C. Embedding in Products

We begin this section with a fundamental theorem that provides sufficient conditions for embedding a space in a Cartesian product of other spaces. We exploit this result to embed a large class of spaces in Cartesian products of simple spaces such as closed intervals. We can then conclude that the spaces in this large class share the properties of subspaces of such Cartesian products. In particular, if the Cartesian products are formed from countably many metrizable factors, then we can conclude that the class of spaces that can be embedded in such Cartesian products are metrizable.

Theorem V.18. A General Embedding Theorem. Let X be a T_1 space. Suppose there is a collection $\{ f_\gamma : X \to Y_\gamma : \gamma \in \Gamma \}$ of maps with domain X and with the property that for every point $x \in X$ and every closed subset C of X such that $x \notin C$, there is a $\gamma \in \Gamma$ such that $f_\gamma(x) \notin \mathrm{cl}(f_\gamma(C))$. Then an embedding $e : X \to \prod_{\gamma \in \Gamma} Y_\gamma$ is defined by the equation

$$(e(x))(\gamma) = f_\gamma(x)$$

for each $x \in X$ and $\gamma \in \Gamma$.

Proof. We invoke Theorem V.5 to establish the continuity of $e : X \to \prod_{\gamma \in \Gamma} Y_\gamma$. For each $\gamma \in \Gamma$, since $\pi_\gamma \circ e(x) = (e(x))(\gamma) = f_\gamma(x)$ for every $x \in X$, then $\pi_\gamma \circ e = f_\gamma$. Thus, $\pi_\gamma \circ e : X \to Y_\gamma$ is continuous for every $\gamma \in \Gamma$. Therefore, the continuity of e follows from Theorem V.5.

To prove that $e : X \to \prod_{\gamma \in \Gamma} Y_\gamma$ is injective, let x and x′ be distinct points of X. Since X is a T_1 space, then $\{ x' \}$ is a closed subset. Then, by hypothesis, there is a $\gamma \in \Gamma$ such that $f_\gamma(x) \notin \mathrm{cl}(f_\gamma(\{ x' \}))$. In particular, $f_\gamma(x) \neq f_\gamma(x')$. Thus, $(e(x))(\gamma) \neq (e(x'))(\gamma)$. Consequently, $e(x) \neq e(x')$. This proves e is injective.

Finally, to prove $e : X \to \prod_{\gamma \in \Gamma} Y_\gamma$ is an embedding, we show that $e : X \to e(X)$ is an open map. Let U be an open subset of X and let $y \in e(U)$. Then there is an $x \in U$ such that $e(x) = y$. Since x is not an element of the closed set $X - U$, then, by hypothesis, there is a $\gamma \in \Gamma$ such that $f_\gamma(x) \notin \mathrm{cl}(f_\gamma(X - U))$. Let $V = Y_\gamma - \mathrm{cl}(f_\gamma(X - U))$. Then V is a neighborhood of $f_\gamma(x)$ in Y_γ such that $V \cap f_\gamma(X - U) = \varnothing$. Recall that $\pi_\gamma \circ e = f_\gamma$. Hence $\pi_\gamma \circ e(x) \in V$ and $V \cap \pi_\gamma \circ e(X - U) = \varnothing$. Thus, $y = e(x) \in \pi_\gamma^{-1}(V)$ and $\pi_\gamma^{-1}(V) \cap e(X - U) = \varnothing$. Let $W = \pi_\gamma^{-1}(V) \cap e(X)$. Then it follows that W is a relatively open subset of e(X) such that $y \in W \subset e(U)$. This proves that e(U) is a relatively open subset of e(X). Thus, $e : X \to e(X)$ is an open map. We have established that $e : X \to \prod_{\gamma \in \Gamma} Y_\gamma$ is an embedding. □

The General Embedding Theorem has a variety of applications in which the factor spaces Y_γ ($\gamma \in \Gamma$) assume different identities. However, in the remainder of this section, we will specialize to situations in which all the Y_γ's are homeomorphic to closed intervals in $\mathbb{R}$.

Notation. Let $\mathbb{I}$ denote the closed interval [0, 1] in $\mathbb{R}$.

Definition. Let X be a topological space. A set Φ of maps from X to $\mathbb{I}$ is a *regularizing family* if for every point $x \in X$ and every closed subset C of X such that $x \notin C$, there is a $\phi \in \Phi$ such that $\phi(x) = 0$ and $\phi(C) = \{ 1 \}$. If there is a regularizing family of maps from X to $\mathbb{I}$, then we call X a *completely regular* space.

