C. Continua and Discontinua

Definition. A compact connected Hausdorff space is called a *continuum*. If X is a continuum which is a subspace of a topological space Y, then X is called a *subcontinuum* of Y. If X is a subset of a space Y such that Y – X is connected, then X is called a *non-separating* subset of Y.

The following simply stated longstanding conjecture about continua is currently unresolved.

The Planar Fixed Point Conjecture. Every non-separating subcontinnum of $\mathbb{R}^2$ has the fixed point property.

Observe that the topologist's sine wave (Example IV.1) is a non-separating subcontinuum of $\mathbb{R}^2$.

Problem IV.7. Prove that the topologist's sine wave has the fixed point property.

Recall that a collection of sets $\mathscr{N}$ is called a nest if for all C, D $\in \mathscr{N}$, either C $\subset$ D or D $\subset$ C.

Theorem IV.18. If $\mathscr{N}$ is a nest of non-empty continua in a Hausdorff space X, then $\bigcap\mathscr{N}$ is a non-empty continuum.

Proof. Let C = $\bigcap\mathscr{N}$. Corollary III.5 and Theorem III.10 imply that C is non-empty and compact. It remains to prove that C is connected.

Assume C is not connected. Then, according to Theorem IV.1, C = D $\cup$ E where D and E are non-empty disjoint relatively closed subsets of C. Theorem III.2 implies that D and E are compact. Hence, Theorem III.3 implies that D and E have disjoint neighborhoods U and V in X. Therefore, U $\cup$ V is a neighborhood of C in X. Corollary III.11 implies that there is a K $\in \mathscr{N}$ such that K $\subset$ U $\cup$ V. Since D and E are non-empty sets such that D = C $\cap$ U $\subset$ K $\cap$ U and E = C $\cap$ V $\subset$ K $\cap$ V, then K $\cap$ U and K $\cap$ V are non-empty. Hence, { K $\cap$ U, K $\cap$ V } is a separation of K. However, since K $\in \mathscr{N}$, then K is connected. We have reached a contradiction. We must conclude that C is connected. Thus, C is a non-empty continuum. ◻

The following theorem is a fundamental and useful criterion for the existence of separations of a space. It's proof is substantial.

Theorem IV.19. Let A and B be non-empty disjoint closed subsets of a compact Hausdorff space X. Then there is a separation { U, V } of X such that A $\subset$ U and B $\subset$ V if and only if no subcontinuum of X intersects both A and B.

Proof. First, suppose there is a separation { U, V } of X such that $A \subset U$ and $B \subset V$. Then Theorem IV.2 implies that every subcontinuum of X is contained in either U or V. Hence, no subcontinuum of X can intersect both A and B. This proves one direction of this theorem. So we focus our attention on proving the converse direction.

Case 1: A and B are one-point sets. Suppose A = { a } and B = { b }. We will establish the assertion – if no subcontinuum contains a and b, then there is a separation of { U, V } of X such that $a \in U$ and $b \in V$ – by proving its contrapositive. To this end, assume there is no separation of { U, V } of X such that $a \in U$ and $b \in V$. We will construct a subcontinuum of X that contains both a and b.

Let $\mathscr{C}$ denote the collection of all closed subsets Y of X with the property that $\{ a , b \} \subset Y$ and there is no separation { U, V } of Y such that $a \in U$ and $b \in V$. $\mathscr{C}$ is non-empty because $X \in \mathscr{C}$. We regard $\mathscr{C}$ as partially ordered by inclusion. We will invoke Zorn's Lemma to produce a minimal element of $\mathscr{C}$. We will then argue that any minimal element of $\mathscr{C}$ is a continuum joining a to b. To establish the hypothesis of Zorn's Lemma, we must show that every nest in $\mathscr{C}$ has a lower bound in $\mathscr{C}$.

Let $\mathscr{N}$ be a nest in $\mathscr{C}$; thus, if $Y, Z \in \mathscr{N}$, then either $Y \subset Z$ or $Z \subset Y$. We must produce an element of $\mathscr{C}$ which is a subset of every element of $\mathscr{N}$. Let $C = \bigcap \mathscr{N}$. Then C is a subset of every element of $\mathscr{N}$. It remains to show that $C \in \mathscr{C}$. Assume $C \notin \mathscr{C}$. Since $\mathscr{N} \subset \mathscr{C}$, then every element of $\mathscr{N}$ is a closed subset of X that contains a and b. Since intersection preserves these properties, then C is also a closed subset of X that contains a and b. Since $C \notin \mathscr{C}$, then there is a separation { D, E } of C such that $a \in D$ and $b \in E$. Then D and E are disjoint relatively closed subsets of C and, hence, of X. Since X is a normal space by Corollary III.6. then D and E have disjoint neighborhoods U and V in X. Therefore, $U \cup V$ is a neighborhood of C in X. Corollary III.11 implies that $U \cup V$ contains an element Y of $\mathscr{N}$. Then $Y \in \mathscr{C}$, $a \in D = C \cap U \subset Y \cap U$ and $b \in E = C \cap V \subset Y \cap V$. Thus, $\{ Y \cap U, Y \cap V \}$ is a separation of C with the property that $a \in Y \cap U$ and $b \in Y \cap V$. This contradicts the fact that $Y \in \mathscr{C}$. We conclude that $C \in \mathscr{C}$. We have now shown that the hypothesis of Zorn's Lemma is satisfied

