V. Product Spaces

A. Fundamental Properties

Definition. For sets X and Y, recall that Y^X denotes the set of all functions from X to Y. The *Cartesian product* of an indexed collection of sets $\{ X_\gamma : \gamma \in \Gamma \}$, denoted $\prod_{\gamma\in\Gamma} X_\gamma$, is the set the set of all functions $x : \Gamma \to \bigcup_{\gamma\in\Gamma} X_\gamma$ such that $x(\gamma) \in X_\gamma$ for every $\gamma \in \Gamma$. Thus, $\prod_{\gamma\in\Gamma} X_\gamma \subset (\bigcup_{\gamma\in\Gamma} X_\gamma)^\Gamma$. An element x of $\prod_{\gamma\in\Gamma} X_\gamma$ can be thought of as a "Γ-tuple" or a "Γ-indexed sequence" $(x_\gamma)_{\gamma\in\Gamma}$ where $x_\gamma = x(\gamma)$ for each $\gamma \in \Gamma$.

We make two observations about the notation for Cartesian products.

1) To define the Cartesian product of a collection of sets, it is not necessary that the collection be indexed. Indeed, we can define the Cartesian product of an unindexed collection of sets $\mathscr{A}$ by letting $\mathscr{A}$ itself play the role of the index set. Specifically, if $\mathscr{A}$ is an unindexed collection of sets, then we define the *Cartesian product* of $\mathscr{A}$ to be the set

$$\prod\mathscr{A} = \{ x \in (\bigcup\mathscr{A})^{\mathscr{A}} : x(A) \in A \text{ for every } A \in \mathscr{A} \}.$$

2) If X and Y are sets, then Y^X is a Cartesian product. Indeed, if we set $Y_x = Y$ for every $x \in X$, then $Y^X = \prod_{x\in X} Y_x$.

Definition. Let $\{ X_\gamma : \gamma \in \Gamma \}$ be a collection of topological spaces. An *open box* in $\prod_{\gamma\in\Gamma} X_\gamma$ is a subset of $\prod_{\gamma\in\Gamma} X_\gamma$ of the form $\prod_{\gamma\in\Gamma} U_\gamma$ where U_γ is an open subset of X_γ for each $\gamma \in \Gamma$. An open box $\prod_{\gamma\in\Gamma} U_\gamma$ is *restricted* if $U_\gamma = X_\gamma$ for all but finitely many $\gamma \in \Gamma$.

Lemma V.1. Let $\{ X_\gamma : \gamma \in \Gamma \}$ be a collection of topological spaces, let $\mathscr{O}$ denote the collection of all open boxes in $\prod_{\gamma\in\Gamma} X_\gamma$, and let $\mathscr{O}_r$ denote the collection of all restricted open boxes in $\prod_{\gamma\in\Gamma} X_\gamma$. Then both $\mathscr{O}$ and $\mathscr{O}_r$ are bases for topologies on $\prod_{\gamma\in\Gamma} X_\gamma$.

Proof. According to the Corollary to Theorem I.2, it suffices to prove that each of $\mathscr{O}$ and $\mathscr{O}_r$ covers $\prod_{\gamma\in\Gamma} X_\gamma$ and is closed under the formation of finite intersections. Since $\prod_{\gamma\in\Gamma} X_\gamma$ is an element of both $\mathscr{O}$ and $\mathscr{O}_r$, then both of these collections cover $\prod_{\gamma\in\Gamma} X_\gamma$. Next suppose $A = \prod_{\gamma\in\Gamma} U_\gamma$ and $B = \prod_{\gamma\in\Gamma} V_\gamma \in \mathscr{O}$. Since $A \cap B = \prod_{\gamma\in\Gamma} (U_\gamma \cap V_\gamma)$, then clearly $A \cap B \in \mathscr{O}$. Finally suppose $A = \prod_{\gamma\in\Gamma} U_\gamma$ and $B = \prod_{\gamma\in\Gamma} V_\gamma$ are elements of $\mathscr{O}_r$. Then the sets $F = \{ \gamma \in \Gamma : U_\gamma \neq X_\gamma \}$ and $G = \{ \gamma \in \Gamma : V_\gamma \neq X_\gamma \}$ are finite. Observe that the set $\{ \gamma \in \Gamma : U_\gamma \cap V_\gamma \neq X_\gamma \}$ is equal to $F \cup G$. Hence, $\{ \gamma \in \Gamma : U_\gamma \cap V_\gamma \neq X_\gamma \}$ is a finite set. Since $A \cap B = \prod_{\gamma\in\Gamma} (U_\gamma \cap V_\gamma)$, then it follows that $A \cap B \in \mathscr{O}_r$. This proves that both $\mathscr{O}$ and $\mathscr{O}_r$ are closed under the formation of finite intersections. Consequently, both $\mathscr{O}$ and $\mathscr{O}_r$ are bases for topologies on $\prod_{\gamma\in\Gamma} X_\gamma$. ◻

