(z\cdot(z\times x))}{\|y-(y\cdot z)z\|\ \|z\times x\|} = \frac{(z\times x)\cdot y}{\|y-(y\cdot z)z\|\ \|z\times x\|} > 0$$

because $(z\times x)\cdot y > 0$.

To summarize: $u, v, x', y' \in V$, $u\cdot y' = 0 = v\cdot x'$, $u\cdot x' > 0$ and $v\cdot y' > 0$.

Now we break the proof into three cases: $u\cdot v = 0$, $u\cdot v > 0$, $u\cdot v < 0$, and consider each case.

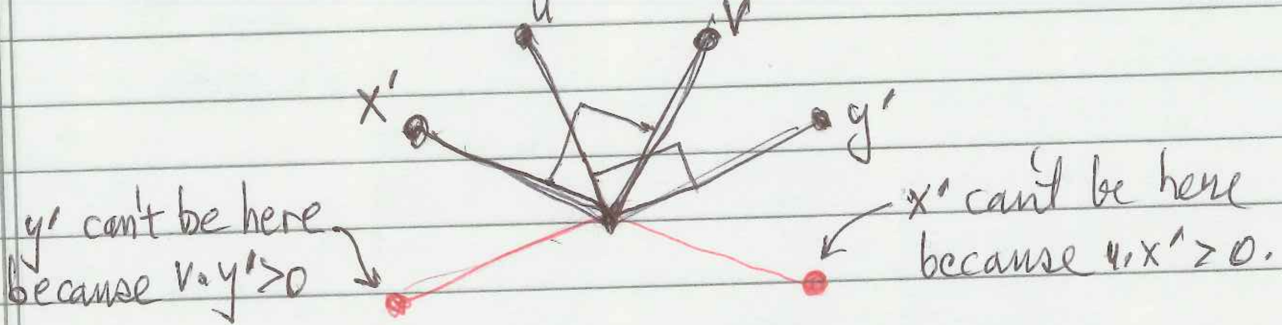
Case 1: $u \cdot v = 0$. Thus $\theta(u,v) = \pi/2$

Since $u \cdot y' = 0 = v \cdot x'$ and V is 2-dimensional, then $y' = \pm v$ and $x' = \pm u$. Thus, $x' \cdot y' = \pm u \cdot v = 0$. So $\theta(x', y') = \pi/2$.

Therefore,

$$\theta(x', y') = \frac{\pi}{2} = \pi - \theta(u,v) = \pi - m(\angle xzy).$$

Case 2: $u \cdot v > 0$.



Hence, $\theta(x', y') = \theta(x'u) + \theta(u,v) + \theta(v, y')$.

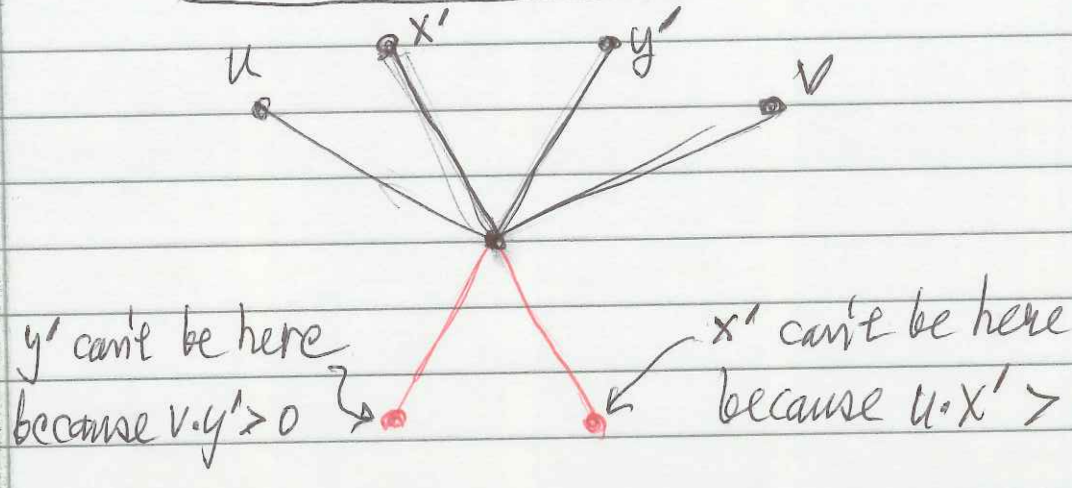
Therefore

$$\theta(x'y') + \theta(uv) = (\theta(x'u) + \theta(uv)) + (\theta(uv) + \theta(vy'))$$