We make several observations about completely regular spaces.

a) Urysohn's Lemma (Theorem II.13) implies that every T_1 normal space is completely regular.

b) Every completely regular space is regular.

c) Every subspace of a completely regular space is completely regular.

d) If X_γ is a completely regular space for each $\gamma \in \Gamma$, then $\prod_{\gamma \in \Gamma} X_\gamma$ is completely regular.

e) There is a completely regular T_1 space that is not normal.

Problem V.9. Prove observations a) through e) above.

Example V.1. Here we construct an example of a regular Hausdorff space that is not completely regular. Let ∞ be a point that is not an element of $\mathbb{R} \times [0, \infty)$ and let $X = (\mathbb{R} \times [0, \infty)) \cup \{ \infty \}$. We define a (non-standard) topology on X by specifying a basis for this topology. For each $x \in \mathbb{R}$, let $J(x) = \{ x \} \times [0, 2]$, let $K(x) = \{ (x + y,y) : 0 \le y \le 2 \}$ and let $V(x) = J(x) \cup K(x)$. For each $n \ge 1$, let $U_n = ((n,\infty) \times [0, \infty)) \cup \{ \infty \}$. Let $\mathscr{B}$ denote the union of the following three collections of subsets of X:

$$\{ \{ p \} : p \in \mathbb{R} \times (0, \infty) \},$$

$$\{ V(x) - F : x \in \mathbb{R} \text{ and } F \text{ is a finite subset of } V(x) \} \text{ and}$$

$$\{ U_n : n \ge 1 \}.$$

Then

a) $\mathscr{B}$ is a basis for a topology on X.

We assign this topology to X.

b) X with this topology is a regular Hausdorff space.

X is not completely regular. To prove this, let $D = (-\infty, 1] \times [0, \infty)$. Then

c) D is a closed subset of X.

Suppose $f : X \to [0, 1]$ is a map such that $f(D) = \{ 1 \}$. We will prove $f(\infty) = 1$. For each $n \geq 1$, let $A_n = \{ x \in [n - 1, n] : f((x,0)) = 1 \}$. We will prove inductively that A_n is an infinite set for each $n \geq 1$.

d) A_1 is infinite.

Let $n \geq 1$ and assume A_n is infinite.

e) If $x \in A_n$, then $K(x) \cap f^{-1}([0, 1 - {}^1/_k])$ is a finite set for each $k \geq 1$.

f) If $x \in A_n$, then $K(x) \cap f^{-1}([0, 1))$ is a countable set.

Let $\pi : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ denote projection to first coordinate. Let B be an countably infinite subset of A_n and let $P = \pi(\cup_{x \in B} (K(x) \cap f^{-1}([0, 1))))$. Then

g) P is a countable subset of $\mathbb{R}$.

Let $C = [n, n + 1] - P$. Then C is an infinite subset of $[n, n + 1]$. We will prove that $C \subset A_{n+1}$. To this end, let $y \in C$. We must prove $f((y,0)) = 1$.

h) $(\cup_{x \in B} K(x)) \cap J(y)$ is an infinite subset of $f^{-1}(\{ 1 \})$.

i) If $f((y,0)) \neq 1$, then $J(y) \cap f^{-1}(\{ 1 \})$ is a finite set. Hence, $f((y,0)) = 1$.

This proves $C \subset A_{n+1}$. Hence, A_{n+1} is infinite. It follows by induction that A_n is infinite for each $n \geq 1$. Then, in particular, $A_n \neq \emptyset$ for each $n \geq 1$.

j) $U_n \cap f^{-1}(\{ 1 \}) \neq \emptyset$ for each $n \geq 1$.

k) $f(\infty) = 1$.

l) X is not completely regular.

Problem V.10. Complete the exposition of Example V.1 by proving assertions a) through l).

Clearly, a regularizing family satisfies the condition on the collection of maps $\{ f_\gamma : X \to Y_\gamma : \gamma \in \Gamma \}$ in the statement of Theorem V.18. Hence, we have:

Corollary V.19. If X is a T_1 space and Φ is a regularizing family of maps from X to $\mathbb{I}$, then an embedding $e : X \to \mathbb{I}^\Phi$ is defined by the equation

$$(e(x))(\phi) = \phi(x)$$

for each $x \in X$ and $\phi \in \Phi$. □

Theorem V.20. Let X be a topological space. There is a set A such that X can be embedded in the Cartesian product $\mathbb{I}^A$ of closed intervals if and only if X is T_1 and completely regular.