Definition. Let $\{ X_\gamma : \gamma \in \Gamma \}$ be a collection of topological spaces. The set of all restricted open boxes in $\prod_{\gamma \in \Gamma} X_\gamma$ is a basis for a topology on $\prod_{\gamma \in \Gamma} X_\gamma$ called the *product topology*. The set of all open boxes (restricted and unrestricted) in $\prod_{\gamma \in \Gamma} X_\gamma$ is a basis for a topology on $\prod_{\gamma \in \Gamma} X_\gamma$ called the *box topology*.

Convention. From now on, if $\{ X_\gamma : \gamma \in \Gamma \}$ is a collection of topological spaces, then the Cartesian product $\prod_{\gamma \in \Gamma} X_\gamma$ will be assigned the product topology, unless otherwise specified.

If the collection of sets $\{ X_\gamma : \gamma \in \Gamma \}$ is finite, then the product and box topologies on $\prod_{\gamma \in \Gamma} X_\gamma$ coincide. However, if $\{ X_\gamma : \gamma \in \Gamma \}$ is an infinite collection, then the product and box topologies are different, and the box topology is, in general, less well behaved than the product topology. This is illustrated by some of the Additional Problems. For example, the Cartesian product of any collection of connected spaces with the product topology is connected, while the Cartesian product of an infinite collection of connected spaces with the box topology might not be connected. For this reason, the box topology is not commonly used for Cartesian products of infinite collections of topological spaces except to construct exotic examples, and it will not be employed in the remainder of these lessons unless otherwise specified.

Definition. Let $\{ X_\gamma : \gamma \in \Gamma \}$ be a collection of topological spaces. For each $\beta \in \Gamma$, a function $\pi_\beta : \prod_{\gamma \in \Gamma} X_\gamma \to X_\beta$, called the *$\beta^{th}$ projection*, is defined by $\pi_\beta(x) = x(\beta)$ for $x \in \prod_{\gamma \in \Gamma} X_\gamma$.

Theorem V.2. If $\{ X_\gamma : \gamma \in \Gamma \}$ is a collection of topological spaces. then for each $\beta \in \Gamma$, the projection $\pi_\beta : \prod_{\gamma \in \Gamma} X_\gamma \to X_\beta$ is a continuous and open map.

Proof. Let $\beta \in \Gamma$.

To prove the continuity of π_β, let U be an open subset of X_β. Then, $\pi_\beta^{-1}(U) = \prod_{\gamma \in \Gamma} V_\gamma$ where $V_\beta = U$ and $V_\gamma = X_\gamma$ for all $\gamma \in \Gamma - \{ \beta \}$. Thus, $\pi_\beta^{-1}(U)$ is a restricted open box in $\prod_{\gamma \in \Gamma} X_\gamma$. Therefore, $\pi_\beta^{-1}(U)$ is an open set. This proves π_β is continuous.

To prove that π_β is an open map, let V be an open subset of $\prod_{\gamma \in \Gamma} X_\gamma$ and let $y \in \pi_\beta(V)$. Then $y = \pi_\beta(x)$ for some $x \in V$. Since $x \in V$ and V is an open subset of $\prod_{\gamma \in \Gamma} X_\gamma$, then there is a restricted open box $\prod_{\gamma \in \Gamma} U_\gamma$ in $\prod_{\gamma \in \Gamma} X_\gamma$ such that $x \in \prod_{\gamma \in \Gamma} U_\gamma \subset V$. Hence, $\pi_\beta(x) \in \pi_\beta(\prod_{\gamma \in \Gamma} U_\gamma) \subset \pi_\beta(V)$. Since $\pi_\beta(x) = y$ and $\pi_\beta(\prod_{\gamma \in \Gamma} U_\gamma) = U_\beta$, then $y \in U_\beta \subset \pi_\beta(V)$. We have proved that every element y of $\pi_\beta(V)$ is contained in an open subset U_β of X_β such that $U_\beta \subset \pi_\beta(V)$. It follows that $\pi_\beta(V)$ is an open subset of X_β. This proves $\pi_\beta : \prod_{\gamma \in \Gamma} X_\gamma \to X_\beta$ is an open map. $\square$

