VI. Complete Metric Spaces

A. Fundamental Properties

Definition. Let (X, ρ) be a metric space. A sequence { x_n } in X is a *Cauchy sequence* if for every $\varepsilon > 0$, there is an $n \in \mathbb{N}$ such that $\rho(x_i,x_j) < \varepsilon$ for all i, j ≥ n. Hence, { x_n } is a Cauchy sequence if and only if

$$\lim_{n \to \infty} \operatorname{diam} (\{ x_i : i \geq n \}) = 0.$$

We can paraphrase the preceding statement by saying "{ x_n } is a Cauchy sequence if and only if the diameters of the tails { x_n } converge to 0".

Lemma VI.1. In a metric space, every converging sequence is Cauchy.

Proof. Let (X, ρ) be a metric space, and let { x_n } be a sequence in X that converges to a point y of X. Let $\varepsilon > 0$. Then there is an $n \in \mathbb{N}$ such that $\rho(x_i,y) < {}^{\varepsilon}/_2$ for all i ≥ n. Hence, for i, j ≥ n, $\rho(x_i,x_j) \leq \rho(x_i,y) + \rho(y,x_j) < 2({}^{\varepsilon}/_2) = \varepsilon$. Consequently, { x_n } is a Cauchy sequence. □

Example VI.1. Consider the subspace (0, ∞) of $\mathbb{R}$, and restrict the standard metric on $\mathbb{R}$ to obtain a metric on (0, ∞). The sequence { ${}^1/_n$ } is Cauchy in (0, ∞) because it converges (to 0) in $\mathbb{R}$. However, { ${}^1/_n$ } doesn't converge in (0, ∞). Thus, a Cauchy sequence need not converge.

Definition. Let (X, ρ) be a metric space. ρ is a *complete metric* on X and (X, ρ) is a *complete metric space* if every Cauchy sequence in X converges to a point of X.

Remark. Completeness indicates that from the perspective of the metric, all sequences which ought to converge (because they are Cauchy) do converge. In other words, completeness implies that from the perspective of the metric there are no points missing from the space.

Example VI.1 continued. Again consider (0, ∞) with the restriction of the standard metric on $\mathbb{R}$. Since { ${}^1/_n$ } is a non-convergent Cauchy sequence in (0, ∞), then (0, ∞) with the restriction of the standard metric is not a complete metric space. Since the standard metric on $\mathbb{R}$ is complete (see below) and since (0, ∞) is homeomorphic to $\mathbb{R}$, then we observe that completeness is not a topological property of a metric space. Instead, completeness is a property of the metric.

Remark. As Example VI.1 reveals, it is possible for the topology on a metrizable space to be induced by two different equivalent metrics one of which is complete and the other of which is not. On every compact metrizable space, the metrics which induce the given topology must be complete. However, on every non-compact metrizable spaces, there is an incomplete metric which induces the given topology. Some metrizable spaces, such as the subspace of $\mathbb{R}$ consisting of all rational numbers, admit no complete metrics. However, every non-compact locally compact metrizable space, such as (0, ∞) admits both complete metrics and incomplete metrics. All of these statements will be justified by results in this chapter.

Recall that according to Theorem III.7 every compact subset of a metric space is closed and bounded. However, the converse of this result is false: closed bounded subsets of a metric space need not be compact.

Example VI.2. Let ρ denote the discrete metric on the set $\mathbb{N}$ of positive integers. ($\rho(x,y) = 0$ if $x = y$, and ($\rho(x,y) = 1$ if $x \neq y$.) Then $\mathbb{N}$ itself is a closed bounded set in $(\mathbb{N},\rho)$, but $\mathbb{N}$ is non-compact. Indeed, $\{ \{ n \} : n \in \mathbb{N} \}$ is an open cover of $\mathbb{N}$ that has no finite subcover.

Despite such examples, there is an important collection of metric spaces in which closed bounded subsets are always compact. Indeed, the Heine-Borel Theorem (III.8) tells us that $\mathbb{R}$ with the standard metric and $\mathbb{R}^n$ with the Euclidean metric (for $n \geq 2$) enjoy this property. We now prove that all such metric spaces are complete. It will follow that $\mathbb{R}$ with the standard metric and $\mathbb{R}^n$ with the Euclidean metric for $n \geq 2$ are complete metric spaces. It will also follow that all compact metric spaces are complete. We first assign a term to metrics which enjoy the property that all closed bounded subsets are compact.

Definition. Let (X, ρ) be a metric space. If every closed subset of X that is bounded with respect to ρ is compact, then we call ρ a *proper metric* on X. An equivalent way to say that ρ is a proper metric on X is to say "(X, ρ) has the *Heine-Borel property*".

Theorem VI.2. If ρ is a proper metric on a metrizable space X, then ρ is a complete metric.