Zorn's Lemma provides a minimal element C_0 of $\mathscr{C}$. We wish to show that C_0 is a subcontinuum of X that contains a and b. To accomplish this, it suffices to prove that C_0 is connected. Assume C_0 is not connected. Then C_0 has a separation { D, E }. Since $C_0 \in \mathscr{C}$, then we can't have $a \in D$ and $b \in E$ or vice versa. So we can assume without loss of generality that $a, b \in D$. Since D is a proper subset of C_0 and C_0 is a minimal element of $\mathscr{C}$, then $D \notin \mathscr{C}$. Hence, there must be a separation { F, G } of D such that $a \in F$ and $b \in G$. But then $\{ F, G \cup E \}$ is a separation of C_0 such that $a \in F$ and $b \in G \cup E$. This contradicts the fact that $C_0 \in \mathscr{C}$. We conclude that C_0 is connected. Thus, C_0 is a subcontinuum of X that contains a and b.

Case 2: A is a one-point set and B is a closed set. Say A = { a }. Assume there is no subcontinuum of X that contains a and intersects B. We will construct a separation { U, V } of X such that $a \in U$ and $B \subset V$.

For each $b \in B$, there is no subcontinuum of X that contains a and b. Hence, the assertion proved in Case 1 implies there is a separation { U_b, V_b } of X with $a \in U_b$ and $b \in V_b$. Hence, { $V_b : b \in B$ } is a cover of B by open subsets of X. Since B is a closed subset of the compact space X, then (by Theorem III.2) B is compact . Hence, there is a finite subset F of B such that { $V_b : b \in F$ } covers B. Set $U = \bigcap_{b \in F} U_b$ and $V = \bigcup_{b \in F} V_b$. Then { U, V } is a separation of X such that $a \in U$ and $B \subset V$. (Verify!)

Case 3: A and B are disjoint closed sets. Assume there is no subcontinuum of X that intersects both A and B. We will construct a separation { U, V } of X such that $A \subset U$ and $B \subset V$.

For each $a \in A$, there is no subcontinuum of X contains a and intersects B. Hence, the assertion proved in Case 2 implies there is a separation { U_a, V_a } of X with $a \in U_b$ and $B \subset V_b$. Hence, { $U_a : a \in A$ } is a cover of A by open subsets of X. Since A is a closed subset of the compact space X, then A is compact. Hence, there is a finite subset F of A such that { $U_a : a \in F$ } covers A. Set $U = \bigcup_{a \in F} U_a$ and $V = \bigcap_{a \in F} V_a$. Then { U, V } is a separation of X such that $A \subset U$ and $B \subset V$. (Verify!) ☐

Theorem IV.19 can be used to prove the following proposition.

Theorem IV.20. Let X be a continuum.

a) If C is a proper closed subset of X, then every component of C intersects fr(C).

b) If U is a proper open subset of X, then the closure (in X) of every component of U intersects fr(U).

Proof of a). Assume that C is a proper closed subset of X and that C has a component D that is disjoint from fr(C). Then C itself is a compact Hausdorff space by Theorems I.27.d and III.2, and D is a closed subset of C by Theorem IV.12. Hence, D and fr(C) are disjoint closed subsets of C. We assert that no subcontinuum of C intersects both D and fr(C). For suppose there exists a subcontinuum E of C that intersected both D and fr(C). Then $D \cup E$ is a connected subset of C by Theorem IV.10, because any two points of $D \cup E$ are joined by a chain of connected sets of length 1 or 2. Also D is a proper subset of $D \cup E$ because E interests fr(C) but D does not. Since D is a component and, hence, a maximal connected subset of C, we have reached a contradiction. We are forced to conclude that no subcontinuum of C intersects both D and fr(C). We now invoke Theorem IV.19 to obtain a separation { F, G } of C such that $D \subset F$ and $fr(C) \subset G$. Since F and G are relatively closed subsets of C, they are closed subsets of X. Since $fr(C) \subset G$, then $F \subset C - fr(C) = int(C)$. Thus, F is disjoint from the closed set X – int(C). It follows that { F, G $\cup$ (X – int(C)) } is a separation of X. (Verify!) However, X, being a continuum, is connected. We have reached a contradiction. We conclude that every component of C intersects fr(C). ☐

Problem IV.8. Prove Theorem IV.20.b.

Defintion. A topological space X is *totally disconnected* if the only non-empty connected subsets of X are single points. Thus, a space is totally disconnected if and only if all of its components are single point sets.

