The Baire Property

Definition. A topological space X has the *Baire property* or is a *Baire space* if the intersection of every countable collection of dense open subsets of X is a dense subset of X.

Theorem VI.14. The Baire Theorem. Every complete metric space has the Baire property.

Proof. Let (X, ρ) be a complete metric space, and let $\{ U_n : n \in \mathbb{N} \}$ be a countable collection of dense open subsets of X. To prove that $\bigcap_{n \in \mathbb{N}} U_n$ is a densce subset of X, let V be a non-empty open subset of X. We must prove that $(\bigcap_{n \in \mathbb{N}} U_n) \cap V \neq \emptyset$.

We will inductively construct a sequence $\{ C_n \}$ of closed subsets of X with non-empty interior such that

- $C_1 \subset U_1 \cap V$ and $\text{diam}(C_1) \leq 2$

and for n > 1,

- $C_n \subset U_n \cap \text{int}(C_{n-1})$ and $\text{diam}(C_n) \leq {}^2/_n$.

To begin, since U_1 is a dense open subset of X, then $U_1 \cap V$ is a non-empty open set. Choose $x_1 \in U_1 \cap V$. Since metric spaces are regular, then is an $\varepsilon_1 > 0$ such that $\varepsilon_1 < 1$ and $\text{cl}(N(x_1,\varepsilon_1) \subset U_1 \cap V$. Let $C_1 = \text{cl}(N(x_1,\varepsilon_1)$. Then $\text{int}(C_1) \neq \emptyset$, $C_1 \subset U_1 \cap V$ and $\text{diam}(C_1) = \text{diam}(N(x_1,\varepsilon_1)) \leq 2\varepsilon_1 \leq 2$.

Next let $n \geq 1$ and assume there is a closed set C_n with non-empty interior. Since U_{n+1} is a dense open subset of X, then $U_{n+1} \cap \text{int}(C_n)$ is a non-empty open set. Choose $x_{n+1} \in U_{n+1} \cap \text{int}(C_n)$. Since metric spaces are regular, then is an $\varepsilon_{n+1} > 0$ such that $\varepsilon_{n+1} < {}^1/_{n+1}$ and $\text{cl}(N(x_{n+1},\varepsilon_{n+1}) \subset U_{n+1} \cap \text{int}(C_n)$. Let $C_{n+1} = \text{cl}(N(x_{n+1},\varepsilon_{n+1})$. Then $\text{int}(C_{n+1}) \neq \emptyset$, $C_{n+1} \subset U_{n+1} \cap \text{int}(C_n)$ and $\text{diam}(C_{n+1}) = \text{diam}(N(x_{n+1},\varepsilon_{n+1})) \leq 2\varepsilon_{n+1} \leq {}^2/_{n+1}$.

This completes the inductive construction of $\{ C_n \}$.

Since each C_n is a non-empty closed subset of X such that $C_1 \supset C_2 \supset C_3 \supset \ldots$ and $\lim_{n \to \infty} \text{diam}(C_n) = 0$, then the Cantor Intersection Theorem VI.5 implies $\bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$. Let $x \in \bigcap_{n \in \mathbb{N}} C_n$. Since $x \in C_1$, then $x \in U_1$ and $x \in V$. For n > 1, since $x \in C_n$, then $x \in U_n$. Thus, $x \in (\bigcap_{n \in \mathbb{N}} U_n) \cap V$. Hence, $(\bigcap_{n \in \mathbb{N}} U_n) \cap V \neq \emptyset$. This proves $\bigcap_{n \in \mathbb{N}} U_n$ is a dense subset of X. □

A similar argument proves:

Theorem VI.15. Every locally compact Hausdorff space has the Baire property.

Problem VI.3. Prove Theorem VI.15.

Problem VI.4. a) Prove that if X is a topological space with the Baire property and $\{ U_n : n \in \mathbb{N} \}$ is a countable collection of dense open subsets of X, then $\bigcap_{n \in \mathbb{N}} U_n$ has the Baire property.

b) Prove that if X is either a complete metric space or a locally compact Hausdorff space, and $\{ U_n : n \in \mathbb{N} \}$ is a countable collection of open subsets of X with non-empty intersection, then $\bigcap_{n \in \mathbb{N}} U_n$ has the Baire property.

Example VI.3. a) Since the standard metric on $\mathbb{R}$ is complete, then Theorem VI.14 implies that $\mathbb{R}$ has the Baire property.

b) Let $\mathbb{Q}$ denote the subspace of $\mathbb{R}$ consisting of all rational numbers. $\mathbb{Q}$ does not have the Baire property. Indeed, $\{ \mathbb{Q} - \{ x \} : x \in \mathbb{Q} \}$ is a countable collection of dense open subsets of $\mathbb{Q}$ such that $\bigcap_{x \in \mathbb{Q}} (\mathbb{Q} - \{ x \}) = \varnothing$.

c) The subspace $\mathbb{R} - \mathbb{Q}$ of $\mathbb{R}$, consisting of all irrational numbers, does have the Baire property. This follows from the result of Problem VI.4 because $\mathbb{R} - \mathbb{Q}$ is the intersection of the countable collection $\{ \mathbb{R} - \{ x \} : x \in \mathbb{Q} \}$ of dense open subsets of $\mathbb{R}$.