Proof. Assume ρ is a proper metric on the space X. Let $\{ x_n \}$ be a Cauchy sequence in X. Then there is an $n \geq 1$ such that $\rho(x_i,x_j) < 1$ for all $i, j \geq n$. Let $M = \max \{ \rho(x_i,x_j) : 1 \leq i \leq j \leq n \}$. Observe that for $i, j \in \mathbb{N}$: $\rho(x_i,x_j) \leq M$ if $i \leq n$ and $j \leq n$, $\rho(x_i,x_j) \leq \rho(x_i,x_N) + \rho(x_N,x_j) < M + 1$ if $i \leq n \leq j$, and $\rho(x_i,x_j) < 1$ if $i \geq n$ and $j \geq n$. Thus, $\text{diam}(\{ x_n \}) \leq M + 1$.

Next we need a lemma.

Lemma VI.3. If A is a subset of a metric space X, then diam(cl(A)) = diam(A).

Proof of Lemma VI.3. Since A ⊂ cl(A), then diam(A) ≤ diam(cl(A)). Let $\varepsilon > 0$. Let x, y ∈ cl(A). Then $N(x,{}^{\varepsilon}/_2) \cap A \neq \emptyset$ and $N(y,{}^{\varepsilon}/_2) \cap A \neq \emptyset$ by Theorem I.16.b. Hence, we can choose points $x' \in N(x,{}^{\varepsilon}/_2) \cap A$ and $y' \in N(y,{}^{\varepsilon}/_2) \cap A$. Therefore,
$\rho(x,y) \leq \rho(x,x') + \rho(x',y') + \rho(y',y) < {}^{\varepsilon}/_2 + \text{diam}(A) + {}^{\varepsilon}/_2 = \text{diam}(A) + \varepsilon$.
It follows that diam(cl(A)) ≤ diam(A) + ε. Since this inequality holds for every ε > 0, then we conclude that diam(cl(A)) ≤ diam(A). Therefore, diam(cl(A)) = diam(A). ☐

Since diam($\{ x_n \}$) ≤ M + 1, then Lemma VI.3 implies diam(cl($\{ x_n \}$)) ≤ M + 1. Thus, cl($\{ x_n \}$) is a closed bounded subset of X. Since ρ is a proper metric on X, then it follows that cl($\{ x_n \}$) is compact. Since cl($\{ x_n \}$) is a metric space, then Theorem III.30 implies that cl($\{ x_n \}$) is sequentially compact. Consequently, there is a subsequence n(1) < n(2) < n(3) < … of ℕ such that $\{ x_{n(i)} \}$ converges to a point y ∈ X.

We will now prove that $\{ x_n \}$ converges to y. Let ε > 0. Since $\{ x_n \}$ is a Cauchy sequence, then there is an m ≥ 1 such that $\rho(x_i,x_j) < {}^{\varepsilon}/_2$ if i, j ≥ m. Since $\{ x_{n(i)} \}$ converges to y, there is a k ≥ 1 such that $\rho(x_{n(i)},y) < {}^{\varepsilon}/_2$ if i ≥ k. Since n(1) < n(2) < n(3) < … are positive integers, then there is a j ≥ k such that n(j) ≥ m. Then i ≥ m implies $\rho(x_i,y) \leq \rho(x_i,x_{n(j)}) + \rho(x_{n(j)},y) < 2({}^{\varepsilon}/_2) = \varepsilon$. We conclude that $\{ x_n \}$ converges to y.

We have proved that every Cauchy sequence in X converges. Therefore, ρ is a complete metric on X. ☐

Corollary VI.4. Every compact metric space is complete.

Proof. Let (X, ρ) be a compact metric space. Since every closed subset of X is compact by Theorem III.2, then ρ is a proper metric. Hence, Theorem VI.2 implies that ρ is a complete metric. ☐

Corollary VI.4. ℝ with the standard metric and $\mathbb{R}^n$ with the Euclidean metric (for n ≥ 2) are complete metric spaces.

Proof. The Heine-Borel Theorem (III.8) tells us that the standard metric on ℝ and the Euclidean metric on $\mathbb{R}^n$ (for n ≥ 2) are proper metrics. Hence, Theorem VI.2 implies that these metrics are complete. ☐

Recall that, according to Theorem III.10, a nested sequence of non-empty compacta has non-empty intersection. Complete metric spaces are characterized by a similar property which we now present.

Theorem VI.5. The Cantor Intersection Theorem. A metric space (X, ρ) is complete if and only if it has the following property. If $C_1 \supset C_2 \supset C_3 \supset \ldots$ is a nested sequence of non-empty closed subsets of X such that $\lim_{n \to \infty} \operatorname{diam}(C_n) = 0$, then $\bigcap_{n \geq 1} C_n \neq \emptyset$.