$$= \theta(x'v) + \theta(uy') = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Hence, $\theta(x', y') = \pi - \theta(uv) = \pi - m(\angle xzy)$.

Case 3: $u \cdot v < 0$.



Hence, $\theta(u,v) = \theta(ux') + \theta(x'y') + \theta(y'v)$.
Therefore,
$$\theta(uv) + \theta(x'y') = (\theta(uv) + \theta(x'y')) + (\theta(x'y') + \theta(y'v))$$
$$= \theta(u,y') + \theta(x'v) = \pi/2 + \pi/2 = \pi.$$
Hence $\theta(x'y') = \pi - \theta(u,v) = \pi - m(\angle xzy)$.

The other two equations — $\theta(y'z') = \pi - m(\angle yxz)$ and $\theta(z'x') = \pi - m(\angle zyx)$ — are proved similarly. $\square$

Corollary 2.50. Let x, y, z be non-collinear points in S^2, and let the spherical triangle $\Delta x'y'z'$ be the polar triangle of the spherical triangle Δxyz. Then
$$m(\angle x'z'y') = \pi - \theta(xy),$$
$$m(\angle y'x'z') = \pi - \theta(y,z) \text{ and}$$
$$m(\angle z'y'x') = \pi - \theta(z,x).$$

Proof Theorem 2.47 implies Δxyz is the polar triangle of $\Delta x'y'z'$. Hence, Theorem 2.49 implies

$$\Theta(x,y) = \pi - m(\angle x'z'y')$$

$$\Theta(y,z) = \pi - m(\angle y'x'z')$$

$$\Theta(z,x) = \pi - m(\angle z'y'x').$$

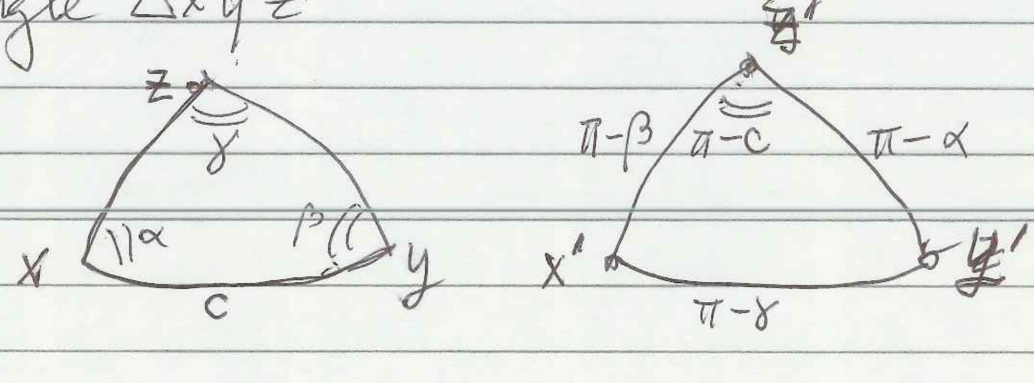
These equations directly imply the corollary. □

The Second Spherical Law of Cosines

2.51. Let x, y, z be non-collinear points in S^n. Let $\alpha = m(\angle yxz)$, $\beta = m(\angle xyz)$, $\gamma = m(\angle xzy)$ and $c = \Theta(x,y)$. Then

$$\cos(c) = \frac{\cos(\alpha)\cos(\beta) + \cos(\gamma)}{\sin(\alpha)\sin(\beta)}$$

