Completions

Definition. Let (X, ρ) and (Y, σ) be metric spaces. A function f : X → Y is *distance preserving* or *isometric* if $\sigma(f(x),f(x'))=\rho(x,x')$ for all x, x′ ∈ X.

Observe that if (X, ρ) and (Y, σ) are metric spaces and f : X → Y is a distance preserving function, then f is injective and $f^{-1}: f(X) \to X$ is also distance preserving. Also distance preserving functions are continuous. Hence, distance preserving functions are embeddings and distance preserving onto functions are homeomorphisms. For this reason, we introduce new terminology for distance preserving functions.

Definition. If (X, ρ) and (Y, σ) are metric spaces and f : X → Y is a distance preserving function, then we say that f : (X, ρ) → (Y, σ) is an *isometric embedding.* If f : X → Y is a distance preserving and onto, then we say that f : (X, ρ) → (Y, σ) is an *isometry*.

Recall that if X is a topological space, then $C(X,\mathbb{R})$ denotes the set of all bounded continuous functions from X to $\mathbb{R}$. If σ denotes the supremum metric on $C(X,\mathbb{R})$ ($\sigma(f,g) = \sup\{\, | f(x) - g(x) | : x \in X \,\}$ for f, g ∈ $C(X,\mathbb{R})$), then according to Theorem VI.13, σ is a complete metric on $C(X,\mathbb{R})$ because the standard metric on $\mathbb{R}$ is complete.

Theorem VI.19. The Isometric Embedding Theorem. If (X, ρ) is a metric space, then there is an isometric embedding of (X, ρ) in ($C(X,\mathbb{R})$, σ).

Proof. We begin by noting a useful variant of the triangle inequality:

∗) $| \rho(x,z) - \rho(y,z) | \le \rho(x,y)$ for all x, y, z ∈ X.

To prove **∗)**, observe that the triangle inequality implies $\rho(x,z) \le \rho(x,z) + \rho(x,z)$ and $\rho(y,z) \le \rho(x,y) + \rho(x,z)$. Hence, $\rho(x,y)$ is an upper bound of both $\rho(x,z) - \rho(y,z)$ and $\rho(y,z) - \rho(x,z)$. Therefore, $| \rho(x,z) - \rho(y,z) | \le \rho(x,y)$.

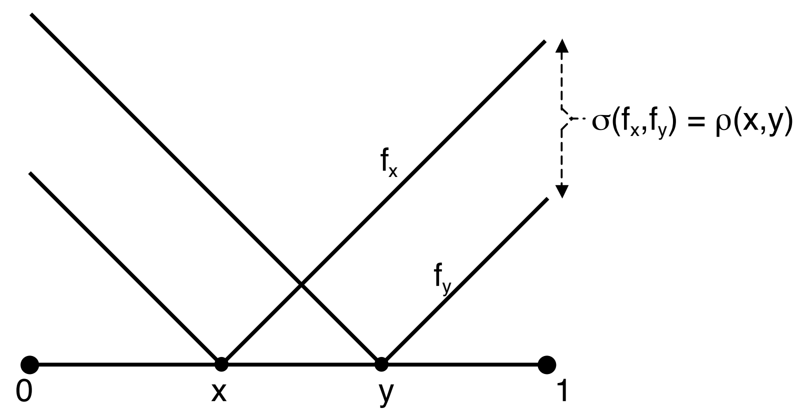
To prove this theorem we must find a bounded map $f_x \in C(X,\mathbb{R})$ for every x ∈ X such that the function $x \mapsto f_x : X \to C(X,\mathbb{R})$ is distance preserving.

As motivation for this proof, we first consider the special case in which the metric ρ on X is bounded. In this case the bounded map f_x associated to x can be defined by the equation $f_x(z) = \rho(x,z)$ for z ∈ X. (When X = [0, 1], this choice of f_x is illustrated in the figure at the end of this paragraph.) In this situation, we must prove that $\sigma(f_x,f_y) = \rho(x,y)$ for all x, y ∈ X. To this end, let x, y ∈ X. First note that variant **∗)** of the triangle inequality implies $| f_x(z) - f_y(z) | = | \rho(x,z) - \rho(y,z) | \le \rho(x,y)$ for all z ∈ X. Hence,

$\sigma(f_x,f_y) \le \rho(x,y)$. Also

$$\sigma(f_x,f_y) \ \ge\ | f_x(y) - f_y(y) | \ =\ | \rho(x,y) - \rho(y,y) | \ =\ | \rho(x,y) | \ =\ \rho(x,y).$$

This proves $\sigma(f_x,f_y) = \rho(x,y)$ for all $x, y \in X$, and finishes the proof in the case that ρ is a bounded metric on X.



