B. The Tychonoff Theorem

The Tychonoff Theorem says that the Cartesian product of every collection of compact spaces is compact. It is one of most important results of set theoretic topology. There are many approaches to the proof, all of which rely on some form of the Axiom of Choice. (In fact, the Tychonoff Theorem is equivalent to the Axiom of Choice.) We will take an approach that invokes Zermelo's Well Ordering Principle.

Before attacking the proof of the Tychonoff Theorem, we establish a useful lemma.

Lemma V.16. Let $\mathscr{B}$ be a basis for a topological space X. Then X is compact if and only if every cover of X by elements of $\mathscr{B}$ has a finite subcover.

Proof. Clearly, if X is compact, then every cover of X by elements of $\mathscr{B}$ has a finite subcover.

Assume every cover of X by elements of $\mathscr{B}$ has a finite subcover. Let $\mathscr{U}$ be any open cover of X. We will prove that $\mathscr{U}$ has a finite subcover. Let

$$\mathscr{C} = \{ B \in \mathscr{B} : B \text{ is contained in an element of } \mathscr{U} \}.$$

We assert that $\mathscr{C}$ covers X. For suppose that $x \in X$. Since $\mathscr{U}$ covers X, there is a $U \in \mathscr{U}$ such that $x \in U$. Then, since $\mathscr{B}$ is a basis for X, there is a $B \in \mathscr{B}$ such that $x \in B \subset U$. Thus, $x \in B \in \mathscr{C}$. Hence, $\mathscr{C}$ covers X.

Since $\mathscr{C}$ is an cover of X by elements of $\mathscr{B}$, then, by hypothesis, there is a finite subset $\{ B_1, B_2, \ldots , B_n \}$ of $\mathscr{C}$ that covers X. For $1 \le i \le n$, since $B_i \in \mathscr{C}$, then it is possible to choose $U_i \in \mathscr{U}$ such that $B_i \subset U_i$. Then $\{ U_1, U_2, \ldots , U_n \}$ is a finite subset of $\mathscr{U}$ that covers X. We conclude that $\mathscr{U}$ has a finite subcover. Thus, X is compact. □

Theorem V.17: The Tychonoff Theorem. The Cartesian product of every collection of compact spaces is compact.

The concept underlying our proof of the Tychonoff Theorem is a generalization of the idea we used to prove that the Cartesian product of finitely many compact spaces is compact. (See the proof of Theorem III.13.) There are other proofs of this theorem based on markedly different ideas. We outline one of these proofs in an Additional Problem.

Proof. Let $\{ X_\gamma : \gamma \in \Gamma \}$ be a collection of compact spaces. We will prove that $\prod_{\gamma \in \Gamma} X_\gamma$ is compact. Since the collection of all restricted open boxes is a basis for the product topology on $\prod_{\gamma \in \Gamma} X_\gamma$, then Lemma V.16 implies that it suffices to prove that every cover of $\prod_{\gamma \in \Gamma} X_\gamma$ by restricted open boxes has a finite subcover. We will proceed by contradiction. Assume that $\mathscr{U}$ is a cover of $\prod_{\gamma \in \Gamma} X_\gamma$ by restricted open boxes that has no finite subcover.

We introduce a simple idea which is useful for working with Cartesian products called the *Subproduct Membership Principle* which will be used three times in this proof. If $A_\gamma \subset X_\gamma$ for each $\gamma \in \Gamma$, then we call the set $\prod_{\gamma \in \Gamma} A_\gamma$ a *subproduct* of $\prod_{\gamma \in \Gamma} X_\gamma$. Hence, a subset A of $\prod_{\gamma \in \Gamma} X_\gamma$ is a subproduct of $\prod_{\gamma \in \Gamma} X_\gamma$ if and only if $A = \prod_{\gamma \in \Gamma} \pi_\gamma(A)$. (Verify!) Observe that every open box in $\prod_{\gamma \in \Gamma} X_\gamma$ is a subproduct of $\prod_{\gamma \in \Gamma} X_\gamma$. In proving the compactness of $\prod_{\gamma \in \Gamma} X_\gamma$, we choose to work with a cover consisting of restricted open boxes rather than arbitrary open sets because open boxes, being subproducts, have a simple structure not found in arbitrary open sets. Now we state:

The Subproduct Membership Principle. Let A be a subproduct of $\prod_{\gamma \in \Gamma} X_\gamma$ and let $x \in \prod_{\gamma \in \Gamma} X_\gamma$. Then $x \in A$ if the following condition holds. There is an $x' \in A$ and a $\Delta \subset \Gamma$ such that $x \mid \Delta = x' \mid \Delta$ and $x(\gamma) \in \pi_\gamma(A)$ for each $\gamma \in \Gamma - \Delta$.