Proof Since x, y, z lie in an isometric copy of S^2 in S^n, we can assume $x, y, z \in S^2$. Let $x', y', z' \in S^2$ so that the spherical triangle $\Delta x'y'z'$ is the polar triangle of the spherical triangle Δxyz



Theorem 2.49 and Corollary 2.50 imply $\Theta(y'z') = \pi - \alpha$, $\Theta(x'z') = \pi - \beta$, $\Theta(x'y') = \pi - \gamma$ and $m(\angle x'z'y') = \pi - c$.
We apply the Spherical Law of Cosines to the spherical triangle $\triangle x'y'z'$ to obtain:

$$\cos(\pi - \gamma) = \cos(\pi - \alpha)\cos(\pi - \beta) + \sin(\pi - \alpha)\sin(\pi - \beta)\cos(\pi - c).$$

Since $\cos(\pi - \theta) = -\cos(\theta)$ and $\sin(\pi - \theta) = \sin(\theta)$, we have:

$$-\cos(\gamma) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\cos(c)$$

Solving for $\cos(c)$, we get:

$$\cos(c) = \frac{\cos(\alpha)\cos(\beta) + \cos(\gamma)}{\sin(\alpha)\sin(\beta)} \quad \square$$

Theorem 2.52. If S and T are subsets of S^n and $f: S \to T$ is an isometry with respect to metric Θ on S^n, then there is an isometry $g: S^n \to S^n$ such that $g|S = f$.

Proof For $x, y \in S^n$,
$\|x-y\|^2 = \|x\|^2 - 2x \cdot y + \|y\|^2 = 2 - 2\cos(\Theta(x,y))$.
Hence, for all $x, y \in S$, since $\Theta(f(x), f(y)) = \Theta(x,y)$, then it follows that $\|f(x) - f(y)\| = \|x - y\|$.

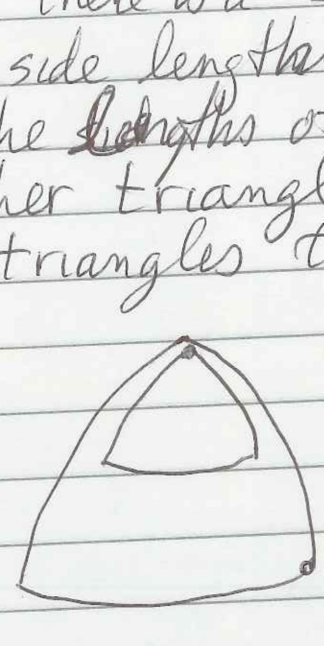
Thus, $f: S \to T$ is an isometry with respect to the Euclidean metric on $\mathbb{E}^{n+1}$.
Define $\bar{f}: S \cup \{0\} \to T \cup \{0\}$ by $\bar{f}|S = f$ and $\bar{f}(0) = 0$. Clearly, $\bar{f}: S \cup \{0\} \to T \cup \{0\}$ is an isometry.

Hence, Theorem 1.51 provides a rigid motion $\bar{g}: \mathbb{E}^{n+1} \to \mathbb{E}^{n+1}$ such that $\bar{g}|S \cup \{0\} = \bar{f}$. Since $\bar{g}(0) = \bar{f}(0) = 0$, then $\bar{g} \in O(\mathbb{E}^{n+1})$. Let $g = \bar{g}|S^n$.
Then Theorem 2.18 implies $g \in \mathcal{I}(S^n)$.
Also $g|S = \bar{g}|S = \bar{f}|S = f$. $\square$

<u>Theorem 2.53.</u> If x_1, x_2, x_3 and y_1, y_2, y_3 are each non-collinear three-point sets in S^n and if $m(\angle x_i x_j x_k) = m(\angle y_i y_j y_k)$ for all $(i, j, k) \in \{(1,2,3), (2,3,1), (3,1,2)\}$, then there is an isometry $g: S^n \to S^n$ such that $g(x_i) = y_i$ for $i = 1, 2, 3$.