With this motivation, we now give the proof in the general case that the metric ρ on X is not necessarily bounded. In this situation, we first choose a point $x_0 \in X$. Then for each $x \in X$, we define the function $f_x : X \to \mathbb{R}$ by $f_x(z) = \rho(x,z) - \rho(x_0,z)$ for all $z \in X$. To prove that f_x is a bounded function, note that variant ∗**)** of the triangle inequality implies that $| f_x(z) | \ =\ | \rho(x,z) - \rho(x_0,z) | \ \le\ \rho(x_0,x)$ for all $z \in X$. (Observe that the term $\rho(x_0,z)$ in the definition of $f_x(z)$ is a "fudge factor" introduced to insure that f_x is a bounded function even if ρ is an unbounded metric.) To prove that f_x is continuous, note that variant ∗**)** of the triangle inequality implies that

$$| f_x(z) - f_x(z') | \ =\ | (\rho(x,z) - \rho(x_0,z)) - (\rho(x,z') - \rho(x_0,z')) | \ =$$

$$| (\rho(x,z) - \rho(x,z')) + (\rho(x_0,z') - \rho(x_0,z)) | \ \le$$

$$| \rho(x,z) - \rho(x,z') | + | \rho(x_0,z') - \rho(x_0,z) | \ \le\ 2\rho(z,z')$$

for all $z, z' \in X$. We conclude that $f_x \in C(X,\mathbb{R})$ for every $x \in X$.

To complete this proof, we must show that the function $x \mapsto f_x : X \to C(X,\mathbb{R})$ is distance preserving. To this end, let $x, y \in X$. First note that variant ∗**)** of the triangle inequality implies

$$| f_x(z) - f_y(z) | \ =\ | (\rho(x,z) - \rho(x_0,z)) - (\rho(y,z) - \rho(x_0,z)) | \ =\ | \rho(x,z) - \rho(y,z) | \ \le\ \rho(x,y)$$

for all $z \in X$. Hence, $\sigma(f_x,f_y) \le \rho(x,y)$. Also

$$\sigma(f_x,f_y) \ \ge\ | f_x(y) - f_y(y) | \ =\ | (\rho(x,y) - \rho(x_0,y)) - (\rho(y,y) - \rho(x_0,y)) | \ =\ | \rho(x,y) | \ =\ \rho(x,y).$$

This proves $\sigma(f_x,f_y) = \rho(x,y)$ for all x, y $\in$ X.

We have proved that the function $x \mapsto f_x : (X, \rho) \to (C(X,\mathbb{R}), \sigma)$ is an isometric embedding. □

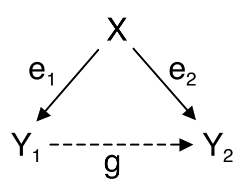
Definition. A *completion* of a metric space (X, ρ) is a pair ((Y, σ), e) satisfying the following three conditions:

- (Y, σ) is a complete metric space,
- $e : (X, \rho) \to (Y, \sigma)$ is an isometric embedding and
- e(X) is a dense subset of Y.

Theorem VI.20. The Existence of Completions. Every metric space has a completion.

Proof. Let (X, ρ) be a metric space. Theorem VI.19 provides an isometric embedding $e : (X, \rho) \to (C(X,\mathbb{R}), \sigma)$ where σ is the supremum metric on $C(X,\mathbb{R})$. Let Y = cl(e(X)) and let σ_Y denote the restriction of the metric σ to Y. Then e(X) is a dense subset of Y and σ_Y is a complete metric on Y by Theorem VI.13 and VI.3. Consequently, ((Y, σ_Y), e) is a completion of (X, ρ). □

Theorem VI.21. The Uniqueness of Completions. The completions of a metric spaces are unique up to isometry. In other words, if ((Y_1, σ_1), e_1) and ((Y_2, σ_2), e_2) are both completions of a metric space (X, ρ), then there is an isometry $g : (Y_1, \sigma_1) \to (Y_2, \sigma_2)$ such that $g \circ e_1 = e_2$.



We will deduce Theorem VI.21 from a more general result which we now state and prove.

Definition. Let (X, ρ) and (Y, σ) be metric spaces. A function $f : X \to Y$ is *uniformly continuous* if for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\rho(x,x') < \delta$ implies $\sigma(f(x),f(x')) < \varepsilon$ for all x, x′ $\in$ X.

Theorem VI.22. If Z is a dense subset of a metric space (X, ρ), (Y, σ) is a complete metric space, and f : Z → Y is a uniformly continuous map, then there is a unique uniformly continuous map g : X → Y which extends f. (Thus, g | Z = f.)

Before proving Theorem VI.22, it is convenient to establish the following two lemmas and a corollary.