Definition. A topological space X is *zero dimensional* if for every point x of X and every neighborhood U of x in X, there is a closed and open subset V of X such that $x \in V \subset U$. A subset of a space that is both closed and open is called a *clopen* set. Thus, a space is zero dimensional if and only if it has a basis of clopen sets.

We make three simple observations about the concepts of total disconnectedness and zero dimensionality.

a) Every zero dimensional space is regular.

Indeed, if X is a zero dimensional space, $x \in X$ and U is a neighborhood of x in X, then there is a clopen set V such that $x \in V \subset U$. Since V is clopen, then cl(V) = V. So $x \in V \subset$ cl(V) $\subset$ U. Thus, X is regular.

b) Every zero dimensional T_1 space is totally disconnected.

Let X be a zero dimensional T_1 space, and let C be a subset of X containing more than one point. Let x and y be distinct points of C. Since { y } is a closed set, then there is a clopen subset V of X such that $x \in V \subset X - \{ y \}$. Hence, { C $\cap$ V, C $\cap$ (X – V) } is a separation of C. Thus, C is not connected. This proves that X is totally disconnected.

c) $\mathbb{R}$ is not zero dimensional but that its subspaces $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are zero dimensional.

If $\mathbb{R}$ were zero dimensional, then there would be a clopen subset V of $\mathbb{R}$ such that $0 \in V \subset (-1,1)$. Then { V, $\mathbb{R}$ – V } would be a separation of $\mathbb{R}$. However, $\mathbb{R}$ is connected by Theorem IV.4. We have reached a contradiction. Hence, $\mathbb{R}$ is not zero dimensional.

To prove that $\mathbb{Q}$ is zero dimensional, let $x \in \mathbb{Q}$ and let V be a neighborhood of x in $\mathbb{Q}$. Then there is an open subset U of $\mathbb{R}$ such that $U \cap \mathbb{Q} = V$. There are irrational numbers $a < b$ such that $x \in (a,b) \subset U$. Hence, $x \in (a,b) \cap \mathbb{Q} \subset U \cap \mathbb{Q} = V$. Furthermore, $(a,b) \cap \mathbb{Q} = [a,b] \cap \mathbb{Q}$. Hence, $(a,b) \cap \mathbb{Q}$ is a clopen subset of $\mathbb{Q}$.

The proof that $\mathbb{R} - \mathbb{Q}$ is zero dimensional is similar. In the preceding argument, replace $\mathbb{Q}$ by $\mathbb{R} - \mathbb{Q}$ and choose a and b to be rational numbers.

Remark. Although zero dimensional T_1 spaces are totally disconnected, totally disconnected spaces need not be zero dimensional. Below we will present an example of a subspace of $\mathbb{R}^2$ that is totally disconnected but not zero dimensional. On the other hand, for compact Hausdorff spaces, the notions of total disconnectedness and zero dimensionality coincide, as the next theorem asserts.

Definition. If $\mathscr{U}$ and $\mathscr{V}$ are collections of sets such that every element of $\mathscr{V}$ is a subset of some element of $\mathscr{U}$, then we call $\mathscr{V}$ a *refinement* of $\mathscr{U}$, and we say that $\mathscr{V}$ *refines* $\mathscr{U}$ and that $\mathscr{U}$ *is refined by* $\mathscr{V}$.

Theorem IV.21. If X is a compact Hausdorff space, then the following statements are equivalent.

a) X is totally disconnected.

b) For any two disjoint closed subsets A and B of X, there is a separation { U, V } of X such that $A \subset U$ and $B \subset V$.

c) X is zero dimensional.

d) Every open cover of X is refined by a pairwise disjoint open cover of X.

Problem IV.9. Prove Theorem IV.21.

The concept of zero dimensionality and the various parts of Theorem IV.21 motivate the following remarks about *dimension theory.* The definition of “zero dimensional” given previously is just the first step in an inductive procedure for assigning a dimension to every topological space. Moreover, statements b), c) and d) of Theorem IV.21 contain ideas that give rise to three competing definitions of dimension. Each of these methods for defining dimension produces the same values on separable metric spaces. Furthermore, they each assign a dimension of n to the space $\mathbb{R}^n$, and thus coincide with our intuitive expectations for a notion of dimension.

The definition of “zero dimensional” given above is part of a scheme for defining a dimension function known as *small inductive dimension*. The small inductive dimension of a topololgical space X is denoted *ind(X)*. For each topological space X, ind(X) is an element of the set $\{-1, 0, 1, 2, \dots\} \cup \{\infty\}$ which is defined inductively as follows. $ind(X) = -1$ if and only if $X = \varnothing$. Next let $n \geq 0$ be an integer and assume that the statement $ind(X) \leq n-1$ is defined for every topological space X (in the sense that for every space X, we can in principle decide whether the statement $ind(X) \leq n-1$ is true). For a topological space X, we define the statement $ind(X) \leq n$ to hold if for every point $x \in X$ and every neighborhood U of x in X, there is a neighborhood V of x in X such that $cl(V) \subset U$ and $ind(fr(V)) \leq n-1$. Now for a topological space X, we define the statement $ind(X) = n$ to hold if $ind(X) \leq n$ is true but $ind(X) \leq n-1$ is false. Finally, for a topological space X, we define the statement $ind(X) = \infty$ to hold if $ind(X) = n$ is false for every integer $n \geq -1$. Observe that a space X is zero dimensional in the sense defined previously if and only if ind(X) = 0.