Lemma V.3. Let $\{ X_\gamma : \gamma \in \Gamma \}$ be a collection of topological spaces. If $A_\gamma \subset X_\gamma$ for each $\gamma \in \Gamma$, and if $F = \{ \gamma \in \Gamma : A_\gamma \neq X_\gamma \}$, then

$$\prod_{\gamma \in \Gamma} A_\gamma \; = \; \cap_{\gamma \in F} \pi_\gamma^{-1}(A_\gamma).$$

Hence, every restricted open box in $\prod_{\gamma \in \Gamma} X_\gamma$ is of the form $\cap_{\gamma \in F} \pi_\gamma^{-1}(U_\gamma)$ where F is a finite subset of Γ and U_γ is an open subset of X_γ for each $\gamma \in \Gamma$.

Proof. Proving the equation $\prod_{\gamma \in \Gamma} A_\gamma = \cap_{\gamma \in F} \pi_\gamma^{-1}(A_\gamma)$ is a set theoretic exercise which we leave to the student. The final sentence of this lemma is an immediate consequence of this equation. ◻

Theorem V.4. If $\{ X_\gamma : \gamma \in \Gamma \}$ is a collection of topological spaces. then the product topology on $\prod_{\gamma \in \Gamma} X_\gamma$ is the smallest topology on $\prod_{\gamma \in \Gamma} X_\gamma$ with the property that for every $\beta \in \Gamma$, $\pi_\beta : \prod_{\gamma \in \Gamma} X_\gamma \to X_\beta$ continuous.

Proof. Let $\mathscr{T}$ be a topology on $\prod_{\gamma \in \Gamma} X_\gamma$ with the property that if $\prod_{\gamma \in \Gamma} X_\gamma$ is given the topology $\mathscr{T}$, then $\pi_\beta : \prod_{\gamma \in \Gamma} X_\gamma \to X_\beta$ is continuous for every $\beta \in \Gamma$. We must prove that $\mathscr{T}$ contains the product topology on $\prod_{\gamma \in \Gamma} X_\gamma$.

First we prove that $\mathscr{T}$ contains all the restricted open boxes in $\prod_{\gamma \in \Gamma} X_\gamma$. Let $\prod_{\gamma \in \Gamma} U_\gamma$ be a restricted open box in $\prod_{\gamma \in \Gamma} X_\gamma$. Let $F = \{ \gamma \in \Gamma : U_\gamma \neq X_\gamma \}$. Then F is a finite subset of Γ and $\prod_{\gamma \in \Gamma} U_\gamma = \cap_{\gamma \in F} \pi_\gamma^{-1}(U_\gamma)$ (by Lemma V.3). For each $\gamma \in F$, since π_γ is continuous with respect to $\mathscr{T}$, then $\pi_\gamma^{-1}(U_\gamma) \in \mathscr{T}$. Since $\mathscr{T}$, being a topology, is closed under the formation of finite intersections, then $\prod_{\gamma \in \Gamma} U_\gamma = \cap_{\gamma \in F} \pi_\gamma^{-1}(U_\gamma) \in \mathscr{T}$. Thus, $\mathscr{T}$ contains all restricted open boxes.

Now let V be any element of the product topology on $\prod_{\gamma \in \Gamma} X_\gamma$. Since the restricted open boxes form a basis for the product topology, then V is a union of restricted open boxes. Since $\mathscr{T}$ contains all restricted open boxes and is closed under the formation of arbitrary unions, then $V \in \mathscr{T}$. Thus, $\mathscr{T}$ contains the product topology on $\prod_{\gamma \in \Gamma} X_\gamma$. ◻

The following theorem gives us a useful criterion for the continuity of a map into a Cartesian product $\prod_{\gamma \in \Gamma} Y_\gamma$ (with the product topology). The function $f : X \to \prod_{\gamma \in \Gamma} Y_\gamma$ is continuous if and only if for each $\gamma \in \Gamma$, the component function $\pi_\gamma \circ f : X \to Y_\gamma$ is continuous.