Lemma VI.23. Uniformly continuous functions preserve Cauchy sequences. In other words, if (X, ρ) and (Y, σ) are metric spaces, f : X → Y is a uniformly continuous function and { x_n } is a Cauchy sequence in (X, ρ), then { $f(x_n)$ } is a Cauchy sequence in (Y, σ).

Proof. Let { x_n } be a Cauchy sequence in (X, ρ). To prove { $f(x_n)$ } is a Cauchy sequence in (Y, σ), let $\varepsilon > 0$. Since f is uniformly continuous, then there is a $\delta > 0$ such that $\rho(x,x') < \delta$ implies $\sigma(f(x),f(x')) < \varepsilon$ for all x, x′ ∈ X. Since { x_n } is Cauchy in (X, ρ), then there is an $n \in \mathbb{N}$ such that $\rho(x_j,x_k) < \delta$ for all j, k ≥ n. Hence, $\sigma(f(x_j),f(x_k)) < \varepsilon$ for all j, k ≥ n. This proves { $f(x_n)$ } is Cauchy in (Y, σ). ◻

Lemma VI.24. If f : X → Y and g : X → Y are both maps from a topological space X to a Hausdorff space Y, then { x ∈ X : f(x) = g(x) } is a closed subset of X.

Proof. Let E = { x ∈ X : f(x) = g(x) }. We will prove that X – E is an open subset of X. Let x ∈ X – E. Then f(x) ≠ g(x). Since Y is a Hausdorff space, then there are disjoint open subsets U and V of Y such that f(x) ∈ U and g(x) ∈ V. Let W = $f^{-1}(U) \cap g^{-1}(V)$. Since f and g are continuous, then W is a neighborhood of x in X such that f(W) ⊂ U and g(W) ⊂ V. Hence f(W) ∩ g(W) ⊂ U ∩ V = ∅. Therefore, W ⊂ X – E. This proves X – E is an open subset of X. We conclude that E is a closed subset of X. ◻

Corollary VI.25. If f : X → Y and g : X → Y are both maps from a topological space X to a Hausdorff space Y and f | Z = g | Z where Z is a dense subset of X, then f = g.

Proof. Let E = { x ∈ X : f(x) = g(x) }. Then Z ⊂ E. Lemma VI.24 implies cl(Z) ⊂ E. Since Z is a dense subset of X, then cl(Z) = X. Therefore, X = E. Hence, f = g. ◻

Proof of Theorem VI.22. For each x ∈ X, choose a sequence { z_n } in Z which converges to x. ({ z_n } exists because Z is a dense subset of X.) Furthermore, if x ∈ Z, then let { z_n } be the constant sequence at x. Since { z_n } converges, then it is a Cauchy sequence in (X, ρ). Since f is uniformly continuous, then Lemma VI.23 implies that { $f(z_n)$ } is a Cauchy sequence in (Y, σ). Since (Y, σ) is a complete metric space, then it follows that { $f(z_n)$ } converges to a point of Y which we call g(x). This defines the function $g : X \to Y$.

If $x \in Z$, then $\{ z_n \}$ was chosen to be the constant sequence at x. Hence, $\{ f(z_n) \}$ is the constant sequence at f(x). Therefore, $\{ f(z_n) \}$ converges to f(x). Since $\{ f(z_n) \}$ also converges to g(x), then g(x) = f(x). Thus, $g \mid Z = f$.

To prove $g : X \to Y$ is uniformly continuous, let $\varepsilon > 0$. Since $f : Z \to Y$ is uniformly continuous, there is a $\delta > 0$ such that $\rho(z,z') < \delta$ implies $\sigma(f(z),f(z')) < \varepsilon/3$ for all $z, z' \in Z$. Let $x, x' \in X$ such that $\rho(x,x') < \delta/3$. We will prove that $\sigma(g(x),g(x')) < \varepsilon$. To define g(x) and g(x′), we chose sequences $\{ z_n \}$ and $\{ z'_n \}$ in Z that converge to x and x′, respectively. Then g(x) and g(x′) were chosen so that $\{ f(z_n) \}$ converges to g(x), and $\{ f(z'_n) \}$ converges to g(x′). Hence, there is a positive integers $n \in \mathbb{N}$ such that $i \geq n$ implies $\rho(x,z_i) < \delta/3$, $\rho(x',z'_i) < \delta/3$, $\sigma(g(x),f(z_i)) < \varepsilon/3$ and $\sigma(g(x'),f(z'_i)) < \varepsilon/3$. Thus, $i \geq n$ implies $\rho(z_i,z'_i) \leq \rho(z_i,x) + \rho(x,x') + \rho(x',z'_i) < \delta$. Hence, $i \geq n$ implies $\sigma(f(z_i),f(z'_i)) < \varepsilon/3$. Combining these inequalities, we obtain:

$$\sigma(g(x),g(x')) \leq \sigma(g(x),f(z_n)) + \sigma(f(z_n),f(z'_n)) + \sigma(f(z'_n),g(x')) < \varepsilon.$$

This proves g is uniformly continuous.