<u>Remark</u> If we define triangles in S^n to be <u>similar</u> if they have corresponding angles of equal measure, then Theorem 2.53

says: similar triangles in S^n are congruent. However, if we define triangles in S^n to be <u>similar</u> if there is a "scale factor" $r>0$ such that the side lengths in one triangle are r times the lengths of corresponding sides in the other triangle, then there exist similar triangles that are not congruent.



We outline two different proofs of <u>Theorem 2.53</u>.

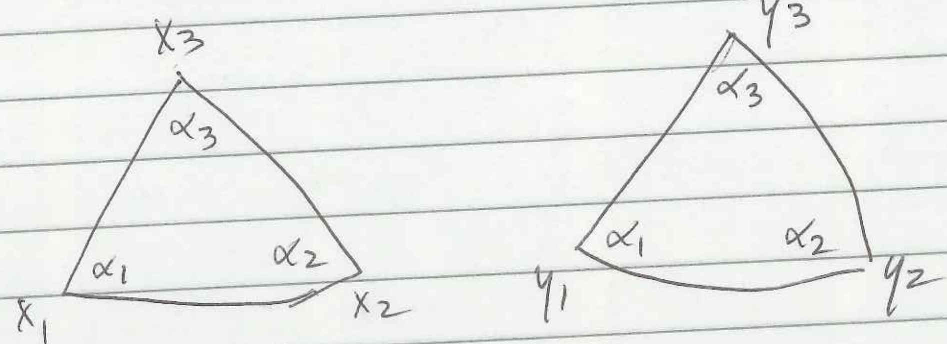
<u>Sketch of first proof</u>. Let V and W be 3-dimensional vector subspaces of $\mathbb{E}^{n+1}$ such that $x_1, x_2, x_3 \in V$ and $y_1\, y_2\, y_3 \in W$. There is an $f \in O(\mathbb{E}^{n+1})$ such that $f(W) = V$. Thus, $x_1, x_2, x_3, f(y_1), f(y_2), f(y_3) \in V$. So we can assume $x_1, x_2, x_3, y_1\, y_2\, y_3 \in V \cap S^n$.

Let $\Sigma^2 = V \cap S^n$. Then Σ^2 is an isometric copy of S^2. Let $x_1', x_2', x_3', y_1', y_2', y_3' \in \Sigma^2$ so that $\Delta x_1' x_2' x_3'$ and

$\Delta y_1' y_2' y_3'$ are the polar triangles of $\Delta x_1 x_2 x_3$ and $\Delta y_1 y_2 y_3$, respectively, in Σ^2. Theorem 2.49 implies $\theta(x_i', x_j') = m(\angle x_i x_k x_j)$ and $\theta(y_i' y_j') = m(\angle y_i y_k y_j)$ for distinct $i, j, k \in \{1,2,3\}$. Since $m(\angle x_i x_k x_j) = m(\angle y_i y_j y_k)$ for distinct i, j, k, then it follows that $\theta(x_i', x_j') = \theta(y_i' y_j')$ for all $i \neq j$. Hence, the function $x_1' \mapsto y_1'$, $x_2' \mapsto y_2'$, $x_3' \mapsto y_3'$ is an isometry with respect to θ. So Theorem 2.52 implies there is an isometry $g: \Sigma^2 \to \Sigma^2$ such that $g(x_i') = y_i'$ for $i = 1,2,3$. Since $\Delta x_1 x_2 x_3$ and $\Delta y_1 y_2 y_3$ are the polar triangles of $\Delta x_1' x_2' x_3'$ and $\Delta y_1' y_2' y_3'$ (by Theorem 2.47), then g must map $\Delta x_1 x_2 x_3$ to $\Delta y_1 y_2 y_3$. (The polar triangle of a given triangle is uniquely define in metric terms.) In particular, $g(x_i) = y_i$ for $i = 1,2,3$. g extends to an element of $O(V)$ which extends to an element $\bar{g}$ of $O(E^{n+1})$. Hence, $\bar{g}/S^n : S^n \to S^n$ is an isometry such that $\bar{g}(x_i) = y_i$ for $i = 1,2,3$. $\square$