Proof. First assume (X, ρ) is a complete metric space. Let $C_1 \supset C_2 \supset C_3 \supset \ldots$ be a nested sequence of non-empty closed subsets of X such that $\lim_{n \to \infty} \operatorname{diam}(C_n) = 0$. For each $n \geq 1$, since C_n is non-empty, we can choose a point $x_n \in C_n$. For $n \geq 1$, since $i \geq n$ implies $x_i \in C_i \subset C_n$, then $\{ x_i : i \geq n \} \subset C_n$. Hence, for $n \geq 1$, $\operatorname{diam}(\{ x_i : i \geq n \}) \leq \operatorname{diam}(C_n)$. Consequently, $\lim_{n \to \infty} \operatorname{diam}(\{ x_i : i \geq n \}) = 0$. Therefore, $\{ x_n \}$ is a Cauchy sequence. Since (X, ρ) is a complete metric space, then $\{ x_n \}$ converges to a point $y \in X$. Let $n \geq 1$. We will prove $y \in C_n$. Let U be a neighborhood of y in X. Then there is an $m \geq 1$ such that $x_i \in U$ for every $i \geq m$. Let $i = \max \{ m, n \}$. Then $x_i \in U$ and $x_i \in C_i \subset C_n$. Thus, $U \cap C_n \neq \emptyset$. Thus, every neighborhood of y intersects C_n. Since C_n is a closed subset of X, then it follows from Theorem I.15 that $y \in C_n$. Hence, $y \in \bigcap_{n \geq 1} C_n$. This proves $\bigcap_{n \geq 1} C_n \neq \emptyset$.

Second assume that if $C_1 \supset C_2 \supset C_3 \supset \ldots$ is a decreasing sequence of non-empty closed subsets of X such that $\lim_{n \to \infty} \operatorname{diam}(C_n) = 0$, then $\bigcap_{n \geq 1} C_n \neq \emptyset$. To prove that (X, ρ) is a complete metric space, let $\{ x_n \}$ be a Cauchy sequence in X. For each $n \geq 1$, let $C_n = \operatorname{cl}(\{ x_i : i \geq n \})$. Lemma VI.3 implies that $\operatorname{diam}(C_n) = \operatorname{diam}(\{ x_i : i \geq n \})$. Since $\{ x_n \}$ is a Cauchy sequence, then $\lim_{n \to \infty} \operatorname{diam}(\{ x_i : i \geq n \}) = 0$. It follows that $\lim_{n \to \infty} \operatorname{diam}(C_n) = 0$. Hence, our hypothesis implies $\bigcap_{n \geq 1} C_n \neq \emptyset$. Therefore, there is a point $y \in \bigcap_{n \geq 1} C_n$. We will now prove that $\{ x_n \}$ converges to y. To this end, let $\varepsilon > 0$. Since $\lim_{n \to \infty} \operatorname{diam}(C_n) = 0$, then there is an $n \geq 1$ such that $\operatorname{diam}(C_n) < \varepsilon$. Since $y \in C_n$ and $x_i \in C_n$ for all $i \geq n$, then $\rho(x_i,y) \leq \operatorname{diam}(C_n) < \varepsilon$ for all $i \geq n$. We conclude that $\{ x_n \}$ converges to y. We have proved that every Cauchy sequence in X converges to a point of X. Thus (X, ρ) is a complete metric space. ☐

Recall that a point x in a topological space X is *isolated* if $\{ x \}$ is an open subset of X. Furthermore recall the result of Problem III.2: if X is a non-empty compact Hausdorff space with no isolated points, then $X \succeq \mathbb{R}$ (i.e., there is a one-to-one function from $\mathbb{R}$ into X). The similarity between compactness and completeness illustrated by the Cantor Intersection Theorem is extended by the following analogue of Problem III.2.

Problem VI.1. Prove that if X is a non-empty complete metric space with no isolated points, then $X \succeq \mathbb{R}$.

Theorem VI.6. Suppose (X, ρ) is a metric space and C is a subset of X.

a) If ρ is a complete metric on X and C is a closed subset of X, then ρ restricts to a complete metric on C.

b) If ρ restricts to a complete metric on C, then C is a closed subset of X.

Proof of a). Assume ρ is a complete metric on X and C is a closed subset of X. Let { x_n } be a Cauchy sequence in C. Then { x_n } converges to a point $y \in X$. In this situation, Corollary I.22.a implies $y \in C$. This proves ρ restricts to a complete metric on C. ◻

Proof of b). Assume ρ restricts to a complete metric on C. Since the metric space X is first countable, then according to Corollary I.22.b, to prove that C is a closed subset of X, it suffices to prove that if { x_n } is a sequence in C that converges to a point $y \in X$, then $y \in C$. So assume { x_n } is a sequence in C that converges to a point $y \in X$. Then { x_n } is a Cauchy sequence by Lemma VI.1. Since ρ restricts to a complete metric on C, then it follows that { x_n } converges to a point $z \in C$. We assert that y = z. For suppose $y \neq z$. Since the metric space X is Hausdorff by Theorem I.25, then there are disjoint neighborhoods U and V of y and z, respectively, in X. Since { x_n } converges to y, then there is an $m \geq 1$ such that $x_i \in U$ for all $i \geq m$; and since { x_n } converges to z, then there is an $n \geq 1$ such that $x_i \in V$ for all $i \geq n$. Let i = max { m, n }. Then $x_i \in U \cap V$. Since U and V are disjoint, we have reached a contradiction. We conclude that y = z. Therefore, $y \in C$. It now follows from Corollary I.22.b that C is a closed subset of X. ◻

Recall that a metric space (X, ρ) is *totally bounded* if for every $\varepsilon > 0$, a finite subset of { $N(x,\varepsilon) : x \in X$ } covers X.