Statement b) of Theorem IV.21 is part of different definition of dimension known as *large inductive dimension*. The large inductive dimension of a space X is denoted *Ind(X)* and is also an element of the set $\{ -1, 0, 1, 2, \dots \} \cup \{ \infty \}$. Before defining large inductive dimension, we must introduce another concept: if A and B are disjoint subsets of a space X, then a *separator* of A and B in X is a closed subset S of X such that there is a separation $\{ U, V \}$ of $X - S$ in which $A \subset U$ and $B \subset V$. The large inductive dimension Ind(X) of a topological space X is defined inductively as follows. Again *Ind(X) = –1* if and only if $X = \varnothing$. Next let $n \geq 0$ be an integer and assume that the statement $\text{Ind}(X) \leq n - 1$ is defined for every topological space X. For a topological space X, we define the statement *Ind(X) ≤ n* to hold if for any two disjoint closed subsets A and B of X, there is a separator S of A and B in X such that $\text{Ind}(S) \leq n - 1$. Now for a topological space X, we define the statement *Ind(X) = n* to hold if $\text{Ind}(X) \leq n$ is true but $\text{Ind}(X) \leq n - 1$ is false. Finally, for a topological space X, we define the statement *Ind(X) = ∞* to hold if Ind(X) = n if false for every integer $n \geq -1$. Observe that if X is a T_1 space, then $\text{ind}(X) \leq \text{Ind}(X)$.

Statement d) of Theorem IV.21 is part of third definition of dimension known as *covering dimension*. The covering dimension of a space X is denoted *dim(X)* and it, too, is an element of the set $\{ -1, 0, 1, 2, \dots \} \cup \{ \infty \}$. Before defining covering dimension, we must introduce the following concept: for an integer $n \geq 1$, the *order* of a collection $\mathscr{C}$ of sets is ≤ n if every element of $\bigcup\mathscr{C}$ belongs to at most n elements of $\mathscr{C}$. Thus, a collection of sets is order ≤ 1 if and only if it is pairwise disjoint. The covering dimension dim(X) of a topological space X is defined (non-inductively) as follows. Again *dim(∅) = –1*. For any non-empty topological space X and any integer $n \geq 0$, define *dim(X) ≤ n* if for every open cover of X is refined by an open cover of X of order $\leq n + 1$. Now for a topological space X, define *dim(X) = n* if $\dim(X) \leq n$ is true but $\dim(X) \leq n - 1$ is false. Finally, for a topological space X, define *dim(X) = ∞* if dim(X) = n is false for every integer $n \geq -1$.

If X is a separable metric space, then ind(X), Ind(X) and dim(X) coincide and provide a very satisfactory theory of dimension. If X is a non-separable metric space, then Ind(X) and dim(X) coincide, but there are examples showing that ind(X) < Ind(X) is possible. Specifically, there is non-separable metric space X such that ind(X) = 0 and Ind(X) = 1. For general (separable and non-separable) metric spaces, the dimension functions Ind and dim agree and provide a satisfactory dimension theory. For larger classes of spaces (for example, normal spaces), the dimension functions ind, Ind and dim may all disagree and they give rise to strange dimension theories. For example there is a compact Hausdorff space with covering dimension 0 that contains a subspace of covering dimension 1.

Next we define *discontinua.* These spaces are in a sense the extreme opposites of continua.

Definition. A non-empty compact totally disconnected Hausdorff space without isolated points is called a *discontinuum.* A metrizable discontinuum is called a *metric discontinuum.*

Observe that Theorem IV.21 implies that each discontinuum is zero dimensional. Also note that the theorem stated in Problem III.2 implies that if X is a discontinuum, then $X \succeq \mathbb{R}$.

Problem IV.10. Prove that every uncountable compact metric space contains a metric discontinuum.

We now describe a well known metric discontinuum – the Cantor set.