To prove the uniqueness of $g : X \to Y$, suppose that $h : X \to Y$ is also a map such that $h \mid Z = f$. Then Z is a dense subset of X such that $g \mid Z = f = h \mid Z$. Therefore, Corollary VI.25 implies that g = h. ◻

Finally we prove Theorem VI.21.

Proof of Theorem VI.21. Suppose $((Y_1, \sigma_1), e_1)$ and $((Y_2, \sigma_2), e_2)$ are both completions of a metric space (X, ρ). Since $e_1 : X \to Y_1$ and $e_2 : X \to Y_2$ are distance preserving functions, so are $e_1^{-1} : e_1(X) \to X$ and $e_2^{-1} : e_2(X) \to X$. Therefore, $e_2 \circ e_1^{-1} : e_1(X) \to Y_2$ is a distance preserving function from a dense subset of Y_1 to the complete metric space Y_2, and $e_1 \circ e_2^{-1} : e_2(X) \to Y_1$ is a distance preserving function from a dense subset of Y_2 to the complete metric space Y_1. Observe that distance preserving functions are uniformly continuous. (Indeed, given a distance preserving function and an $\varepsilon > 0$, one can choose $\delta = \varepsilon$ to verify uniform continuity.) Hence, we can apply Theorem VI.22 to $e_2 \circ e_1^{-1} : e_1(X) \to Y_2$ and $e_1 \circ e_2^{-1} : e_2(X) \to Y_1$ to obtain uniformly continuous maps $g : Y_1 \to Y_2$ and $h : Y_2 \to Y_1$ such that g extends $e_2 \circ e_1^{-1}$ and h extends $e_1 \circ e_2^{-1}$. Therefore, $g \circ e_1 = e_2 \circ e_1^{-1} \circ e_1 = e_2$ and $h \circ e_2 = e_1 \circ e_2^{-1} \circ e_2 = e_1$.

It remains to prove that $g : (Y_1, \sigma_1) \to (Y_2, \sigma_2)$ is an isometry.

First we prove that $g : Y_1 \to Y_2$ is onto. Since $g \circ h \circ e_2 = g \circ e_1 = e_2$, then $g \circ h \mid e_2(X) = id_{Y_2} \mid e_2(X)$. Since $e_2(X)$ is a dense subset of Y_2, then Corollary VI.25 implies that

$g \circ h = id_{Y_2}$. It follows that $g : Y_1 \to Y_2$ is onto.

Now we prove that $g : Y_1 \to Y_2$ is distance preserving. Define the function $D : Y_1 \times Y_1 \to \mathbb{R}$ by $D(y,y') = \sigma_1(y,y') - \sigma_2(g(y),g(y'))$ for $(y,y') \in Y_1 \times Y_1$. Since the metrics σ_1 and σ_2 and the functions g are continuous, then D is clearly continuous. Obviously, to prove that g is distance preserving, it suffices to prove that $D(y,y') = 0$ for all $(y,y') \in Y_1 \times Y_1$. We will first show that $D \mid e_1(X) \times e_1(X) = 0$. Let $(y,y') \in e_1(X) \times e_1(X)$. Then there are elements x, x′ of X such that $y = e_1(x)$ and $y' = e_1(x')$. Thus, $g(y) = g \circ e_1(x) = e_2(x)$ and $g(y') = g \circ e_1(x') = e_2(x')$. Therefore, $D(y,y') = \sigma_1(e_1(x),e_1(x')) - \sigma_2(e_2(x),e_2(x'))$. Since $e_1 : (X, \rho) \to (Y_1, \sigma_1)$ and $e_2 : (X, \rho) \to (Y_2, \sigma_2)$ are isometric embeddings, then $\sigma_1(e_1(x),e_1(x')) = \rho(x,x') = \sigma_2(e_2(x),e_2(x'))$. Consequently, $D(y,y') = 0$. Thus, $D \mid e_1(X) \times e_1(X) = 0$. Since $e_1(X)$ is a dense subset of Y_1, then $e_1(X) \times e_1(X)$ is a dense subset of $Y_1 \times Y_1$. (Verify this assertion.) It now follows from Corollary VI.25 that $D(y,y') = 0$ for all $(y,y') \in Y_1 \times Y_1$. We conclude that $g : Y_1 \to Y_2$ is distance preserving.

We have now proved that $g : (Y_1, \sigma_1) \to (Y_2, \sigma_2)$ is an isometry. □

Corollary VI.26. Up to isometry, $\mathbb{R}$ with the standard metric is the unique completion of $\mathbb{Q}$ with the standard metric. □