Theorem VI.7. A metric space is compact if and only if it is complete and totally bounded.

Problem VI.2. Prove Theorem VI.7.

Recall that if $\rho : X \times X \to [\, 0. \infty)$ is a metric on a set X, then according to Theorem I.12, an equivalent metric $\bar{\rho} : X \times X \to [\, 0. \infty)$ is defined by the equation $\bar{\rho}(x,y) = \min \{ \rho(x,y), 1 \}$ for all x, $y \in X$.

Theorem VI.8. If (X, ρ) is a metric space, then ρ is complete if and only if $\bar{\rho}$ is complete.

Proof. Since $\bar{\rho}(x,y) \leq \rho(x,y)$ for all x, $y \in X$, then every sequence in X which is Cauchy with respect to ρ is also Cauchy with respect to $\bar{\rho}$. Also if { x_n } is a sequence in X that is Cauchy with respect to $\bar{\rho}$, then for every $\varepsilon > 0$, there is an $n \geq 1$ such that

$\bar{\rho}(x_i,x_j) < \min\{ \varepsilon, 1 \}$ whenever $i, j \geq n$. Then $\rho(x_i,x_j) = \bar{\rho}(x_i,x_j) < \varepsilon$ whenever $i, j \geq n$. Thus, $\{ x_n \}$ is Cauchy with respect to ρ. We conclude that every sequence in X which is Cauchy with respect to $\bar{\rho}$ is also Cauchy with respect to ρ.

Now assume ρ is a complete metric on X. Let $\{ x_n \}$ be a sequence in X that is Cauchy with respect to $\bar{\rho}$. Then $\{ x_n \}$ is Cauchy with respect to ρ. Hence, $\{ x_n \}$ converges with respect to ρ. Since ρ and $\bar{\rho}$ are equivalent metrics on X, then $\{ x_n \}$ converges with respect to $\bar{\rho}$. It follows that $\bar{\rho}$ is a complete metric on X.

We simply interchange the roles of ρ and $\bar{\rho}$ in the preceding paragraph to obtain a proof that if $\bar{\rho}$ is a complete metric on X, then so is ρ. ◻

Theorem VI.9. Let $(X_1, \rho_1), (X_2, \rho_2), \ldots , (X_n, \rho_n)$ be metric spaces. Define the three metrics σ_1, σ_2 and σ_∞ on $X_1 \times X_2 \times \ldots \times X_n$ by the formulas:

$$\sigma_1(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} \rho_i(x_i,y_i),$$

$$\sigma_2(\mathbf{x},\mathbf{y}) = \left(\sum_{i=1}^{n} (\rho_i(x_i,y_i))^2\right)^{1/2},$$

$$\sigma_\infty(\mathbf{x},\mathbf{y}) = \max\{ \rho_i(x_i,y_i) : 1 \leq i \leq n \}$$

for $\mathbf{x} = (x_1, x_2, \ldots , x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots , y_n) \in X_1 \times X_2 \times \ldots \times X_n$. (Theorem I.32 implies that σ_1, σ_2 and σ_∞ are equivalent metrics that induce the product topology on $X_1 \times X_2 \times \ldots \times X_n$.) Then σ_1, σ_2 and σ_∞ are complete metrics on $X_1 \times X_2 \times \ldots \times X_n$ if and only if ρ_i is a complete metric on X_i for $1 \leq i \leq n$.

Proof. For $1 \leq i \leq n$, let $\pi_i : X_1 \times X_2 \times \ldots \times X_n \rightarrow X_i$ denote the i^{th} projection map; thus, $\pi_i(\mathbf{x}) = x_i$ for $\mathbf{x} = (x_1, x_2, \ldots , x_n) \in X_1 \times X_2 \times \ldots \times X_n$. For $\mathbf{a} = (a_1, a_2, \ldots , a_n) \in X_1 \times X_2 \times \ldots \times X_n$ and $1 \leq i \leq n$, let $e_{\mathbf{a},i} : X_i \rightarrow X_1 \times X_2 \times \ldots \times X_n$ denote the i^{th} injection map; thus, $e_{\mathbf{a},i}(x) = (a_1, \ldots , a_{i-1}, x, a_{i+1}, \ldots , a_n)$ for $x \in X_i$.

We make two observations:

i) For $1 \leq i \leq n$ and $r \in \{ 1, 2, \infty \}$, $\rho_i(\pi_i(\mathbf{x}),\pi_i(\mathbf{y})) \leq \sigma_r(\mathbf{x},\mathbf{y})$ for $\mathbf{x}$ and $\mathbf{y} \in X_1 \times X_2 \times \ldots \times X_n$.

ii) For $\mathbf{a} \in X_1 \times X_2 \times \ldots \times X_n$, $1 \leq i \leq n$ and $r \in \{ 1, 2, \infty \}$, $\sigma_r(e_{\mathbf{a},i}(x),e_{\mathbf{a},i}(y)) = \rho_i(x,y)$ for x and $y \in X_i$.