Theorem V.5. Let X be a topological space, and let $\{ Y_\gamma : \gamma \in \Gamma \}$ be a collection of topological spaces. Then a function $f : X \to \prod_{\gamma\in\Gamma} Y_\gamma$ is continuous if and only if $\pi_\gamma \circ f : X \to Y_\gamma$ is continuous for every $\gamma \in \Gamma$.

Proof. Clearly, if $f : X \to \prod_{\gamma\in\Gamma} Y_\gamma$ is continuous, then $\pi_\gamma \circ f : X \to Y_\gamma$ is continuous for every $\gamma \in \Gamma$.

Now assume $\pi_\gamma \circ f : X \to Y_\gamma$ is continuous for every $\gamma \in \Gamma$. To prove that $f : X \to \prod_{\gamma\in\Gamma} Y_\gamma$ is continuous, let $x \in X$ and let U be a neighborhood of f(x) in $\prod_{\gamma\in\Gamma} Y_\gamma$. Then there is a restricted open box $\prod_{\gamma\in\Gamma} V_\gamma$ in $\prod_{\gamma\in\Gamma} Y_\gamma$ such that $f(x) \in \prod_{\gamma\in\Gamma} V_\gamma \subset U$. Let $W = f^{-1}(\prod_{\gamma\in\Gamma} V_\gamma)$. Hence, $x \in W$ and $f(W) \subset \prod_{\gamma\in\Gamma} V_\gamma \subset U$. Let $F = \{ \gamma \in \Gamma : V_\gamma \neq Y_\gamma \}$. Then F is a finite subset of Γ and $\prod_{\gamma\in\Gamma} V_\gamma = \cap_{\gamma\in F} \pi_\gamma^{-1}(V_\gamma)$ (by Lemma V.3). Hence, W = $f^{-1}(\prod_{\gamma\in\Gamma} V_\gamma) = f^{-1}(\cap_{\gamma\in F} \pi_\gamma^{-1}(V_\gamma)) = \cap_{\gamma\in F} f^{-1}(\pi_\gamma^{-1}(V_\gamma)) = \cap_{\gamma\in F} (\pi_\gamma \circ f)^{-1}(V_\gamma)$. For each $\gamma \in F$, since $\pi_\gamma \circ f$ is continuous, then $(\pi_\gamma \circ f)^{-1}(V_\gamma)$ is an open subset of X. Since F is a finite set, it follows that $\cap_{\gamma\in F} (\pi_\gamma \circ f)^{-1}(U_\gamma)$ is an open subset of X. Thus, W is an open subset of X such that $x \in W$ and $f(W) \subset U$. This proves the continuity of $f : X \to \prod_{\gamma\in\Gamma} Y_\gamma$. □

Theorem V.6. Let X be a topological space, and let $\{ Y_\gamma : \gamma \in \Gamma \}$ be a collection of topological spaces. Then a function $f : X \to \prod_{\gamma\in\Gamma} Y_\gamma$ is continuous if and only if $\pi_\gamma \circ f : X \to Y_\gamma$ is continuous for every $\gamma \in \Gamma$.

Proof. Clearly, if $f : X \to \prod_{\gamma\in\Gamma} Y_\gamma$ is continuous, then $\pi_\gamma \circ f : X \to Y_\gamma$ is continuous for every $\gamma \in \Gamma$.