Example IV.3. We now describe the subspace of $\mathbb{R}$ known as the *standard deleted middle thirds Cantor set*. For each integer n ≥ 0, let $\mathscr{I}_n$ denote the collection

$$\left\{ \left[0,\frac{1}{3^n}\right], \left[\frac{1}{3^n},\frac{2}{3^n}\right], \left[\frac{2}{3^n},\frac{3}{3^n}\right], \ldots, \left[\frac{3^n-1}{3^n}, 1\right] \right\}.$$

Thus, $\mathscr{I}_n$ is a cover of [0,1] by closed intervals of length $1/3^n$. We inductively define a nested sequence $C_0 \supset C_1 \supset C_2 \supset \ldots$ of closed subsets of [0,1] as follows. Let $C_0 =$ [0,1]. For each integer n ≥ 1, assume C_{n-1} is already defined and define

$$C_n = C_{n-1} - (\bigcup \{ \mathrm{int}(J) : J \in \mathscr{I}_n \text{ and } J \subset \mathrm{int}(C_{n-1}) \}).$$

The *standard deleted middle thirds Cantor set* is the subspace $\bigcap_{n=1}^{\infty} C_n$ of $\mathbb{R}$. More succinctly, the standard deleted middle thirds Cantor set is the subspace

$$[0,1] - \left(\bigcup_{n=1}^{\infty} \bigcup_{i=1}^{3^{n-1}} \left(\frac{3i-2}{3^n}, \frac{3i-1}{3^n} \right) \right)$$

of $\mathbb{R}$.

Exercise. Verify that the standard deleted middle thirds Cantor set is a metric discontinuum.

Definition. Any topological space that is homeomorphic to the standard deleted middle thirds Cantor set is called a *Cantor set.*

The following theorem gives a strikingly simple topological characterization of the Cantor set.

Theorem IV.22. A topological space is a Cantor set if and only if it is a metric discontinuum.

Problem IV.11. Prove Theorem IV.22.

According to Theorem IV.21, every totally disconnected compact Hausdorff space is zero dimensional. The following example shows, among other things, that without the hypothesis of compactness, a totally disconnected subspace of $\mathbb{R}^2$ need not be zero dimensional.

Example IV.4. We construct a subspace Z of $\mathbb{R}^2$ with the following properties.

a) Z is connected.

b) There is a point **v** in Z such that $Z - \{ \mathbf{v} \}$ is totally disconnected. (The point **v** is called an *explosion point* of Z.)

c) $Z - \{ \mathbf{v} \}$ is not zero dimensional.

Before starting this construction, we need to show that every well ordered set can be given an optimal well ordering called a *best well ordering.* Suppose $<$ is a well ordering of a set X. We call $<$ a *best well ordering* of X if for every $x \in X$, $(-\infty,x) \prec X$. (Recall that for sets A and B, $A \prec B$ means there is an injective function from A to B but there is no bijective function from A to B.) Observe that the well ordering on Ω is a best well ordering.

Lemma IV.23. Every well ordered set has a best well ordering.

Proof. Let $(X,<)$ be a well ordered set. Observe that for each $x \in X$, since $(-\infty,x) \subset X$, then either $(-\infty,x) \prec X$ or $(-\infty,x) \approx X$. If $(-\infty,x) \prec X$ for every $x \in X$, then the given well ordering is a best well ordering, and we're done. So assume there is an $x \in X$ for which $(-\infty,x) \approx X$. Let $A = \{ x \in X : (-\infty,x) \approx X \}$. Then A is non-empty. Hence, A has a least element x_0. Thus, $(-\infty,x_0) \approx X$ and $(-\infty,x) \prec X$ for each $x \in (-\infty,x_0)$. Since $(-\infty,x_0) \approx X$, then there is a bijection $f : X \to (-\infty,x_0)$. Use f to "pull back" the well ordering on $(-\infty,x_0)$ to obtain a new well ordering on X. In other words, define a relation $<_2$ on X by declaring $x <_2 y$ if and only if $f(x) < f(y)$. Then f is an order preserving bijections from $(X,<_2)$ to $((-\infty,x_0),<)$. Since $<$ restricts to a well ordering on $(-\infty,x_0)$, then it follows that $<_2$ is a well ordering of X. For each $x \in X$, let $(-\infty,x)_2 = \{ y \in X : y <_2 x \}$. Then for each $x \in X$, f restricts to a bijection from $(-\infty,x)_2$ to $(-\infty,f(x))$. Furthermore, for each $x \in X$, since $f(x) \in (-\infty,x_0)$, then $(-\infty,f(x)) \prec X$. Consequently, $(-\infty,x)_2 \prec X$ for each $x \in X$. Hence, $<_2$ is a best well ordering of X. ◻

We now begin the construction of Example IV.4. Let $\mathbb{C}$ denote the standard deleted middle thirds Cantor set. Then $\mathbb{C} \subset [0,1]$. Let $\mathbf{v} = (^1/_2,1) \in \mathbb{R}^2$. For any two points **p** and **q** of $\mathbb{R}^2$, let **p**∗**q** denote the line segment $\{ (1-t)\mathbf{p} + t\mathbf{q} : 0 \le t \le 1 \}$ joining **p** to **q**. Let $K = \bigcup_{x \in \mathbb{C}} \mathbf{v}*(x,0)$. Then K is the *cone* over the Cantor set $\mathbb{C} \times \{ 0 \}$ with *vertex* **v**.