Proof. Assume X is T_1 and completely regular. Then X has a regularizing family of maps Φ from X to $\mathbb{I}$. Hence, according to Corollary V.19, X can be embedded in $\mathbb{I}^\Phi$.

The proof of the converse statement is left as a problem.

Problem V.11. Let X be a topological space. Prove that if there is a set A such that X can be embedded in $\mathbb{I}^A$, then X is T_1 and completely regular.

Corollary V.19 implies that any T_1 space with a countable regularizing family Φ can be embedding in a countable Cartesian product of closed intervals $\mathbb{I}^\Phi$. Moreover, since Φ is countable, then $\mathbb{I}^\Phi$ is metrizable by Theorem V.14. Hence, the class of T_1 spaces with a countable regularizing families is of particular interest, and it is the next focus of our attention.

Lemma V.21. Every second countable T_1 regular space has a countable regularizing family.

Proof. Let X be second countable T_1 regular space. Since X is second countable, then Theorem III.26 implies X is Lindelöf. Since X is regular and Lindelöf, then Theorem III.27 implies that X is normal.

Let $\mathscr{B}$ be a countable basis for the topology of X. Let

$$\mathscr{P} = \{ (U, V) \in \mathscr{B} \times \mathscr{B} : \mathrm{cl}(V) \subset U \}.$$

$\mathscr{P}$ is countable because it is a subset of the countable set $\mathscr{B} \times \mathscr{B}$. For each element (U, V) of $\mathscr{P}$, since cl(V) and X – U are disjoint closed subsets of the normal space X, then Urysohn's Lemma provides a map $f_{(U, V)} : X \to \mathbb{I}$ such that $f_{(U, V)}(\mathrm{cl}(V)) = \{ 0 \}$ and $f_{(U, V)}(X - U) = \{ 1 \}$.

We will now prove that the countable collection $\{ f_{(U, V)} : (U, V) \in \mathscr{P} \}$ is a regularizing family for X. To this end, assume that $x \in X$ and C is a closed subset of X such that $x \notin C$. Then X – C is a neighborhood of x in X. Therefore, there is a $U \in \mathscr{B}$ such that $x \in U \subset X - C$. Since X is regular, then there is a neighborhood W of x in X such that $\mathrm{cl}(W) \subset U$. Next there is a $V \in \mathscr{B}$ such that $x \in V \subset W$. Since $\mathrm{cl}(V) \subset \mathrm{cl}(W) \subset U$, then $(U, V) \in \mathscr{P}$. Since $x \in V$ and $f_{(U, V)}(\mathrm{cl}(V)) = \{ 0 \}$, the $f_{(U, V)}(x) = 0$. Since $f_{(U, V)}(X - U) = \{ 1 \}$ and $C \subset X - U$, then $f_{(U, V)}(C) = \{ 1 \}$. This proves $\{ f_{(U, V)} : (U, V) \in \mathscr{P} \}$ is a countable regularizing family for X. ◻

We now combine the preceding results to obtain a fundamental characterization of separable metrizable spaces.

Theorem V.23. Urysohn's Metrization Theorem. A topological space X is separable metrizable if and only if X is a second countable T_1 and regular.

Proof. If X is a separable metrizable space, then previous results (Theorems I.13 and I.21) imply X is second countable T_1 and regular.

On the other hand, if X is a second countable T_1 regular space, then Lemma V.21 imlies that X has a countable regularizing family Φ. It follows by Corollary V.19 that X embeds in $\mathbb{I}^{\Phi}$. Since $\mathbb{I}$ is metrizable and Φ is countable, then $\mathbb{I}^{\Phi}$ is metrizable by Theorem V.14. Recall that every subspace of a metrizable space is metrizable by Theorem I.27.f. We conclude that X is metrizable. ◻

Below we will find that it is convenient to be able to change the index set of a Cartesian product. The following elementary observation provides this ability.

Definition. If $f : A \to B$ is a function from a set A to a set B and Y is a topological space, then define the function $f^* : Y^B \to Y^A$ by $f^*(x) = x \circ f$ for $x \in X^B$.

Lemma V.24. Let Y be a topological space.

a) If A is a set, then $(id_A)^* = id_{Y^A}$.

b) If $f : A \to B$ and $g : B \to C$ are functions, then $(g \circ f)^* = f^* \circ g^*$.

c) If $f : A \to B$ is a function, then $f^* : Y^B \to Y^A$ is continuous.

d) If $f : A \to B$ is a bijection, then $f^* : Y^B \to Y^A$ is a homeomorphism.