Sketch of second proof. For distinct $i, j, k \in \{1,2,3\}$, let

$$\alpha_j = m(\angle x_i x_j x_k) = m(\angle y_i y_j y_k)$$



Then for distinct $i, j, k \in \{1,2,3\}$, the Second Spherical Law of Cosines implies

$$\theta(x_i x_k) = \frac{\cos(\alpha_i)\cos(\alpha_k) - \cos(\alpha_j)}{\sin(\alpha_i)\,\sin(\alpha_k)} = \theta(y_i, y_k).$$

Thus, the function $x_1 \mapsto y_1,\ x_2 \mapsto y_2,\ x_3 \to y_3$ is an isometry with respect to θ. Hence, Theorem 2.52 provides an isometry $g: S^n \to S^n$ such that $g(x_i) = y_i$ for $i = 1, 2, 3$. $\square$

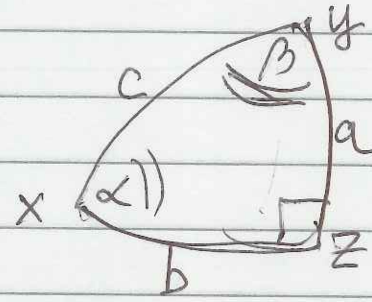
Using the approach of the second proof of Theorem 2.53, we can generalize this theorem to the following.

<u>Theorem 2.54</u> Suppose S and T are subsets of S^n and $f: S \to T$ is a bijection with the following property. For all distinct $x_1, x_2 \in S$, there is an $x_3 \in S$ such that x_1, x_2, x_3 and $f(x_1), f(x_2), f(x_3)$ are non-collinear three-point sets in S^n and $m(\angle x_i x_j x_k) = m(\angle f(x_i) f(x_j) f(x_k))$ for all distinct $i, j, k \in \{1,2,3\}$. Then there is an isometry $g: S^n \to S^n$ such that $g|S = f$.

<u>Exercise</u> Prove Theorem 2.54.

The next theorem states several identities relating the sides and angles of spherical right triangles. These identities are known as Napier's Rules of Circular Parts

Theorem 2.55. Let x, y, z be non-collinear points in S^n such that $m(\angle xzy) = \pi/2$. Let $a = \theta(y,z)$, $b = \theta(x,z)$, $c = \theta(x,y)$, $\alpha = m(\angle yxz)$ and $\beta = m(\angle xyz)$.



Then:

a) $\cos(a)\cos(b) = \cos(c)$

b) $\sin(\beta) = \dfrac{\sin(b)}{\sin(c)}$ and $\sin(\alpha) = \dfrac{\sin(a)}{\sin(c)}$

c) ~~$\cos(\beta) = \dfrac{\tan(a)}{\tan(c)}$ and $\cos(\alpha) = \dfrac{\tan(b)}{\tan(c)}$~~

d) ~~$\tan(\beta) = \dfrac{\tan(b)}{\sin(a)}$ and $\tan(\alpha) = \dfrac{\tan(a)}{\sin(b)}$~~

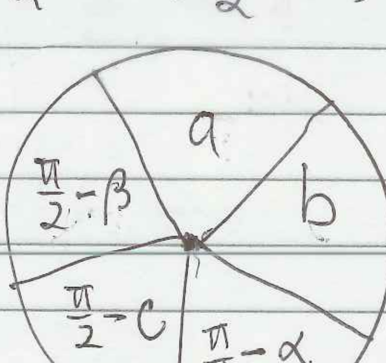
e) $\cot(\alpha)\cot(\beta) = \cos(c)$

f) ~~$\cos(\beta) = \sin(\alpha)\cos(b)$ and $\cos(\alpha) = \sin(\beta)\cos(a)$~~.

We have already proved statements a) and b) of this theorem, a) is the Spherical Pythagorean Theorem and b) is the Spherical Law of Sines for a Right Triangle.