Let $\mathscr{D}$ denote the collection of all closed subsets D of $\mathbb{R}^2$ with the property that $\mathbf{v} \notin D$ and $\{ x \in \mathbb{C} : D \cap (\mathbf{v}*(x,0)) \neq \varnothing \} \approx \mathbb{R}$. Thus, $D \in \mathscr{D}$ if and only if $\mathbf{v} \notin D$ and D intersects $\mathbb{R}$–many different line segments joining **v** to points of $\mathbb{C} \times \{ 0 \}$. We assert that $\mathscr{D} \approx \mathbb{R}$. Here is a proof. Let $\mathscr{T}$ denote the standard topology on $\mathbb{R}^2$. Since $\mathbb{R}^2$ is a second countable space, then $\mathscr{T}$ has a countable basis $\mathscr{B}$. Since $\mathscr{B}$ is countable, then $\mathscr{P}(\mathscr{B}) \approx \mathbb{R}$ (by Theorems 0.13, 0.20 and 0.22). Since $\mathscr{B}$ is a basis for $\mathscr{T}$, then the function $\mathscr{U} \mapsto \cup\mathscr{U} : \mathscr{P}(\mathscr{B}) \to \mathscr{T}$ is surjective. Therefore, $\mathscr{T} \preceq \mathscr{P}(\mathscr{B})$. Let $\mathscr{C}$ denote the collections of all closed subsets of $\mathbb{R}^2$. Since the function $C \mapsto \mathbb{R}^2 - C : \mathscr{C} \to \mathscr{T}$ is a bijection, then $\mathscr{C} \approx \mathscr{T}$. Since $\mathscr{D} \subset \mathscr{C}$, then $\mathscr{D} \preceq \mathscr{C}$. Combining these observations, we have $\mathscr{D} \preceq \mathscr{C} \approx \mathscr{T} \preceq \mathscr{P}(\mathscr{B}) \approx \mathbb{R}$. Thus, $\mathscr{D} \preceq \mathbb{R}$. On the other hand, for each $t \in [0,1)$, the horizontal line $\mathbb{R} \times \{ t \}$ intersects every line segment joining **v** to a point of $\mathbb{C} \times \{ 0 \}$. It is well known that $\mathbb{C} \approx \mathbb{R}$. Hence, for each $t \in [0,1)$, $\mathbb{R} \times \{ t \} \in \mathscr{D}$. Moreover, the function $t \mapsto \mathbb{R} \times \{ t \} : [0,1) \to \mathscr{D}$ is clearly injective. Hence, $[0,1) \preceq \mathscr{D}$. Since $\mathbb{R} \approx [0,1)$, then we have $\mathbb{R} \preceq \mathscr{D}$. We have shown that $\mathscr{D} \preceq \mathbb{R}$ and $\mathbb{R} \preceq \mathscr{D}$. The Schroder-Bernstein Theorem 0.15 now implies our assertion: $\mathscr{D} \approx \mathbb{R}$.

Zermelo's Well Ordering Principle (or equivalently the Axiom of Choice) together with Lemma IV.23 imply that $\mathscr{D}$ has a best well order <. Guided by this best well order, we inductively choose points $x_D \in \mathbb{C}$ and $\mathbf{p}_D \in D \cap (\mathbf{v}*(x_D,0))$ for each $D \in \mathscr{D}$ so that $x_D \notin \{ x_E : E \in \mathscr{D} \text{ and } E < D \}$. We explain why these choices are possible. For each $D \in \mathscr{D}$, the fact that < is a best well ordering insures that $\{ E \in \mathscr{D} : E < D \} \prec \mathscr{D}$. Since $\{ x_E : E \in \mathscr{D} \text{ and } E < D \} \preceq \{ E \in \mathscr{D} : E < D \}$, then $\{ x_E : E \in \mathscr{D} \text{ and } E < D \} \prec \mathscr{D} \approx \mathbb{R}$. On the other hand, since $D \in \mathscr{D}$, then $\{ x \in \mathbb{C} : D \cap (\mathbf{v}*(x,0)) \neq \varnothing \} \approx \mathbb{R}$. Therefore, the set $\{ x \in \mathbb{C} : D \cap (\mathbf{v}*(x,0)) \neq \varnothing \} - \{ x_E : E \in \mathscr{D} \text{ and } E < D \}$ is never empty. Thus, it is always possible to choose $x_D \in \mathbb{C}$ so that $D \cap (\mathbf{v}*(x,0)) \neq \varnothing$ and $x_D \notin \{ x_E : E \in \mathscr{D} \text{ and } E < D \}$. Now let $Z = \{ \mathbf{p}_D : D \in \mathscr{D} \} \cup \{ \mathbf{v} \}$.

To proceed further, we need to introduce another concept and prove a lemma about it. We say that a space X is *completely normal* if every pair of subsets A and B of X that satisfy $A \cap \mathrm{cl}(B) = \varnothing = \mathrm{cl}(A) \cap B$ have disjoint neighborhoods.

Lemma IV.24. Every metric space is completely normal.