Problem V.12. Prove Lemma V.24.

Terminology. The space $\mathbb{I}^{\mathbb{N}}$ (with the product topology) is known as the *Hilbert cube.*

Observe that since $\mathbb{I}$ is separable and metrizable, then Theorems V.12.c and V.14 imply that the Hilbert cube is a separable metrizable space.

Definition. Let $\mathscr{C}$ be a collection of topological spaces. An element X of $\mathscr{C}$ is called a *universal element* of $\mathscr{C}$ if every other element of $\mathscr{C}$ can be embedded in X.

Theorem V.25. Urysohn's Embedding Theorem. The Hilbert cube is a universal element of the collection of all separable metrizable spaces. In other words, every separable metrizable space can be embedded in the Hilbert cube.

Proof. Let X be a separable metrizable space. Then X is second countable T_1 and regular by Theorems I.13 and I.21. Hence, X has a countable regularizing family Φ by Lemma V.21. Therefore, there is an embedding $e : X \to \mathbb{I}^\Phi$ by Corollary V.19. Since Φ is countable, then Lemma V.23.d provides a homeomorphism $h : \mathbb{I}^\Phi \to \mathbb{I}^\mathbb{N}$. Consequently, $h \circ e : X \to \mathbb{I}^\mathbb{N}$ is an embedding of X in the Hilbert cube. □

We have just explored the advantages of finding a small (i.e., countable) regularizing family of maps from a space to an interval. Spaces with countable regularizing families are metrizable and embed in the Hilbert cube. Next we investigate the potential that arises from considering the largest possible regularizing family on a space – the collection of all maps from the space to the closed unit interval.

Recall that a *compactification* of a topological space X is a pair (Y, e) such that Y is a compact Hausdorff space, $e : X \to Y$ is an embedding and e(X) is a dense subset of Y. We will first consider the smallest possible compactification of a space – the so-called *one-point compactification.* This investigation will be a warm-up for an exploration of the largest possible compactification of a space – the so-called *Stone-Čech compactification*. The construction of the Stone-Čech compactification of a space starts by considering the largest possible regularizing family on the space.

Definition. Let X be a topological space. A compactification (Y, e) of X is a *one-point compactification* if Y – e(X) is a one-point set.

Theorem V.26. Let X be a topological space.

a) X has a one-point compactification if and only if X is a non-compact locally compact Hausdorff space.

b) If (Y_1, e_1) and (Y_2, e_2) are both one-point compactifications of X, then there is a unique homeomorphism $h : Y_1 \to Y_2$ such that $h \circ e_1 = e_2$.

c) If (Y, e) is a one-point compactification of X and (Y´, e´) is any compactification of X, then there is a unique map $f : Y' \to Y$ such that $h \circ e' = e$.

Problem V.13. Prove Theorem V.26.

Remark. Theorem V.26.a clarifies the existence of one-point compactifications. Theorem V.26.b asserts the uniqueness of one-point compactifications. Theorem V.26.c expresses the fact that one-point compactifications are the smallest possible compactifications in the sense that all other compactifications map onto them.

Definition. Let X be a topological space. A compactification (C, e) of X is a *Stone-Čech compactification* of X if for every map f : X → D from X to a compact Hausdorff space D, there is a unique map g : C → D such that g∘e = f.

Definition. Let X be a completely regular T_1 space. Let $C(X,\mathbb{I})$ denote the set of all maps from X to $\mathbb{I}$. Since X is completely regular, then $C(X,\mathbb{I})$ is a regularizing family for X. Hence, Corollary V.19 defines an embedding $e : X \to \mathbb{I}^{C(X,\mathbb{I})}$. $\mathbb{I}^{C(X,\mathbb{I})}$ is compact and Hausdorff by Theorems V.10.b and V.17 (the Tychonoff Theorem). Let $\beta(X)$ denote the closure of e(X) in $\mathbb{I}^{C(X,\mathbb{I})}$. Then $\beta(X)$ is compact and Hausdorff by Theorems I.27.d and III.2. Hence, ($\beta(X)$, e) is a compactification of X.

Theorem V.27. The Existence of the Stone-Čech Compactifications. If X is a completely regular T_1 space, then ($\beta(X)$, e) is a Stone-Čech compactification of X.