Proof of the Subproduct Membership Principle. Suppose there is an $x' \in A$ and a $\Delta \subset \Gamma$ such that $x \mid \Delta = x' \mid \Delta$ and $x(\gamma) \in \pi_\gamma(A)$ for each $\gamma \in \Gamma - \Delta$. Since $x' \in A$, then for every $\gamma \in \Delta$, $x(\gamma) = x'(\gamma) = \pi_\gamma(x') \in \pi_\gamma(A)$. Hence, $x(\gamma) \in \pi_\gamma(A)$ for every $\gamma \in \Gamma$. Therefore, $x \in \prod_{\gamma \in \Gamma} \pi_\gamma(A) = A$. $\square$

Next we define a family of subsets of $\prod_{\gamma \in \Gamma} X_\gamma$ that plays an essential role in this proof. Suppose $\Delta \subset \Gamma$ and $x \in \prod_{\gamma \in \Delta} X_\gamma$. (Thus, x is a function with domain Δ.) Define

$$\prod(x) = \{ y \in \prod_{\gamma \in \Gamma} X_\gamma : y \mid \Delta = x \}.$$

Thus, $\prod(x)$ is the subset of $\prod_{\gamma \in \Gamma} X_\gamma$ consisting of all $y \in \prod_{\gamma \in \Gamma} X_\gamma$ such that $y(\gamma) = x(\gamma)$ for all $\gamma \in \Delta$ and $y(\gamma)$ ranges freely throughout X_γ for all $\gamma \in \Gamma - \Delta$. Observe that there is a natural identification between $\prod(x)$ and the Cartesian product $\{ x \} \times \prod_{\gamma \in \Gamma - \Delta} X_\gamma$. Furthermore, if we define $A_\gamma = \{ x(\gamma) \}$ for every $\gamma \in \Delta$ and $A_\gamma = X_\gamma$ for every $\gamma \in \Gamma - \Delta$, then $\prod(x)$ equals the subproduct $\prod_{\gamma \in \Gamma} A_\gamma$ of $\prod_{\gamma \in \Gamma} X_\gamma$.

We must consider what this notation means in the degenerate case that $\Delta = \varnothing$ and $x \in \prod_{\gamma \in \Delta} X_\gamma$. In this situation, $x = \varnothing$ because $\varnothing$ is the one and only set that satisfies the definition of a function with empty domain. In this case, every element y of $\prod_{\gamma \in \Gamma} X_\gamma$ satisfies the vacuous restriction $y \mid \Delta = x$. So, in this case, $\prod(x) = \prod_{\gamma \in \Gamma} X_\gamma$.

We now invoke Zermelo's Well Ordering Principle to obtain a well ordering $<$ of Γ.

The goal of the remainder of the proof is the construction of an element z of $\prod_{\gamma\in\Gamma} X_\gamma$ with the property that for each $\beta \in \Gamma$, no finite subset of $\mathscr{U}$ covers $\prod(z \mid (-\infty,\beta])$. Once we have constructed z, we will obtain a contradiction as follows. z lies in an element U of $\mathscr{U}$. We will show that it is possible to choose $\beta \in \Gamma$ so large that $\prod(z \mid (-\infty,\beta])$ is contained in U. Thus a single element of $\mathscr{U}$ covers $\prod(z \mid (-\infty,\beta])$, yielding a contradiction.

We will construct z by "transfinite induction" as follows. For each $\beta \in \Gamma$, we will construct $z_\beta \in \prod_{\gamma\in(-\infty,\beta]} X_\gamma$ so that no finite subset of $\mathscr{U}$ covers $\prod(z_\beta)$, and so that if α and $\beta \in \Gamma$ and $\alpha < \beta$, then $z_\beta \mid (-\infty,\alpha] = z_\alpha$. Then we will define $z \in \prod_{\gamma\in\Gamma} X_\gamma$ by letting $z(\gamma) = z_\gamma(\gamma)$ for each $\gamma \in \Gamma$.