First assume ρ_i is a complete metric on X_i for $1 \leq i \leq n$. Let $r \in \{ 1, 2, \infty \}$. Suppose $\{ \mathbf{x}_k \}$ is a Cauchy sequence in $(X_1 \times X_2 \times \ldots \times X_n, \sigma_r)$. Then observation i) above implies that $\{ \pi_i(\mathbf{x}_k) \}$ is a Cauchy sequence in (X_i, ρ_i) for $1 \leq i \leq n$. For $1 \leq i \leq n$, since (X_i, ρ_i) is a complete metric space, then $\{ \pi_i(\mathbf{x}_k) \}$ converges in X_i to a point $y_i \in X_i$.

Let $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in X_1 \times X_2 \times \ldots \times X_n$. Then Theorem V.7 implies $\{ \mathbf{x}_k \}$ converges to $\mathbf{y}$ in $X_1 \times X_2 \times \ldots \times X_n$. Hence, σ_r is a complete metric on $X_1 \times X_2 \times \ldots \times X_n$.

Second let $r \in \{ 1, 2, \infty \}$ and assume σ_r is a complete metric on $X_1 \times X_2 \times \ldots \times X_n$. Let $1 \le i \le n$. Suppose $\{ x_k \}$ is a Cauchy sequence in X_i. Let $\mathbf{a} \in X_1 \times X_2 \times \ldots \times X_n$. Then observation ii) above implies that $\{ e_{\mathbf{a},i}(x_k) \}$ is a Cauchy sequence in $(X_1 \times X_2 \times \ldots \times X_n, \sigma_r)$. Since $(X_1 \times X_2 \times \ldots \times X_n, \sigma_r)$ is a complete metric space, then $\{ e_{\mathbf{a},i}(x_k) \}$ converges in $X_1 \times X_2 \times \ldots \times X_n$ to a point $\mathbf{y} \in X_1 \times X_2 \times \ldots \times X_n$. Since π_i is continuous, then Theorem II.7 implies that $\{ \pi_i(e_{\mathbf{a},i}(x_k)) \}$ converges to $\pi_i(\mathbf{y})$ in X_i. Since $\pi_i(e_{\mathbf{a},i}(x_k)) = x_k$ for $k \ge 1$, then we conclude that $\{ x_k \}$ converges in X_i. Hence, ρ_i is a complete metric on X_i. □

Corollary VI.10. For $n \ge 2$, $\mathbb{R}^n$ with the taxicab metric and $\mathbb{R}^n$ with the supremum metric are complete metric spaces.

Theorem VI.11. Let $\{ (X_n,\rho_n) : n \in \mathbb{N} \}$ be a countable collection of metric spaces. For each $n \in \mathbb{N}$, define the metric $\bar{\rho}_n : X_n \times X_n \to [0,\infty)$ by $\bar{\rho}_n(x,y) = \min \{ \rho_n(x,y), 1 \}$ for $x, y \in X_n$. (Theorem I.12 implies that $\bar{\rho}_n$ is equivalent to ρ_n and $\bar{\rho}_n \le 1$.) Define the three metrics σ_1, σ_2 and σ_∞ on $\prod_{n \in \mathbb{N}} X_n$ by the formulas:

a) $\sigma_1(x,y) = \sum_{n \in \mathbb{N}} 2^{-n} \bar{\rho}_n(x(n), y(n))$,

b) $\sigma_2(x,y) = \left(\sum_{n \in \mathbb{N}} (2^{-n} \bar{\rho}_n(x(n), y(n)))^2 \right)^{1/2}$, and

c) $\sigma_\infty(x,y) = \sup \{ 2^{-n} \bar{\rho}_n(x(n), y(n)) : n \in \mathbb{N} \}$.

(Theorem V.14 implies that σ_1, σ_2 and σ_∞ are equivalent metrics that induce the product topology on $\prod_{n \in \mathbb{N}} X_n$.) Then σ_1, σ_2 and σ_∞ are complete metrics on $\prod_{n \in \mathbb{N}} X_n$ if and only if ρ_n is a complete metric on X_n for each $n \in \mathbb{N}$.

Proof. For each $m \in \mathbb{N}$, let $\pi_m : \prod_{n \in \mathbb{N}} X_n \to X_m$ denote the m^{th} projection map; thus, $\pi_m(x) = x(m)$ for $x \in \prod_{n \in \mathbb{N}} X_n$. For $a \in \prod_{n \in \mathbb{N}} X_n$ and $m \in \mathbb{N}$, let $e_{\mathbf{a},m} : X_m \to \prod_{n \in \mathbb{N}} X_n$ denote the the m^{th} injection map; thus, $e_{a,m}(x)(i) = x$ if $i = m$ and $e_{a,m}(x)(i) = a(i)$ if $i \ne m$ for $x \in X_m$.