Proof. Let f : X → D be a map from X to a compact Hausdorff space D. Define the function $f^* : C(D,\mathbb{I}) \to C(X,\mathbb{I})$ by the formula $f^*(\psi) = \psi \circ f$ for each $\psi \in C(D,\mathbb{I})$. Next define the function $f^{**} : \mathbb{I}^{C(X,\mathbb{I})} \to \mathbb{I}^{C(D,\mathbb{I})}$ by $f^{**}(y)(\psi) = y(f^*(\psi)) = y(\psi \circ f)$ for every $y \in \mathbb{I}^{C(X,\mathbb{I})}$ and every $\psi \in C(D,\mathbb{I})$.

We will prove that $f^{**} : \mathbb{I}^{C(X,\mathbb{I})} \to \mathbb{I}^{C(D,\mathbb{I})}$ is continuous. For each $\psi \in C(D,\mathbb{I})$, let $\pi_\psi : \mathbb{I}^{C(D,\mathbb{I})} \to \mathbb{I}$ denote projection onto the ψ^{th} coordinate. In other words, for $\psi \in C(D,\mathbb{I})$, $\pi_\psi(z) = z(\psi)$ for each $z \in \mathbb{I}^{C(D,\mathbb{I})}$. According to Theorem V.5, to prove $f^{**} : \mathbb{I}^{C(X,\mathbb{I})} \to \mathbb{I}^{C(D,\mathbb{I})}$ is continuous, it suffices to prove $\pi_\psi \circ f^{**} : \mathbb{I}^{C(X,\mathbb{I})} \to \mathbb{I}$ is continuous for each $\psi \in C(D,\mathbb{I})$. Let $\psi \in C(D,\mathbb{I})$. Then for every $y \in \mathbb{I}^{C(X,\mathbb{I})}$, $\pi_\psi \circ f^{**}(y) = f^{**}(y)(\psi) = y(f^*(\psi)) = y(\psi \circ f) = \pi_{\psi \circ f}(y)$. Hence, $\pi_\psi \circ f^{**} = \pi_{\psi \circ f}$. Since $\psi \circ f \in C(X,\mathbb{I})$ and $\pi_{\psi \circ f} : \mathbb{I}^{C(X,\mathbb{I})} \to \mathbb{I}$ is a continuous function (by Theorem V.2), then $\pi_\psi \circ f^{**} : \mathbb{I}^{C(X,\mathbb{I})} \to \mathbb{I}$ is continuous. The continuity of $f^{**} : \mathbb{I}^{C(X,\mathbb{I})} \to \mathbb{I}^{C(D,\mathbb{I})}$ now follows by Theorem V.5.

The embedding $e : X \to \mathbb{I}^{C(X,\mathbb{I})}$ specified in Corollary V.19 satisfies the equation $e(x)(\varphi) = \varphi(x)$ for each $x \in X$ and each $\varphi \in C(X,\mathbb{I})$. Since D is a compact Hausdorff space, it is T_1 and normal by Theorem I.24.a and Corollary III.6. Hence, Theorem II.13 (Urysohn's Lemma) implies that D is completely regular. Thus, Corollary V.19 provides an embedding $e' : D \to \mathbb{I}^{C(D,\mathbb{I})}$ which satisfies the equation $e'(z)(\psi) = \psi(z)$ for each $z \in D$ and each $\psi \in C(D,\mathbb{I})$.

Next we will prove that $f^{**} \circ e = e' \circ f$. Let $x \in X$. Then $f^{**} \circ e(x)$ and $e' \circ f(x)$ are both elements of $\mathbb{I}^{C(D,\mathbb{I})}$. Let $\psi \in C(D,\mathbb{I})$. Then $f^{**} \circ e(x)(\psi) = f^{**}(e(x))(\psi) = e(x)(\psi \circ f) = \psi \circ f(x) =$

$\psi(f(x))$ and $e'\circ f(x)(\psi) = e'(f(x))(\psi) = \psi(f(x))$. Thus, $f^{**}\circ e(x)(\psi) = e'\circ f(x)(\psi)$ for every $\psi \in C(D,\mathbb{I})$. Therefore, $f^{**}\circ e(x) = e'\circ f(x)$ for every $x \in X$. We conclude that $f^{**}\circ e = e'\circ f$.