According to Theorem VI.14, every complete metric space has the Baire property. However, the converse is false: there is a metric space X with the Baire property such that no metric which induces the given topology on X is complete. We remark that the space $\mathbb{R} - \mathbb{Q}$ of irrational numbers is not such a space because, although $\mathbb{R} - \mathbb{Q}$ has the Baire property and the restriction to $\mathbb{R} - \mathbb{Q}$ of the standard metric on $\mathbb{R}$ is not complete, none the less there is a complete metric on $\mathbb{R} - \mathbb{Q}$ which induces the given topology. (The existence of a complete metric on will be explained insection VI.D.)

Example VI.4. Let X denote the subspace $(\mathbb{Q} \times \{ 0 \}) \cup (\mathbb{R} \times (0, \infty))$ of $\mathbb{R}^2$. We assert that X has the Baire property and no metric on X which induces the given topology is complete. To prove that X has the Baire property, let $\{ U_n : n \in \mathbb{N} \}$ be a countable collection of dense relatively open subsets of X. Since $\mathbb{R} \times (0, \infty)$ is a dense open subset of X, then $U_n \cap (\mathbb{R} \times (0, \infty))$ is a dense open subset of $\mathbb{R} \times (0, \infty)$ for each $n \in \mathbb{N}$. $\mathbb{R} \times (0, \infty)$ has the Baire property because it is homeomorphic to $\mathbb{R}^2$, and $\mathbb{R}^2$ has the Baire property by Theorem VI.14 because the Euclidean metric on $\mathbb{R}^2$ is complete. Thus, $\bigcap_{n\in\mathbb{N}}(U_n \cap (\mathbb{R} \times (0, \infty))) = (\bigcap_{n\in\mathbb{N}} U_n) \cap (\mathbb{R} \times (0, \infty))$ is a dense subset of $\mathbb{R} \times (0, \infty)$. Since $\mathbb{R} \times (0, \infty)$ is a dense subset of X, then $(\bigcap_{n\in\mathbb{N}} U_n) \cap (\mathbb{R} \times (0, \infty))$ is a dense subset of X. Since $(\bigcap_{n\in\mathbb{N}} U_n) \cap (\mathbb{R} \times (0, \infty)) \subset \bigcap_{n\in\mathbb{N}} U_n$, then it follows that $\bigcap_{n\in\mathbb{N}} U_n$ is a dense subset of X. This proves X has the Baire property. Now we argue that no metric on X that induces the given topology is complete. For suppose there is a complete metric ρ on X that induces the given topology. Since $\mathbb{R} \times \{0\}$ is a closed subset of $\mathbb{R}^2$, then $(\mathbb{R} \times \{0\}) \cap X = \mathbb{Q} \times \{0\}$ is a closed subset of X. Hence, ρ restricts to a complete metric on $\mathbb{Q} \times \{0\}$ by Theorem VI.6.a. Therefore, $\mathbb{Q} \times \{0\}$ has the Baire property by Theorem VI.14. However, $\mathbb{Q} \times \{0\}$ is homeomorphic to $\mathbb{Q}$ which does not have the Baire property according to Example VI.3.b. We have reached a contradiction. We conclude that no metric on X that induces the given topology is complete.

Definition. A subset A of a topological space X is *nowhere dense* if int(cl(A)) = $\emptyset$.

Let A be a subset of a topological space X. Observe that the following three statements are equivalent.

- A is a nowhere dense subset of X.
- cl(A) contains no non-empty open subset of X.
- If U is a non-empty open subset of X, then U – cl(A) $\neq \emptyset$.
- X – cl(A) is a dense open subset of X.

Definition. A topological space X is *of the first category* if it is the union of a countable collection of nowhere dense subsets. X is *of the second category* if it is not of the first category.

Theorem VI.16. Every space which has the Baire property is of the second category.