Proof. Let A and B be subsets of a metric space X such that $A \cap \mathrm{cl}(B) = \varnothing = \mathrm{cl}(A) \cap B$. For each $x \in A$, since $x \notin \mathrm{cl}(B)$, then there is a $\delta_x > 0$ such that $N(x,2\delta_x) \cap B = \varnothing$. Similarly, for each $y \in B$, since $y \notin \mathrm{cl}(A)$, then there is an $\varepsilon_y > 0$ such that

$N(y,2\varepsilon_y) \cap A = \varnothing$. Let $U = \cup_{x \in A} N(x,\delta_x)$ and let $V = \cup_{y \in B} N(y,\varepsilon_y)$. Then U is a neighborhood of A and V is a neighborhood of B. We assert that $U \cap V = \varnothing$. For assume $U \cap V \neq \varnothing$. Let $z \in U \cap V$. Then there are points $x \in A$ and $y \in B$ such that $z \in N(x,\delta_x) \cap N(y,\varepsilon_y)$. We may assume without loss of generality that $\delta_x \leq \varepsilon_y$. Then $d(x,y) \leq d(x,z) + d(z,y) \leq \delta_x + \varepsilon_y \leq 2\varepsilon_y$. Therefore, $x \in N(y,2\varepsilon_y) \cap A$. We have reached a contradiction. We must conclude that $U \cap V = \varnothing$. □

We now turn to the proof that Z is connected. This is the most difficult aspect of this example. Assume Z is not connected. Then Z has a separation { A, B }. We may assume without loss of generality that $\mathbf{v} \in B$. Since A and B are relatively open subsets of X, then there are open subsets G and H of $\mathbb{R}^2$ such that $G \cap Z = A$ and $H \cap Z = B$. Since A and B are disjoint subsets of Z, then it follows that $G \cap B = \varnothing = H \cap A$. Hence, G is a neighborhood in $\mathbb{R}^2$ of each point of A such that $G \cap B = \varnothing$, and H is a neighborhood in $\mathbb{R}^2$ of each point of B such that $H \cap A = \varnothing$. Therefore, $A \cap cl(B) = \varnothing = cl(A) \cap B$ (where "cl" indicates closure *in* $\mathbb{R}^2$). Since $\mathbb{R}^2$ is metrizable, then Lemma IV.24 implies that A and B have disjoint neighborhood U and V in $\mathbb{R}^2$.

Let $E = \mathbb{R}^2 - (U \cup V)$. Then E is a closed subset of $\mathbb{R}^2$. We assert that $E \in \mathscr{D}$. Note that since $\mathbf{v} \in B$, then $\mathbf{v} \in V$. Therefore, $\mathbf{v} \notin E$. Hence, to prove that $E \in \mathscr{D}$, it remains to prove that $\{ x \in \mathbb{C} : E \cap (\mathbf{v}*(x,0)) \neq \varnothing \} \approx \mathbb{R}$. Since A is non-empty subset of Z, there is a $D \in \mathscr{D}$ such that $\mathbf{p}_D \in A$. Then $\mathbf{p}_D \in U$. The description of Z guarantees that there is a point $x_D \in \mathbb{C}$ such that $\mathbf{p}_D \in D \cap (\mathbf{v}*(x_D,0))$. Hence there is a $t_D \in [0,1)$ such that

$$\mathbf{p}_D = (1 - t_D)(x_D,0) + t_D\mathbf{v}.$$

(Since $\mathbf{p}_D \in D$ and $\mathbf{v} \notin D$, then $\mathbf{p}_D \neq \mathbf{v}$. Hence, $t \neq 1$.) Since the function