Assume $\pi_\gamma \circ f : X \to Y_\gamma$ is continuous for every $\gamma \in \Gamma$. To prove that $f : X \to \prod_{\gamma\in\Gamma} Y_\gamma$ is continuous, let $x \in X$ and let U be a neighborhood of f(x) in $\prod_{\gamma\in\Gamma} Y_\gamma$. Then there is a restricted open box $\prod_{\gamma\in\Gamma} V_\gamma$ in $\prod_{\gamma\in\Gamma} Y_\gamma$ such that $f(x) \in \prod_{\gamma\in\Gamma} V_\gamma \subset U$. Let W = $f^{-1}(\prod_{\gamma\in\Gamma} V_\gamma)$. Hence, $x \in W$ and $f(W) \subset \prod_{\gamma\in\Gamma} V_\gamma \subset U$. Let $F = \{ \gamma \in \Gamma : V_\gamma \neq Y_\gamma \}$. Then F is a finite subset of Γ and $\prod_{\gamma\in\Gamma} V_\gamma = \cap_{\gamma\in F} \pi_\gamma^{-1}(V_\gamma)$ (by Lemma V.3.a). Hence, W = $f^{-1}(\cap_{\gamma\in F} \pi_\gamma^{-1}(V_\gamma)) = \cap_{\gamma\in F} f^{-1}(\pi_\gamma^{-1}(V_\gamma)) = \cap_{\gamma\in F} (\pi_\gamma \circ f)^{-1}(V_\gamma)$. For each $\gamma \in F$, since $\pi_\gamma \circ f$ is continuous, then $(\pi_\gamma \circ f)^{-1}(V_\gamma)$ is an open subset of X. Since F is a finite set, it follows that $\cap_{\gamma\in F} (\pi_\gamma \circ f)^{-1}(U_\gamma)$ is an open subset of X. Thus, W is an open subset of X such that $x \in W$ and $f(W) \subset U$. This proves the continuity of $f : X \to \prod_{\gamma\in\Gamma} Y_\gamma$. □

Theorem V.7. Let $\{ X_\gamma : \gamma \in \Gamma \}$ be a collection of topological spaces. Then a sequence $\{ x_n \}$ in $\prod_{\gamma\in\Gamma} X_\gamma$ converges to a point $y \in \prod_{\gamma\in\Gamma} X_\gamma$ if and only if for every $\gamma \in \Gamma$, the sequence $\{ x_n(\gamma) \}$ converges to $y(\gamma)$ in X_γ.

Proof. First assume that the sequence $\{ x_n \}$ in $\prod_{\gamma\in\Gamma} X_\gamma$ converges to the point $y \in \prod_{\gamma\in\Gamma} X_\gamma$. Let $\beta \in \Gamma$. Since $\pi_\beta : \prod_{\gamma\in\Gamma} X_\gamma \to X_\beta$ is continuous, and continuous functions preserve sequential convergence (by Theorem II.7), then $\{ \pi_\beta(x_n) \}$ converges to $\pi_\beta(y)$. Since $\pi_\beta(x_n) = x_n(\beta)$ for $n \geq 1$ and $\pi_\beta(y) = y(\beta)$, then it follows that $\{ x_n(\beta) \}$ converges to $y(\beta)$.

Second assume that $\{ x_n \}$ is a sequence in $\prod_{\gamma\in\Gamma} X_\gamma$ and $y \in \prod_{\gamma\in\Gamma} X_\gamma$ such that $\{ x_n(\gamma) \}$ converges to $y(\gamma)$ in X_γ for every $\gamma \in \Gamma$. To prove that $\{ x_n \}$ converges to y in $\prod_{\gamma\in\Gamma} X_\gamma$, let U be a neighborhood of y in $\prod_{\gamma\in\Gamma} X_\gamma$. Then there is a restricted open box $\prod_{\gamma\in\Gamma} V_\gamma$ in $\prod_{\gamma\in\Gamma} X_\gamma$ such that $y \in \prod_{\gamma\in\Gamma} V_\gamma \subset U$. Let $F = \{ \gamma \in \Gamma : V_\gamma \neq Y_\gamma \}$. Then F is a finite subset of Γ and $\prod_{\gamma\in\Gamma} V_\gamma = \cap_{\gamma\in F} \pi_\gamma^{-1}(V_\gamma)$ (by Lemma V.3.a). Thus, $y \in \cap_{\gamma\in F} \pi_\gamma^{-1}(V_\gamma) \subset U$. Let $\gamma \in F$. Then $y \in \pi_\gamma^{-1}(V_\gamma)$. Therefore, $y(\gamma) = \pi_\gamma(y) \in V_\gamma$. Since $\{ x_n(\gamma) \}$ converges to $y(\gamma)$ in X_γ, then there is an $N_\gamma \geq 1$ such that $x_n(\gamma) \in V_\gamma$ for each $n \geq N_\gamma$. Since $\pi_\gamma(x_n) = x_n(\gamma)$ for $n \geq 1$, then $\pi_\gamma(x_n) \in V_\gamma$ for each $n \geq N_\gamma$. Hence, $x_n \in \pi_\gamma^{-1}(V_\gamma)$ for each $n \geq N_\gamma$. Let $N = \max \{ N_\gamma : \gamma \in F \}$. Then for each $\gamma \in F$, $x_n \in \pi_\gamma^{-1}(V_\gamma)$ for each $n \geq N$. Thus, $x_n \in \cap_{\gamma\in F} \pi_\gamma^{-1}(V_\gamma)$ for each $n \geq N$. Since $\cap_{\gamma\in F} \pi_\gamma^{-1}(V_\gamma) = \prod_{\gamma\in\Gamma} V_\gamma \subset U$, then $x_n \in U$ for each $n \geq N$. This proves $\{ x_n \}$ converges to y in $\prod_{\gamma\in\Gamma} X_\gamma$. ◻