Since $f^{**}\circ e = e'\circ f$, then $f^{**}(e(X)) = e'(f(X)) \subset e'(D)$. Therefore, $e(X) \subset (f^{**})^{-1}(e'(D))$. Since D is compact and e' is continuous, then $e'(D)$ is compact by Theorem III.15. Hence, $e'(D)$ is a closed subset of $\mathbb{I}^{C(D,\mathbb{I})}$ by Corollary III.4. Since f^{**} is continuous, then $(f^{**})^{-1}(e'(D))$ is a closed subset of $\mathbb{I}^{C(X,\mathbb{I})}$. Since $e(X) \subset (f^{**})^{-1}(e'(D))$, then it follows that $\beta(X) = cl(e(X)) \subset (f^{**})^{-1}(e'(D))$. Therefore, $f^{**}(\beta(X)) \subset e'(D)$. Since $e' : D \to \mathbb{I}^{C(D,\mathbb{I})}$ is an embedding, then $(e')^{-1} : e'(D) \to D$ is continuous. Therefore a map $g : \beta(X) \to D$ is defined by $g = (e')^{-1}\circ f^{**} | \beta(X)$.

Observe that $g\circ e = ((e')^{-1}\circ f^{**})\circ e = (e')^{-1}\circ(f^{**}\circ e) = (e')^{-1}\circ(e'\circ f) = ((e')^{-1}\circ e')\circ f = f$. Thus, $g\circ e = f$.

It remains to prove the uniqueness of g. Assume $g' : \beta(X) \to D$ is also a map satisfying $g'\circ e = f$ such that $g' \neq g$. Then there is a $y \in \beta(X)$ such that $g(y) \neq g'(y)$. Since D is Hausdorff there are disjoint neighborhoods U and U´ of g(y) and g´(y) in D. Since g and g´ are continuous, then $V = g^{-1}(U) \cap g'^{-1}(U')$ is a neighborhood of y in $\beta(X)$ such that $g(V) \subset U$ and $g'(V) \subset U'$. Since $\beta(X) = cl(e(X))$, then $V \cap e(X) \neq \emptyset$ by Theorem I.16.b. Hence, there is an $x \in X$ such that $e(x) \in V$. Hence, $g\circ e(x) \in U$ and $g'\circ e(x) \in U'$. Since $U \cap U' = \emptyset$, then $g\circ e(x) \neq g'\circ e(x)$. However, $g\circ e(x) = f(x) = g'\circ e(x)$. We have reached a contradiction. We conclude that if $g' : \beta(X) \to D$ is a map satisfying $g'\circ e = f$, then $g' = g$. □

The following lemma and corollary reveal the sense in which the Stone-Čech compactification of a space is larger than every other compactification of the space. It says that the Stone-Čech compactification of a space maps onto every other compactification of the space.

Lemma V.28. If $f : X \to Y$ is a map between topological spaces, (C, e) is a Stone-Čech compactification of X and (D, e´) is a compactification of Y, then there is a unique map $g : C \to D$ such that $g\circ e = e'\circ f$.

Proof. Observe that $e'\circ f : X \to D$ is a map from X to a compact Hausdorff space. Since (C, e) is a Stone-Čech compactification of X, it follow from the definition of "Stone-Čech compactification" that there is a unique map $g : C \to D$ such that $g\circ e = e'\circ f$. □

Corollary V.29. If (C, e) is a Stone-Čech compactification of a topological space X, and (D, e´) is another compactification of X, then there is a unique map $g : C \to D$ such that $g \circ e = e'$.

Proof. Let $f = id_X$ and apply the previous lemma. ☐

Corollary V.30. The Uniqueness of Stone-Čech Compactifications. If (C, e) and (D, e´) are both Stone-Čech compactifications of a topological space X, then there is a unique homeomorphism $g : C \to D$ such that $g \circ e = e'$.

Proof. Corollary V.29 provides unique maps $g : C \to D$ and $h : D \to C$ such that $g \circ e = e'$ and $h \circ e' = e$. We must prove that g is a homeomorphism. To this end, observe that $h \circ g : C \to C$ and $g \circ h : D \to D$ are maps such that $(h \circ g) \circ e = h \circ (g \circ e) = h \circ e' = e$ and $(g \circ h) \circ e' = g \circ (h \circ e') = g \circ e = e'$. Also $id_C : C \to C$ and $id_D : D \to D$ are maps such that $id_C \circ e = e$ and $id_D \circ e' = e'$. Since (C, e) and (D, e´) are Stone-Čech compactifications of X, then maps $G : C \to C$ and $H : D \to D$ satisfying $G \circ e = e$ and $H \circ e' = e'$ are unique. Hence, $h \circ g = id_C$ and $g \circ h = id_D$. Consequently, $g : C \to D$ is a homeomorphism. ☐

Thus, every Stone-Čech compactification of a space X is homeomorphic to $\beta(X)$.