We make two observations:

i) For $m \in \mathbb{N}$ and $r \in \{ 1, 2, \infty \}$, $2^{-m} \bar{\rho}_m(\pi_m(x),\pi_m(y)) \le \sigma_r(x,y)$ for x and $y \in \prod_{n \in \mathbb{N}} X_n$.

ii) For $a \in \prod_{n \in \mathbb{N}} X_n$, $m \in \mathbb{N}$, and $r \in \{ 1, 2, \infty \}$, $\sigma_r(e_{a,m}(x),e_{a,m}(y)) = 2^{-m} \bar{\rho}_m(x,y)$ for x and $y \in X_m$.

First assume ρ_n is a complete metric on X_n for each $n \in \mathbb{N}$. Then Theorem VI.8 implies that $\bar{\rho}_n$ is a complete metric on X_n for each $n \in \mathbb{N}$. Let $r \in \{ 1, 2, \infty \}$. Suppose $\{ x_k \}$ is a Cauchy sequence in $(\prod_{n\in\mathbb{N}} X_n, \sigma_r)$. Let $m \in \mathbb{N}$. We assert that $\{ \pi_m(x_k) \}$ is a Cauchy sequence in $(X_m, \bar{\rho}_m)$. Let $\varepsilon > 0$. Then there is an $k \geq 1$ such that $\sigma_r(x_i,x_j) < 2^{-m}\varepsilon$ whenever $i, j \geq k$. Then observation i) above implies that $\bar{\rho}_m(\pi_i(x_i),\pi_i(x_j)) < \varepsilon$ whenever $i, j \geq k$. This proves our assertion: $\{ \pi_m(x_k) \}$ is a Cauchy sequence in $(X_m, \bar{\rho}_m)$. Since $(X_m, \bar{\rho}_m)$ is a complete metric space, then $\{ \pi_m(x_k) \}$ converges in X_m to a point $y_m \in X_m$. Define the point $y \in \prod_{n\in\mathbb{N}} X_n$ by $y(m) = y_m$ for each $m \in \mathbb{N}$. Then $\{ \pi_m(x_k) \}$ converges to $\pi_i(y) = y_m$ for each $m \in \mathbb{N}$. Therefore Theorem V.7 implies $\{ x_k \}$ converges to y in $\prod_{n\in\mathbb{N}} X_n$. This proves that σ_r is a complete metric on $\prod_{n\in\mathbb{N}} X_n$.

Second let $r \in \{ 1, 2, \infty \}$ and assume σ_r is a complete metric on $\prod_{n\in\mathbb{N}} X_n$. Let $m \in \mathbb{N}$. Suppose $\{ x_k \}$ is a Cauchy sequence in $(X_m, \bar{\rho}_m)$. Let $a \in \prod_{n\in\mathbb{N}} X_n$. Observation ii) above implies that $\sigma_r(e_{a,m}(x),e_{a,m}(y)) \leq \bar{\rho}_m(x,y)$ for $x, y \in X_m$. Consequently, $\{ e_{a,m}(x_k) \}$ is a Cauchy sequence in $(\prod_{n\in\mathbb{N}} X_n, \sigma_r)$. Since $(\prod_{n\in\mathbb{N}} X_n, \sigma_r)$ is a complete metric space, then $\{ e_{a,m}(x_k) \}$ converges in $\prod_{n\in\mathbb{N}} X_n$ to a point $y \in \prod_{n\in\mathbb{N}} X_n$. Since π_m is continuous, then Theorem II.7 implies that $\{ \pi_m(e_{a,m}(x_k)) \}$ converges to $\pi_m(y)$ in X_i. Since $\pi_m(e_{a,m}(x_k)) = x_k$ for $k \geq 1$, then we conclude that $\{ x_k \}$ converges in X_m. Hence, $\bar{\rho}_m$ is a complete metric on X_m. It follows via Theorem VI.8 that ρ_m is a complete metric on X_m. ◻

Definition. Let X be a set and let (Y, ρ) be a metric space. A function $f : X \to Y$ is *bounded* if $\mathrm{diam}(f(X)) < \infty$. Let $B(X,Y)$ denote the set of all bounded functions from X to Y. Furthermore, if X is a topological space, let $C(X,Y)$ denote the set of all continuous bounded functions from X to Y; thus, $C(X,Y) \subset B(X,Y)$.

Theorem VI.12. Let X be a set and let (Y, ρ) be a metric space. Define the function $\sigma : B(X,Y) \times B(X,Y) \to [0,\infty)$ by $\sigma(f,g) = \sup \{ \rho(f(x),g(x)) : x \in X \}$. Then σ is a metric on $B(X,Y)$ which is called the *supremum metric* on $B(X,Y)$. Furthermore, σ is a complete metric on $B(X,Y)$ if and only if ρ is a complete metric on Y.