To begin this inductive construction, let $\beta \in \Gamma$ and assume that for each $\alpha \in (-\infty,\beta)$, we have already obtained $z_\alpha \in \prod_{\gamma\in(-\infty,\alpha]} X_\gamma$ so that no finite subset of $\mathscr{U}$ covers $\prod(z_\alpha)$, and so that if α and $\gamma \in (-\infty,\beta)$ and $\alpha < \gamma$, then $z_\gamma \mid (-\infty,\alpha] = z_\alpha$. Since any two z_α's (for $\alpha \in (-\infty,\beta)$) agree on the intersection of their domains, then we can define an element y of $\prod_{\gamma\in(-\infty,\beta)} X_\gamma$ by $y(\alpha) = z_\alpha(\alpha)$ for each $\alpha \in (-\infty,\beta)$. It follows that for each $\alpha \in (-\infty,\beta)$, $y \mid (-\infty,\alpha] = z_\alpha$. Therefore, y has the property that for each $\alpha \in (-\infty,\beta)$, no finite subset of $\mathscr{U}$ covers $\prod(y \mid (-\infty,\alpha])$. (In the case that $\beta = \min(\Gamma)$, $y = \emptyset$ and this paragraph simply reaffirms the fact that no finite subset of $\mathscr{U}$ covers $\prod(\emptyset) = \prod_{\gamma\in\Gamma} X_\gamma$.)

We assert that no finite subset of $\mathscr{U}$ covers $\prod(y)$. (In the case that $\beta = \min(\Gamma)$ and $y = \emptyset$, there is nothing to prove.) For suppose that a finite subset $\{ U_1, U_2, \dots , U_n \}$ of $\mathscr{U}$ covers $\prod(y)$. For $1 \le i \le n$, since U_i is a restricted open box, then there is a finite subset F_i of Γ such that $\pi_\gamma(U_i) = X_\gamma$ for each $\gamma \in \Gamma - F_i$. Let α be the maximal element of the finite set $(F_1 \cup F_2 \cup \dots \cup F_n) \cap (-\infty,\beta)$. Thus, if $\gamma \in (\alpha,\beta)$, then $\pi_\gamma(U_i) = X_\gamma$ for $1 \le i \le n$. We will now argue that $\{ U_1, U_2, \dots , U_n \}$ covers $\prod(y \mid (-\infty,\alpha])$, which contradicts the conclusion of the preceding paragraph. To show that $\{ U_1, U_2, \dots , U_n \}$ covers $\prod(y \mid (-\infty,\alpha])$, let $x \in \prod(y \mid (-\infty,\alpha])$. Then $x \mid (-\infty,\alpha] = y \mid (-\infty,\alpha]$. Define $x' \in \prod_{\gamma\in\Gamma} X_\gamma$ by changing the coordinates $x(\gamma)$ of x so that they agree with $y(\gamma)$ for $\gamma \in (\alpha,\beta)$. In other words, define $x' \in \prod_{\gamma\in\Gamma} X_\gamma$ as follows:

$$x' \mid (-\infty,\alpha] \cup [\beta,\infty) = x \mid (-\infty,\alpha] \cup [\beta,\infty) \quad \text{and}$$

$$x' \mid (\alpha,\beta) = y \mid (\alpha,\beta).$$

Since $x' \mid (-\infty,\alpha] = x \mid (-\infty,\alpha] = y \mid (-\infty,\alpha]$ and $x' \mid (\alpha,\beta) = y \mid (\alpha,\beta)$, then $x' \mid (-\infty,\beta) = y$. Thus, $x' \in \prod(y)$. Therefore, $x' \in U_i$ for some i between 1 and n. To summarize the situation:

- U_i is a subproduct of $\prod_{\gamma\in\Gamma} X_\gamma$ because it is an open box,
- $x' \in U_i$,

- $x \mid (-\infty,\alpha] \cup [\beta,\infty) = x' \mid (-\infty,\alpha] \cup [\beta,\infty)$ and
- $x(\gamma) \in \pi_\gamma(U_i)$ for each $\gamma \in (\alpha,\beta)$ because $\pi_\gamma(U_i) = X_\gamma$ for each $\gamma \in (\alpha,\beta)$.