Observe that Theorem V.7 implies that if X is a set and Y is a topological space, then a sequence $\{ f_n \}$ in Y^X converges $g \in Y^X$ if and only the sequence $\{ f_n(x) \}$ converges to g(x) in Y for every $x \in X$. This reveals why the product topology is also called the "topology of pointwise convergence".

Lemma V.8. If $\{ X_\gamma : \gamma \in \Gamma \}$ is a collection of topological spaces, and if C_γ is a closed subset of X_γ for each $\gamma \in \Gamma$, then $\prod_{\gamma\in\Gamma} C_\gamma$ is a closed subset of $\prod_{\gamma\in\Gamma} X_\gamma$.

Proof. We assert that $(\prod_{\gamma\in\Gamma} X_\gamma) - (\prod_{\gamma\in\Gamma} C_\gamma) = \cup_{\gamma\in\Gamma} \pi_\gamma^{-1}(X_\gamma - C_\gamma)$. Indeed, if $x \in (\prod_{\gamma\in\Gamma} X_\gamma) - (\prod_{\gamma\in\Gamma} C_\gamma)$, then $x(\beta) \notin C_\beta$ for some $\beta \in \Gamma$. Then $\pi_\beta(x) = x(\beta) \in X_\beta - C_\beta$. So $x \in \pi_\beta^{-1}(X_\beta - C_\beta) \subset \cup_{\gamma\in\Gamma} \pi_\gamma^{-1}(X_\gamma - C_\gamma)$. This proves $(\prod_{\gamma\in\Gamma} X_\gamma) - (\prod_{\gamma\in\Gamma} C_\gamma) \subset \cup_{\gamma\in F} \pi_\gamma^{-1}(X_\gamma - C_\gamma)$. On the other hand, if $x \in \cup_{\gamma\in F} \pi_\gamma^{-1}(X_\gamma - C_\gamma)$, then $x \in \pi_\beta^{-1}(X_\beta - C_\beta)$ for some $\beta \in \Gamma$. So $\pi_\beta(x) \in X_\beta - C_\beta$. Therefore, $\pi_\beta(x) \notin C_\beta = \pi_\beta(\prod_{\gamma\in\Gamma} C_\gamma)$. Hence, $x \notin \prod_{\gamma\in\Gamma} C_\gamma$. Thus, $x \in (\prod_{\gamma\in\Gamma} X_\gamma) - (\prod_{\gamma\in\Gamma} C_\gamma)$. This proves $\cup_{\gamma\in F} \pi_\gamma^{-1}(X_\gamma - C_\gamma) \subset (\prod_{\gamma\in\Gamma} X_\gamma) - (\prod_{\gamma\in\Gamma} C_\gamma)$. Our assertion follows.

Since each π_γ is continuous, then each $\pi_\gamma^{-1}(X_\gamma - C_\gamma)$ is an open subset of $\prod_{\gamma\in\Gamma} X_\gamma$. Hence, $\cup_{\gamma\in\Gamma} \pi_\gamma^{-1}(X_\gamma - C_\gamma)$ is an open subset of $\prod_{\gamma\in\Gamma} X_\gamma$. Thus, the assertion in the preceding paragraph implies $(\prod_{\gamma\in\Gamma} X_\gamma) - (\prod_{\gamma\in\Gamma} C_\gamma)$ is an open subset of $\prod_{\gamma\in\Gamma} X_\gamma$. Consequently, $\prod_{\gamma\in\Gamma} C_\gamma$ is a closed subset of $\prod_{\gamma\in\Gamma} X_\gamma$. ◻