Proof. First we show that the function $\sigma : B(X,Y) \times B(X,Y) \to [0,\infty)$ is well defined by proving that $\sigma(f,g) < \infty$ for all $f, g \in B(X.Y)$. Let $f, g \in B(X.Y)$. Then $\mathrm{diam}(f(X)) < \infty$ and $\mathrm{diam}(g(X)) < \infty$. Choose $x_0 \in X$. Then for each $x \in X$, $\rho(f(x),g(x)) \leq \rho(f(x),f(x_0)) + \rho(f(x_0),g(x_0)) + \rho(g(x_0),g(x)) \leq \mathrm{diam}(f(X)) + \rho(f(x_0),g(x_0)) + \mathrm{diam}(g(X))$. Thus, $\sigma(f,g) \leq \mathrm{diam}(f(X)) + \rho(f(x_0),g(x_0)) + \mathrm{diam}(g(X)) < \infty$.

To verify that σ is a metric, let f, g and $h \in B(X.Y)$.

- Clearly: $f = g \Leftrightarrow f(x) = g(x)$ for every $x \in X \Leftrightarrow \rho(f(x),g(x)) = 0$ for every $x \in X \Leftrightarrow \sigma(f,g) = 0$.

- Since $\rho(f(x),g(x)) = \rho(g(x),f(x))$ for every $x \in X$, then $\sigma(f,g) = \sigma(g,f)$.
- Since for every $x \in X$, $\rho(f(x),h(x)) \leq \rho(f(x),g(x)) + \rho(g(x),h(x)) \leq \sigma(f,g) + \sigma(g,h)$, then $\sigma(f,h) \leq \sigma(f,g) + \sigma(g,h)$.

This completes the proof that σ is a metric on B(X,Y).

Now assume that ρ is a complete metric on Y. Let $\{ f_n \}$ be a Cauchy sequence in $(B(X,Y), \sigma)$. Let $x \in X$. Since $\rho(f_i(x),f_j(x)) \leq \sigma(f_i,f_j)$ for $i, j \geq 1$, then $\{ f_n(x) \}$ is a Cauchy sequence in Y. Therefore, $\{ f_n(x) \}$ converges to a point g(x) in Y. This defines a function $g : X \to Y$. We will prove that $g \in B(X,Y)$ and $\{ f_n \}$ converges to g in $(B(X,Y), \sigma)$.

Since $\{ f_n \}$ is a Cauchy sequence in $(B(X,Y), \sigma)$, then there is a $k \geq 1$ such that $\sigma(f_i,f_j) < 1$ whenever $i, j \geq k$. Let $x, x' \in X$. Since $\{ f_n(x) \}$ converges to g(x) in Y, there is an $m \geq 1$ such that $\rho(f_i(x),g(x)) < 1$ whenever $i \geq m$. Similarly, since $\{ f_n(x') \}$ converges to $g(x')$ in Y, there is an $m' \geq 1$ such that $\rho(f_i(x'),g(x')) < 1$ whenever $i \geq m'$. Let $i = \max \{ k, m, m' \}$. Then

$$\rho(g(x),g(x')) \leq \rho(g(x),f_i(x)) + \rho(f_i(x),f_k(x)) + \rho(f_k(x),f_k(x')) + \rho(f_k(x'),f_i(x')) + \rho(f_i(x'),g(x')) \leq$$

$$\rho(g(x),f_i(x)) + \sigma(f_i,f_k) + \text{diam}(f_k(X)) + \sigma(f_k,f_i) + \rho(f_i(x'),g(x')) < 4 + \text{diam}(f_k(X)).$$

Thus, $\text{diam}(g(X)) \leq 4 + \text{diam}(f_k(X)) < \infty$. Consequently, $g \in B(X,Y)$.

To prove that $\{ f_n \}$ converges to g, let $\varepsilon > 0$. Since $\{ f_n \}$ is a Cauchy sequence in $(B(X,Y), \sigma)$, then there is a $k \geq 1$ such that $\sigma(f_i,f_j) < \varepsilon/3$ whenever $i, j \geq k$. We will prove that $\sigma(f_i,g) < \varepsilon$ whenever $i \geq k$. Let $x \in X$. Since $\{ f_n(x) \}$ converges to g(x) in Y, there is an $m \geq 1$ such that $\rho(f_i(x),g(x)) < \varepsilon/3$ whenever $i \geq m$. Let $i \geq k$ and let $j = \max \{ k, m \}$. Then

$$\rho(f_i(x),g(x)) \leq \rho(f_i(x),f_j(x)) + \rho(f_j(x),g(x)) \leq \sigma(f_i,f_j) + \rho(f_j(x),g(x)) < 2\varepsilon/3.$$

Since the choice of $k \geq 1$ is independent of the choice of $x \in X$, then $\rho(f_i(x),g(x)) < 2\varepsilon/3$ for every $x \in X$ whenever $i \geq k$. Thus, $\sigma(f_i,g) \leq 2\varepsilon/3 < \varepsilon$ whenever $i \geq k$. This proves $\{ f_n \}$ converges to g in $(B(X,Y), \sigma)$. We conclude that σ is a complete metric on B(X,Y).