<u>Homework Problem 2.17.</u> Prove parts c) through f) of Theorem 2.55. All one needs to know for these proofs is the Spherical Law of Cosines and parts a) and b) of this theorem.

The identities in Theorem 2.55 first appeared in a paper by Napier (the inventor of logarithms) published in 1614. In the paper, Napier provided a mnemonic device for remembering the identities. He divided a circle into five sectors labeled $a, b, \frac{\pi}{2}-\alpha, \frac{\pi}{2}-c, \frac{\pi}{2}-\beta$ as shown.



Choose one of the sectors and call it the <u>middle part</u>. Then form two equations:

- the sine of the middle part = the product of the tangents of the adjacent parts.
- the sine of the middle part = the product of the cosines of the opposite parts.

Napier called this procedure the <u>Rules of Circular Parts</u>. It generates 10 equations which are equivalent to the 10 identities stated in Theorem 2. For instance, if we choose the sector labeled "$\pi/2 - c$" to be the middle part, then the equation "sin of middle part = product of cosines of opposite parts" becomes $\sin(\pi/2 - c) = \cos(a)\cos(b)$. Since $\sin(\pi/2 - c) = \cos(c)$, we have the identity appearing in a): $\cos(c) = \cos(a)\cos(b)$.

<u>Exercise</u> Verify that the 10 equations produced by Napier's Rules of Circular Parts correspond to the 10 identities in Theorem 2.55.

Theorem 2.56 (A. Girard, T. Harriot)
If T is a spherical triangle in S^2 with angle measures $\alpha_1, \alpha_2, \alpha_3$ and area A, then

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi + A.$$

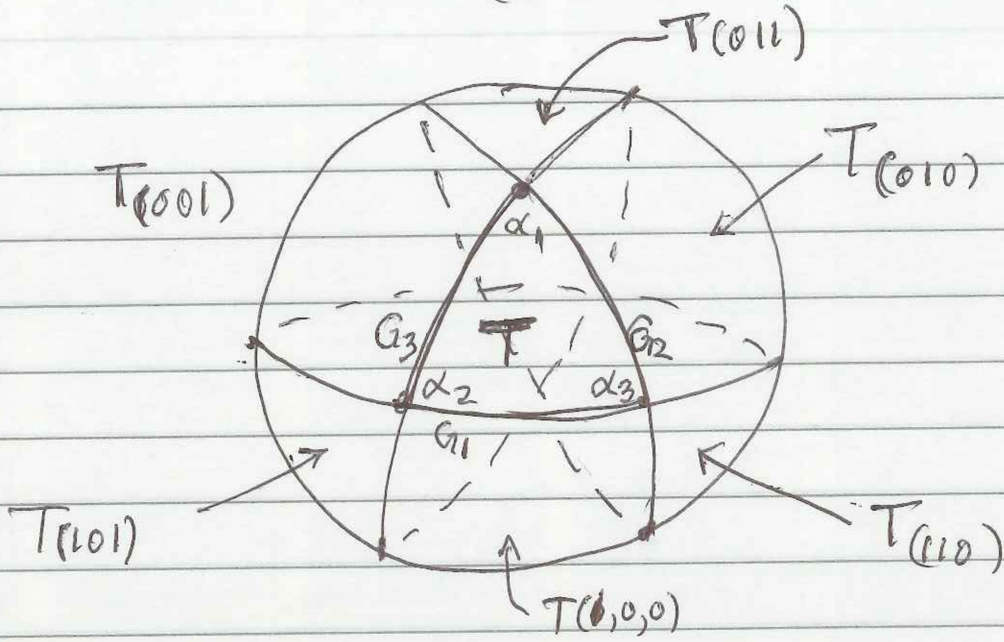
Proof Extend the sides of T to great circles G_1, G_2, G_3 labeled so that G_i contains the side of T that is opposite the angle of measure α_i. G_i divides S^2 into two closed hemispheres H_{i0} and H_{i1} labeled so that $T \subset H_{i0}$. Thus, $T = H_{10} \cap H_{20} \cap H_{30}$. For any ordered triple $(i,j,k) \in \{0,1\}^3$, $T(ijk) = H_{1i} \cap H_{2j} \cap H_{3k}$ is a spherical triangle with the following properties.