Hence, the subproduct membership principle implies $x \in U_i$. This proves $\{ U_1, U_2, \dots , U_n \}$ covers $\prod(y \mid (-\infty,\alpha])$. As we noted earlier, we have now reached a contradiction. We are forced to conclude that no finite subset of $\mathscr{U}$ covers $\prod(y)$.

Next we assert that there is an element z_β of $\prod_{\gamma \in (-\infty,\beta]} X_\gamma$ such that $z_\beta \mid (-\infty,\beta) = y$ and no finite subset of $\mathscr{U}$ covers $\prod(z_\beta)$. Assume that this assertions is false. For each $p \in X_\beta$, define the element w_p of $\prod_{\gamma \in (-\infty,\beta]} X_\gamma$ by $w_p \mid (-\infty,\beta) = y$ and $w_p(\beta) = p$. Since we have assumed that our assertion is false, then it follows that for every $p \in X_\beta$, a finite subset of $\mathscr{U}$ covers $\prod(w_p)$. Let $p \in X_\beta$ and let $\{ U_1, U_2, \dots , U_n \}$ be a finite subset of $\mathscr{U}$ that covers $\prod(w_p)$. If one of the U_i's is disjoint from $\prod(w_p)$, then we can delete it from the set $\{ U_1, U_2, \dots , U_n \}$. So we may assume that each U_i intersects $\prod(w_p)$. Thus, if $1 \le i \le n$, then there is an $x \in \prod(w_p) \cap U_i$ which implies that $p = w_p(\beta) = x(\beta) = \pi_\beta(x) \in \pi_\beta(U_i)$. Let $V_p = \pi_\beta(U_1) \cap \pi_\beta(U_2) \cap \dots \cap \pi_\beta(U_n)$. Then V_p is a neighborhood of p in X_β. We will now prove that $\{ U_1, U_2, \dots , U_n \}$ covers $\{ x \in \prod(y) : x(\beta) \in V_p \}$. To this end, suppose $x \in \prod(y)$ and $x(\beta) \in V_p$. Define $x' \in \prod_{\gamma \in \Gamma} X_\gamma$ as follows:

$$x' \mid (-\infty,\beta) \cup (\beta,\infty) \ = \ x \mid (-\infty,\beta) \cup (\beta,\infty)$$

$$x'(\beta) \ = \ p.$$

Then $x' \mid (-\infty,\beta) = x \mid (-\infty,\beta) = y = w_p \mid (-\infty,\beta)$ and $x'(\beta) \ = \ p \ = \ w_p(\beta)$. Hence, $x' \mid (-\infty,\beta] = w_p$. Therefore, $x' \in \prod(w_p)$. Hence, $x' \in U_i$ for some i between 1 and n. To summarized the situation:

- U_i is a subproduct of $\prod_{\gamma \in \Gamma} X_\gamma$ because it is an open box,
- $x' \in U_i$,
- $x \mid (-\infty,\beta) \cup (\beta,\infty) = x' \mid (-\infty,\beta) \cup (\beta,\infty)$ and
- $x(\beta) \in V_p \subset \pi_\beta (U_i)$.

Hence, the subproduct membership principle implies $x \in U_i$. We have proved that $\{ U_1, U_2, \dots , U_n \}$ covers $\{ x \in \prod(y) : x(\beta) \in V_p \}$. Hence, for each $p \in X_\beta$, a finite subset of $\mathscr{U}$ covers $\{ x \in \prod(y) : x(\beta) \in V_p \}$. Since X_β is compact and $\{ V_p : p \in X_\beta \}$ is an open cover of X_β, then there is a finite subset $\{ p(1), p(2), \dots , p(m) \}$ of X_β such that $\{ V_{p(1)}, V_{p(2)}, \dots , V_{p(m)} \}$ covers X_β. Since a finite subset of $\mathscr{U}$ covers $\{ x \in \prod(y) : x(\beta) \in V_{p(i)} \}$ for $1 \le i \le m$, then a finite subset of $\mathscr{U}$ covers $\bigcup_{1 \le i \le m} \{ x \in \prod(y) : x(\beta) \in V_{p(i)} \}$. Since $\{ V_{p(1)}, V_{p(2)}, \dots , V_{p(m)} \}$ covers X_β, then for each $x \in \prod(y)$, $x(\beta) \in V_{p(i)}$ for some i between 1 and m. Hence,

$$\prod(y) \ = \ \bigcup_{1 \le i \le m} \{ x \in \prod(y) : x(\beta) \in V_{p(i)} \}.$$

Consequently, some finite subset of $\mathscr{U}$ covers $\prod(y)$. This contradicts the conclusion of the previous paragraph. It follows that there is an element z_β of $\prod_{\gamma \in (-\infty,\beta]} X_\gamma$ such that $z_\beta \mid (-\infty,\beta) = y$ and no finite subset of $\mathscr{U}$ covers $\prod(z_\beta)$.