Definition. Let $\{ X_\gamma : \gamma \in \Gamma \}$ be a collection of topological spaces. For each $a \in \prod_{\gamma \in \Gamma} X_\gamma$ and each $\beta \in \Gamma$, a function $e_{a,\beta} : X_\beta \to \prod_{\gamma \in \Gamma} X_\gamma$, called a *$\beta^{th}$ injection function*, is defined as follows. For each $x \in X_\beta$, we determine the point $e_{a,\beta}(x) \in \prod_{\gamma \in \Gamma} X_\gamma$ by specifying its coordinates by the conditions:

$$(e_{a,\beta}(x))(\beta) = x \quad \text{and} \quad (e_{a,\beta}(x))(\gamma) = a(\gamma) \text{ for every } \gamma \in \Gamma - \{ \beta \}.$$

Theorem V.9. Let $\{ X_\gamma : \gamma \in \Gamma \}$ be a collection of topological spaces. Then for each $a \in \prod_{\gamma \in \Gamma} X_\gamma$ and each $\beta \in \Gamma$, $\pi_\beta \circ e_{a,\beta} = id_{X_\beta}$ and $e_{a,\beta} : X_\beta \to \prod_{\gamma \in \Gamma} X_\gamma$ is an embedding.

Problem V.1. Prove Theorem V.9.

Theorem V.10. Let $\{ X_\gamma : \gamma \in \Gamma \}$ be a collection of topological spaces. Then:

a) $\prod_{\gamma \in \Gamma} X_\gamma$ is T_1 if and only if each X_γ is T_1.

b) $\prod_{\gamma \in \Gamma} X_\gamma$ is Hausdorff if and only if each X_γ is Hausdorff.

c) $\prod_{\gamma \in \Gamma} X_\gamma$ is regular if and only if each X_γ is regular.

d) If $\prod_{\gamma \in \Gamma} X_\gamma$ is normal, then each X_γ is normal.

Problem V.2. Prove Theorem V.10.

The converse of Theorem V.10.d is false. Indeed, recall the results of Problems I.15(7) and I.22: $\mathbb{R}_{bad}$ is normal, but $\mathbb{R}_{bad} \times \mathbb{R}_{bad}$ is not.

The Cartesian product of countably many second countable spaces is second countable. Similary, the Cartesian product of countably many first countable spaces is first countable, and the Cartesian product of countably many separable spaces is separable. The following lemma can be used as a tool to proves these results.

Lemma V.11. Let $\{ X_\gamma : \gamma \in \Gamma \}$ be a collection of topological spaces.

a) For each $\gamma \in \Gamma$, suppose $\mathscr{B}_\gamma$ is a basis for X_γ. For each finite subset F of Γ, let

$$\mathscr{B}_F = \{ \cap_{\gamma \in F} \pi_\gamma^{-1}(B_\gamma) : B_\gamma \in \mathscr{B}_\gamma \text{ for each } \gamma \in F \},$$

and let

$$\mathscr{B} = \cup \{ \mathscr{B}_F : F \text{ is a finite subset of } \Gamma \}.$$

Then $\mathscr{B}$ is a basis for the product topology on $\prod_{\gamma \in \Gamma} X_\gamma$.

b) Let $x \in \prod_{\gamma \in \Gamma} X_\gamma$. For each $\gamma \in \Gamma$, suppose $\mathscr{B}_\gamma$ is a basis for X_γ as $x(\gamma)$. For each finite subset F of Γ, let

$$\mathscr{B}_F = \{ \cap_{\gamma \in F} \pi_\gamma^{-1}(B_\gamma) : B_\gamma \in \mathscr{B}_\gamma \text{ for each } \gamma \in F \},$$

and let

$$\mathscr{B} = \cup \{ \mathscr{B}_F : F \text{ is a finite subset of } \Gamma \}.$$

Then $\mathscr{B}$ is a basis for the product topology on $\prod_{\gamma \in \Gamma} X_\gamma$ at x.

c) For each $\gamma \in \Gamma$, suppose D_γ is a dense subset of X_γ. Let $x \in \prod_{\gamma \in \Gamma} X_\gamma$. For each finite subset F of Γ, let

$$D_F = \{ y \in \prod_{\gamma \in \Gamma} X_\gamma : y(\gamma) \in D_\gamma \text{ for each } \gamma \in F, \text{ and } y(\gamma) = x(\gamma) \text{ for each } \gamma \in \Gamma - F \}.$$

Let

$$D = \cup \{ D_F : F \text{ is a finite subset of } \Gamma \}.$$

Then D is a dense subset of $\prod_{\gamma \in \Gamma} X_\gamma$.