Proof. Assume X is a space of the first category. We will prove that X does not have the Baire property. Since X is of the first category, then $X = \bigcup_{n\in\mathbb{N}} A_n$ where each A_n is a nowhere dense subset of X. Then $X = \bigcup_{n\in\mathbb{N}} \mathrm{cl}(A_n)$; and for each $n \in \mathbb{N}$, $\mathrm{cl}(A_n)$ contains no non-empty open subset of X. For each $n \in \mathbb{N}$, let $U_n = X - \mathrm{cl}(A_n)$. Then each U_n is a dense open subset of X. Observe that

$$\bigcap_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} (X - \mathrm{cl}(A_n)) = X - (\bigcup_{n\in\mathbb{N}} \mathrm{cl}(A_n)) = \emptyset.$$

Thus, $\bigcap_{n \in \mathbb{N}} U_n$ is not a dense subset of X. It follows that X does not have the Baire property. □

Theorems 6.14 and 6.16 imply:

Corollary VI.17. The Baire Category Theorem. Every complete metric space is of the second category. □

The converse of Theorem VI.16 is false: there is a metric space of the second category which does not have the Baire property.

Example VI.5. Let X denote the subspace $(\mathbb{R} \times \{ 0 \}) \cup (\mathbb{Q} \times \{ 1 \})$ of $\mathbb{R}^2$. We assert that X is of the second category but X does not have the Baire property. To prove that X is of the second category, assume X is of the first category. We will derive a contradiction. Since X is of the first category, then $X = \bigcup_{n \in \mathbb{N}} A_n$ where each A_n is a nowhere dense subset of X. For each $n \in \mathbb{N}$, let C_n denote the relative closure of A_n in X. Then $X = \bigcup_{n \in \mathbb{N}} C_n$; and for each $n \in \mathbb{N}$, C_n is a relatively closed subset of X that contains no non-empty relatively open subset of X. Hence, $\mathbb{R} \times \{ 0 \} = \bigcup_{n \in \mathbb{N}} (C_n \cap (\mathbb{R} \times \{ 0 \}))$. Also each $C_n \cap (\mathbb{R} \times \{ 0 \})$ is a closed subset of $\mathbb{R} \times \{ 0 \}$. Since $\mathbb{R} \times \{ 0 \}$ is a relatively open subset of X, then any non-empty relatively open subset of $\mathbb{R} \times \{ 0 \}$ is also non-empty relatively open subset of X. Hence, for each $n \in \mathbb{N}$, C_n contains no non-empty relatively open subset of $\mathbb{R} \times \{ 0 \}$. Thus, for each $n \in \mathbb{N}$, $C_n \cap (\mathbb{R} \times \{ 0 \})$ is a closed subset of $\mathbb{R} \times \{ 0 \}$ that contains no non-empty relatively open subset of $\mathbb{R} \times \{ 0 \}$. Thus, $C_n \cap (\mathbb{R} \times \{ 0 \})$ is a nowhere dense subset of $\mathbb{R} \times \{ 0 \}$. Since $\mathbb{R} \times \{ 0 \} = \bigcup_{n \in \mathbb{N}} (C_n \cap (\mathbb{R} \times \{ 0 \}))$, then it follows that $\mathbb{R} \times \{ 0 \}$ is of the first category. However, since $\mathbb{R} \times \{ 0 \}$ is homeomorphic to $\mathbb{R}$, and $\mathbb{R}$ is of the second category because the standard metric on $\mathbb{R}$ is complete, then $\mathbb{R} \times \{ 0 \}$ is of the second category. We have reached a contradiction. We conclude that X is of the second category. To prove that X does not have the Baire property observe that $\{ X - \{ (x,1) \} : x \in \mathbb{Q} \}$ is a countable collection of dense open subsets of X such that $\bigcap_{x \in \mathbb{Q}} (X - \{ (x,1) \}) = X - (\mathbb{Q} \times \{ 1 \}) = \mathbb{R} \times \{ 0 \}$. Therefore, $\mathbb{Q} \times \{ 1 \}$ is a non-empty relatively open subset of X that is disjoint from $\bigcap_{x \in \mathbb{Q}} (X - \{ (x,1) \})$. Consequently, $\bigcap_{x \in \mathbb{Q}} (X - \{ (x,1) \})$ is not a dense subset of X. It follows that X does not have the Baire property. □

A more precise relationship between spaces with the Baire property and spaces of the second category than Theorem VI.16 is established by the following result.

Theorem VI.18. A topological space X has the Baire property if and only if every non-empty open subset of X is of the second category.

Problem VI.5. Prove Theorem VI.18.

The fact that complete metric spaces have the Baire property and are of the second category has many applications both in topology and in other areas of mathematics. We close this section with two simple applications of these ideas.

Problem VI.6. Prove that $\mathbb{R} - \mathbb{Q}$ can't be expressed as the union of a countable collection of closed subsets of $\mathbb{R}$.

Problem VI.7. Let A be a nowhere dense subset of $\mathbb{R}$. Use the fact that $\mathbb{R}$ is of the second category to prove that there is an $x \in \mathbb{R}$ such that $A + x \subset \mathbb{R} - \mathbb{Q}$, where $A + x = \{ a + x : a \in A \}$.