Since $z_\beta \mid (-\infty,\beta) = y$ and $y \mid (-\infty,\alpha] = z_\alpha$ for each $\alpha \in (-\infty,\beta)$, then $z_\beta \mid (-\infty,\alpha] = z_\alpha$ for each $\alpha \in (-\infty,\beta)$. Thus, we have accomplished our goal of performing a (transfinitely) inductive construction of elements $z_\beta \in \prod_{\gamma \in (-\infty,\beta]} X_\gamma$ for each $\beta \in \Gamma$ with the following properties:

- no finite subset of $\mathscr{U}$ covers $\prod(z_\beta)$, and
- if α and $\beta \in \Gamma$ and $\alpha < \beta$, then $z_\beta \mid (-\infty,\alpha] = z_\alpha$.

Finally define $z \in \prod_{\gamma \in \Gamma} X_\gamma$ by $z(\beta) = z_\beta(\beta)$ for each $\beta \in \Gamma$. Then for each $\beta \in \Gamma$, $z \mid (-\infty,\beta] = z_\beta$ and, hence, no finite subset of $\mathscr{U}$ covers $\prod(z \mid (-\infty,\beta])$. Since $\mathscr{U}$ covers $\prod_{\gamma \in \Gamma} X_\gamma$, then z lies in some element U of $\mathscr{U}$. Since U is a restricted open box, then there is a finite subset F of Γ such that $\pi_\gamma(U) = X_\gamma$ for each $\gamma \in \Gamma - F$. Let $\beta = \max(F)$. Then $\pi_\gamma(U) = X_\gamma$ for each $\gamma \in (\beta,\infty)$. We now argue that $\prod(z \mid (-\infty,\beta]) \subset U$. To accomplish this, let $x \in \prod(z \mid (-\infty,\beta])$. Then:

- U is a subproduct of $\prod_{\gamma \in \Gamma} X_\gamma$ because it is an open box,
- $z \in U$,
- $x \mid (-\infty,\beta] = z \mid (-\infty,\beta]$ and
- $x(\gamma) \in \pi_\gamma(U)$ for each $\gamma \in (\beta,\infty)$ because $\pi_\gamma(U) = X_\gamma$ for each $\gamma \in (\beta,\infty)$.

Hence, the Subproduct Membership Principle implies $x \in U$. We have proved that $\prod(z \mid (-\infty,\beta]) \subset U$. Thus, a one-element subset of $\mathscr{U}$ covers $\prod(z \mid (-\infty,\beta])$. We have reached our final contradiction. We conclude that a finite subset of $\mathscr{U}$ covers $\prod_{\gamma \in \Gamma} X_\gamma$. In other words, $\mathscr{U}$ has a finite subcover. Since $\mathscr{U}$ is an arbitrary cover of $\prod_{\gamma \in \Gamma} X_\gamma$ by restricted open boxes, then it follows by Lemma V.16 that $\prod_{\gamma \in \Gamma} X_\gamma$ must be compact. □

Problem III.6 Revisited. In Problem III.6, we let $\Sigma = \{ 0, 1 \}^{\mathbb{N}}$, the set of all functions from $\mathbb{N}$ to $\{ 0, 1 \}$, and we defined a topology on the space $\{ 0, 1 \}^\Sigma$ of all functions from Σ to $\{ 0, 1 \}$. This topology is determined by specifying a basis $\mathscr{B}$ which consists of all sets of the form

$$N(f,A) = \{ g \in \{ 0, 1 \}^\Sigma : g|A = f|A \}$$

where $f \in \{ 0, 1 \}^\Sigma$ and A is a finite subset of Σ. In the statement of Problem III.6, we asserted that this topology makes $\{ 0, 1 \}^\Sigma$ compact. We now justify this assertion by using the Tychonoff Theorem.