Next assume σ is a complete metric on B(X,Y). To prove that ρ is a complete metric on Y, let $\{ y_n \}$ be a Cauchy sequence in Y. For each $n \geq 1$, define the function $f_n : X \to Y$ by $f_n(x) = y_n$ for all $x \in X$. Then for each $n \geq 1$, $\text{diam}(f_n(X)) = \text{diam}(\{ y_n \}) = 0$. Hence, $f_n \in B(X,Y)$ for every $n \geq 1$. For $i, j \geq 1$ and for each $x \in X$, $\rho(f_i(x),f_j(x)) = \rho(y_i,y_j)$. Thus, $\sigma(f_i,f_j) = \rho(y_i,y_j)$ for $i, j \geq 1$. Therefore, the fact that $\{ y_n \}$ is a Cauchy sequence in (Y, ρ) implies that $\{ f_n \}$ is a Cauchy sequence in $(B(X,Y), \sigma)$. Since σ is a complete metric on B(X,Y), then there is a $g \in B(X,Y)$ such that $\{ f_n \}$ converges g in $(B(X,Y), \sigma)$. Let $x_0 \in X$. Then $\rho(y_n,g(x_0)) = \rho(f_n(x_0),g(x_0)) \leq \sigma(f_n,g)$. Since $\{ f_n \}$ converges g with respect to σ, then it follows that $\{ y_n \}$ converges to $g(x_0)$. We conclude that ρ is a complete metric on Y. □

Definition. Let X be a topological space and let (Y, ρ) be a metric space. Let C(X,Y) denote the set of all continuous bounded functions from X to Y. Thus, $C(X,Y) \subset B(X,Y)$. We restrict the supremum metric σ on B(X,Y) to C(X,Y) to obtain a metric on C(X,Y) which is called the *supremum metric* on C(X,Y).

Theorem VI.13. Let X be a topological space, let (Y, ρ) be a metric space, and assign B(X,Y) the supremum metric σ. Then C(X,Y) is a closed subset of B(X,Y). Furthermore, σ restricts to a complete metric on C(X,Y) if and only if ρ is a complete metric on Y.

Proof. To prove that C(X,Y) is a closed subset of B(X,Y), let $f \in B(X,Y) - C(X,Y)$. Then f fails to be continuous at some point $x_0 \in X$. Hence, there is an $\varepsilon > 0$ such that for every neighborhood U of x_0 in X, there is an $x \in U$ such that $\rho(f(x_0),f(x)) \geq \varepsilon$. Let V denote the $\varepsilon/3$–neighborhood of f in $(B(X,Y), \sigma)$. Thus, V is a neighborhood of f in B(X,Y) such that $g \in V$ if and only if $\sigma(f,g) < \varepsilon/3$.

We assert that $V \cap C(X,Y) = \emptyset$. Let $g \in V$. Let U be a neighborhood of x_0 in X. Then there is an $x \in U$ such that $\rho(f(x_0),f(x)) \geq \varepsilon$. Therefore,

$$\varepsilon \leq \rho(f(x_0),f(x)) \leq \rho(f(x_0),g(x_0)) + \rho(g(x_0),g(x)) + \rho(g(x),f(x)) \leq$$

$$\sigma(f,g) + \rho(g(x_0),g(x)) + \sigma(f,g) < 2\varepsilon/3 + \rho(g(x_0),g(x)).$$

Therefore, $\rho(g(x_0),g(x)) > \varepsilon/3$. Hence, there is no neighborhood U of x_0 in X such that $\rho(g(x_0),g(x)) < \varepsilon/3$ for all $x \in U$. Thus, g is discontinuous at x_0. Therefore, $g \notin C(X,Y)$. This proves our assertion: $V \cap C(X,Y) = \emptyset$.

Since every $f \in B(X,Y) - C(X,Y)$ has a neighborhood which is disjoint from C(X,Y), then C(X,Y) is a closed subset of B(X,Y).

If ρ is a complete metric on Y, then Theorem VI.12 implies that σ is a complete metric on B(X,Y). Since C(X,Y) is a closed subset of B(X,Y), then Theorem VI.6.a implies that σ restricts to a complete metric on C(X,Y).

If σ restricts to a complete metric on C(X,Y), then the argument given in the final paragraph of the proof of Theorem VI.12 applies here to prove that ρ is a complete metric on Y. Beginning with a Cauchy sequence $\{ y_n \}$ in Y, we define the constant maps $f_n : X \rightarrow \{ y_n \}$. Since these functions are continuous, $\{ f_n \}$ is a Cauchy sequence in C(X,Y). Therefore, $\{ f_n \}$ converges to some $g \in C(X,Y)$. It then follows as before that $\{ y_n \}$ converges to $g(x_0)$ for any $x_0 \in X$. This proves ρ is a complete metric on Y. □