Problem V.3. Prove Lemma V.11.

Theorem V.12. Let $\{ X_n : n \in \mathbb{N} \}$ be a countable collection of topological spaces. Then:

a) $\prod_{n \in \mathbb{N}} X_n$ is second countable if and only if each X_n is second countable.

b) $\prod_{n \in \mathbb{N}} X_n$ is first countable if and only if each X_n is first countable.

c) $\prod_{n \in \mathbb{N}} X_n$ is separable if and only if each X_n is separable.

Problem V.4. Prove Theorem V.12.

Cartesian products of uncountable collections are generally not first countable anywhere and, therefore, are not second countable.

Definition. A set is *non-degenerate* if it has more than one point.

Theorem V.13. If $\{ X_\gamma : \gamma \in \Gamma \}$ is an uncountable collection of non-degenerate T_1 spaces, then $\prod_{\gamma \in \Gamma} X_\gamma$ fails to be first countable at each of its points.

Problem V.5. Prove Theorem V.13.

Surprisingly, separability, unlike first and second countability, can persist in Cartesian products of uncountable collections. One of the Additional Problems asks for a proof that if $\{ X_\gamma : \gamma \in \Gamma \}$ is a collection of separable spaces such that $\Gamma \preceq \mathbb{R}$, then $\prod_{\gamma \in \Gamma} X_\gamma$ is separable. The Problem also ask for a proof of a converse which asserts that if $\{ X_\gamma : \gamma \in \Gamma \}$ is a collection of Hausdorff spaces such that $\prod_{\gamma \in \Gamma} X_\gamma$ is separable, then $\Gamma \preceq \mathbb{R}$.

Next we consider the question of whether Cartesian products of metrizable spaces are metrizable. Since Cartesian products of uncountable collections are not first countable and, hence, not metrizable, we consider only Cartesian products of countable collections of metrizable spaces.

Theorem V.14. Let $\{ (X_n,\rho_n) : n \in \mathbb{N} \}$ be a countable collection of metric spaces. For each $n \in \mathbb{N}$, define $\bar{\rho}_n : X_n \times X_n \to [0,\infty)$ by $\bar{\rho}_n(x,y) = \min \{ \rho_n(x,y), 1 \}$ for $x, y \in X_n$, and recall that $\bar{\rho}_n$ is a metric on X_n which is equivalent to ρ_n such that $\bar{\rho}_n \leq 1$. (See Theorem I.12.) Then three metrics σ_1, σ_2 and σ_∞ on $\prod_{n \in \mathbb{N}} X_n$ are defined by the following formulas. For $x, y \in \prod_{n \in \mathbb{N}} X_n$, let

a) $\sigma_1(x,y) = \sum_{n \in \mathbb{N}} 2^{-n} \bar{\rho}_n(x(n), y(n))$,

b) $\sigma_2(x,y) = \left(\sum_{n \in \mathbb{N}} (2^{-n} \bar{\rho}_n(x(n), y(n)))^2 \right)^{1/2}$, and

c) $\sigma_\infty(x,y) = \sup \{ 2^{-n} \bar{\rho}_n(x(n), y(n)) : n \in \mathbb{N} \}$.

Furthermore, σ_1, σ_2 and σ_∞ are equivalent metrics on $\prod_{n \in \mathbb{N}} X_n$ which induce the product topology on $\prod_{n \in \mathbb{N}} X_n$.

Problem V.6. Prove Theorem V.14.

Theorem V.15. The Cartesian product of every collection of connected spaces is connected.

Problem V.7. Prove Theorem V.15.

Hint. Recall the equivalence of statements a) and c) of Theorem IV.10: a topological space X is connected if and only if every pair of non-empty open subsets of X is joined by a chain of connected subsets of X.

Problem V.8. a) Prove that $[0, 1]^{\mathbb{N}}$ with the box topology is not connected.

b) Let $x \in [0, 1]^{\mathbb{N}}$. Characterize the component containing x in $[0, 1]^{\mathbb{N}}$ with the box topology.