Proof. We will show that $\{ 0, 1 \}^{\Sigma}$ is a Cartesian product and $\mathscr{B}$ is the collection of all restricted open boxes in this Cartesian product. It will then follow that the topology we have assigned to $\{ 0, 1 \}^{\Sigma}$ is simply the product topology.

To begin, let $X_\sigma = \{ 0, 1 \}$ for each $\sigma \in \Sigma$. Then clearly $\{ 0, 1 \}^{\Sigma} = \prod_{\sigma \in \Sigma} X_\sigma$.

Next consider an $N(f,A) \in \mathscr{B}$ where $f \in \{ 0, 1 \}^{\Sigma}$ and A is a finite subset of Σ. Then N(f,A) is clearly equal to the restricted open box $\prod_{\sigma \in \Sigma} U_\sigma$ where $U_\sigma = \{ f(\sigma) \}$ if $\sigma \in A$ and $U_\sigma = X_\sigma$ if $\sigma \in \Sigma - A$. (Verify!) Thus, every element of $\mathscr{B}$ is a restricted open box.

Now consider a restricted open box $\prod_{\sigma \in \Sigma} U_\sigma$ in $\prod_{\sigma \in \Sigma} X_\sigma = \{ 0, 1 \}^{\Sigma}$. Then the set $A = \{ \sigma \in \Sigma : U_\sigma \neq X_\sigma \}$ is finite, and U_σ is a one-point subset of $X_\sigma = \{ 0, 1 \}$ for each $\sigma \in A$. Thus, we can define function $f \in \{ 0, 1 \}^{\Sigma}$ by specifying that $f(\sigma) \in U_\sigma$ for each $\sigma \in A$ and $f(\sigma) = 0$ for each $\sigma \in \Sigma - A$. Then clearly $\prod_{\sigma \in \Sigma} U_\sigma = N(f,A)$. (Verify!) Hence, every restricted open box in $\{ 0, 1 \}^{\Sigma}$ is an element of $\mathscr{B}$.

We have shown that the basis $\mathscr{B}$ for the topology on $\{ 0, 1 \}^{\Sigma}$ is precisely the collection of all restricted open boxes in $\{ 0, 1 \}^{\Sigma}$. Consequently, the topology we have defined on the Cartesian product $\{ 0, 1 \}^{\Sigma}$ is the product topology. Since each factor $X_\sigma = \{ 0, 1 \}$ of this Cartesian product is compact, then the Tychonoff Theorem implies that $\{ 0, 1 \}^{\Sigma}$ is compact. This completes our justification of the assertion made in the statement of Problem III.6. ◻

Definition. For each $n \in \mathbb{N}$, let X_n be a topological space and let $f_n : X_{n+1} \to X_n$ be a map. Then the countable collection $\{ (X_n, f_n) : n \in \mathbb{N} \}$ is called an *inverse sequence* of topological spaces and maps. This collection is usually denoted more briefly as $\{ X_n, f_n : n \in \mathbb{N} \}$. The *inverse limit* of $\{ X_n, f_n : n \in \mathbb{N} \}$ is the subspace of the Cartesian the Cartesian product $\prod_{n \in \mathbb{N}} X_n$ (with the product topology) defined by

$$\varprojlim \{ X_n, f_n \} = \{ x \in \textstyle\prod_{n \in \mathbb{N}} X_n : f_n(x(n+1)) = x(n) \text{ for every } n \in \mathbb{N} \}.$$

Problem V.9. a) Suppose $\{ X_n, f_n : n \in \mathbb{N} \}$ is an inverse sequence such that each $f_n : X_{n+1} \to X_n$ is a homeomorphism. Prove: $\varprojlim \{ X_n, f_n \}$ is homeomorphic to X_1.

b) For each $n \in \mathbb{N}$, let $Y_n = [0,\infty)$ and define the map $g_n : Y_{n+1} \to Y_n$ by $g_n(y) = y + 1$ for each $x \in Y_{n+1}$. Prove: $\varprojlim \{ Y_n, g_n \}$ is homeomorphic to a familiar space.

c) Prove: if $\{ X_n, f_n : n \in \mathbb{N} \}$ is an inverse sequence of non-empty compact Hausdorff spaces and maps, then $\varprojlim \{ X_n, f_n \}$ is a non-empty compact Hausdorff space.