- $T(000) = T$
- $\bigcup_{(i,j,k) \in \{0,1\}^3} T(i,j,k) = S^2$
- $(\mathrm{int}(T(i,j,k))) \cap T(lmn) = \emptyset$ for $(i,j,k) \neq (l,m,n)$.

Let $A(i,j,k) = \mathrm{Area}(T(i,j,k))$ for $(i,j,k) \in \{0,1\}^3$. Then $A(0,0,0) = A$ and

$$\sum_{(i,j,k) \in \{0,1\}^3} A(i,j,k) = \mathrm{Area}(S^2) = 4\pi.$$

Let $f: S^2 \to S^2$ denote the antipodal map. Clearly, $f(H_{ij}) = H_{i,1-j}$. Therefore,



$f(T(i,j,k)) = T(1-i, 1-j, 1-k)$. Since f is an isometry, it preserves area. Thus $A(i,j,k) = A(1-i,1-j,1-k)$. $\{0,1\}^3$ is the disjoint union of the two sets $J = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1)\}$ and $K = \{(1,1,1), (0,1,1), (1,0,1), (1,1,0)\}$, and $(i,j,k) \mapsto (1-i, 1-j, 1-k)$ is a bijection between J and K. Hence,

$$\sum_{(i,j,k)\in\{0,1\}^3} A(i,j,k) = 2\sum_{(i,j,k)\in J} A(i,j,k)$$

Therefore, $\sum_{(i,j,k)\in J} A(i,j,k) = 2\pi$.

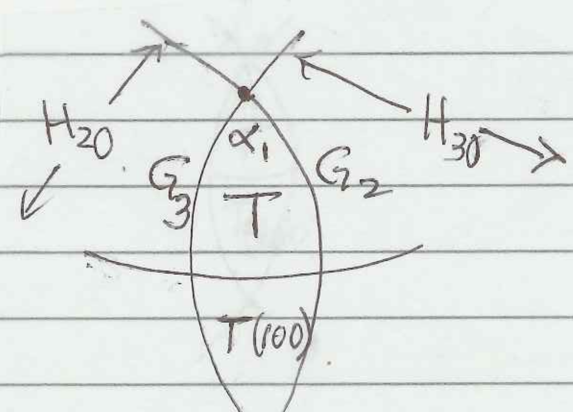
Let $J' = J - \{(0,0,0)\} = \{(1,0,0), (0,1,0), (0,0,1)\}$. Since $A(0,0,0) = A$, then

$$A + \sum_{(i,j,k)\in J'} A(i,j,k) = 2\pi.$$

Observe that

$$T \cup T(1,0,0) = T(0,0,0) \cup T(1,0,0) = H_{20} \cap H_{30}.$$

This set is a "lune" whose boundary is the union of two semi-circular arcs that lie in G_2 and G_3. The measure of the angle between these two arcs is α_1. Under the appropriate Archimedian projection, this lune is the image of a rectangle of width α_1 and height 2. Since Archimedian projections onto S^2 are area preserving, then the area of this lune must be $2\alpha_1$. Hence,
$A + A(1,0,0) = 2\alpha_1$.



By similarly considering the lunes $H_{10} \cap H_{30} = T \cup T(0,1,0)$ and $H_{10} \cap H_{20} = T \cup T(0,0,1)$, we obtain the equations

$A + A(0,1,0) = 2\alpha_2$ and $A + A(0,0,1) = 2\alpha_3$.

Thus,

$$2(\alpha_1 + \alpha_2 + \alpha_3) = 3A + \sum_{(i,j,k) \in J'} A(i,j,k)$$

Hence,

$$2(\alpha_1 + \alpha_2 + \alpha_3) = 2A + 2\pi.$$

Therefore,

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi + A. \ \square$$