$$x \mapsto (1 - t_D)(x,0) + t_D\mathbf{v} : \mathbb{R} \to \mathbb{R}^2$$

is continuous and U is neighborhood of $\mathbf{p}_D$ in $\mathbb{R}^2$, then there is a $\delta > 0$ such that $(1 - t_D)(x,0) + t_D\mathbf{v} \in U$ for all $x \in (x_D - \delta, x_D + \delta)$. Choose an integer $n \geq 1$ such that $1/3^n < \delta$. Since $x_D \in \mathbb{C}$, then there is a closed interval J in [0,1] of the form $\left[\frac{i-1}{3^n}, \frac{i}{3^n}\right]$ (where $1 \leq i \leq 3^n$) such that $x_D \in J$ and int(J) contains points of $\mathbb{C}$. (Thus, J is one of the *non-deleted* intervals of length $1/3^n$ used in the construction of $\mathbb{C}$.) Let $\mathbb{C}_J = J \cap \mathbb{C}$. Then $x_D \in \mathbb{C}_J$. Furthermore, $\mathbb{C}_J$ is a "small copy" of $\mathbb{C}$; in fact, there is a homeomorphism from $\mathbb{C}_J$ onto $\mathbb{C}$ that stretches distances by a factor of 3^n. Thus, $\mathbb{C}_J \approx \mathbb{C} \approx \mathbb{R}$. Since $x_D \in \mathbb{C}_J \subset J$ and $1/3^n < \delta$, then $\mathbb{C}_J \subset (x_D - \delta, x_D + \delta)$. Thus, $(1 - t_D)(x,0) + t_D\mathbf{v} \in U$ for every $x \in \mathbb{C}_J$. Since $\mathbf{v} \in V$, then the line segment $\mathbf{v}*(x,0)$ intersects both U and V for every $x \in \mathbb{C}_J$. Hence, for each $x \in \mathbb{C}_J$, if $\mathbf{v}*(x,0)$ were a subset of $U \cup V$, then $\{ (\mathbf{v}*(x,0)) \cap U, (\mathbf{v}*(x,0)) \cap V \}$ would be a separation of $\mathbf{v}*(x,0)$. Since each line segment $\mathbf{v}*(x,0)$ is connected, it has no separation. We conclude that $\mathbf{v}*(x,0)$ must intersect $\mathbb{R}^2 - (U \cup V) = E$ for each $x \in \mathbb{C}_J$. Thus, $\mathbb{C}_J \subset \{ x \in \mathbb{C} : E \cap (\mathbf{v}*(x,0)) \neq \emptyset \}$. It follows that $\{ x \in \mathbb{C} : E \cap (\mathbf{v}*(x,0)) \neq \emptyset \} \approx \mathbb{R}$. This proves our assertion: $E \in \mathscr{D}$.

Since $E \in \mathscr{D}$, then $\mathbf{p}_E \in Z \cap E$. Hence, $Z \cap E \neq \emptyset$. However, since $Z = A \cup B \subset U \cup V$ and $E = \mathbb{R}^2 - (U \cup V)$, then $Z \cap E = \emptyset$. We have reached a contradiction from which we are forced to conclude that Z must be connected.

Next we prove that $Z - \{ \mathbf{v} \}$ is totally disconnected by showing that no connected subset of $Z - \{ \mathbf{v} \}$ contains two distinct points. To this end, consider any two distinct points $\mathbf{p}_D$ and $\mathbf{p}_E$ of $Z - \{ \mathbf{v} \}$. Then D and E are distinct elements of $\mathscr{D}$, and x_D and x_E are distinct elements of $\mathbb{C}$ such that $p_D \in \mathbf{v}*(x_D,0)$ and $p_E \in \mathbf{v}*(x_E,0)$. We may assume $x_D < x_E$. Because $\mathbb{C}$ is totally disconnected, it doesn't contain the interval (x_D,x_E). Hence, there is a point $y \in (x_D,x_E) - \mathbb{C}$. Let L be the line in $\mathbb{R}^2$ that passes through $\mathbf{v}$ and $(y,0)$. Then L intersects the cone on the Cantor set $K = \bigcup_{x \in \mathbb{C}} \mathbf{v}*(x,0)$ only in the point $\mathbf{v}$. Since $Z \subset K$, then it follows that $(Z - \{ \mathbf{v} \}) \cap L = \emptyset$. Also, since $x_D < y < x_E$, then the line segment joining $\mathbf{p}_D$ to $\mathbf{p}_E$ crosses L. Hence, $\mathbf{p}_D$ and $\mathbf{p}_E$ lie on opposite sides of L. Therefore, $\mathbb{R}^2 - L$ is the union of two disjoint open sets U and V such that $Z - \{ \mathbf{v} \} \subset U \cup V$, $\mathbf{p}_D \in U$ and $\mathbf{p}_E \in V$. Therefore, $\{ (Z - \{ \mathbf{v} \}) \cap U, (Z - \{ \mathbf{v} \}) \cap V \}$ is a separation of $Z - \{ \mathbf{v} \}$ such that $\mathbf{p}_D \in (Z - \{ \mathbf{v} \}) \cap U$ and $\mathbf{p}_E \in (Z - \{ \mathbf{v} \}) \cap V$. Theorem IV.2 tells us that every connected subset of $Z - \{ \mathbf{v} \}$ must lie in either $(Z - \{ \mathbf{v} \}) \cap U$ or $(Z - \{ \mathbf{v} \}) \cap V$. It follows that no connected subset of $Z - \{ \mathbf{v} \}$ can contain both $\mathbf{p}_D$ and $\mathbf{p}_E$. We have proved that no connected subset of $Z - \{ \mathbf{v} \}$ contains two distinct points. We conclude that $Z - \{ \mathbf{v} \}$ is totally disconnected.

Finally we prove that $Z - \{ \mathbf{v} \}$ is not zero dimensional. Assume $Z - \{ \mathbf{v} \}$ is zero dimensional. We will argue that this assumption implies that Z is not connected, contradicting a previously proved assertion. Let $\mathbf{p}_D$ be a point of $Z - \{ \mathbf{v} \}$. Then $\mathbf{p}_D$ and $\mathbf{v}$ have disjoint neighborhoods N_D and N_v in $\mathbb{R}^2$. Therefore, $N_D \cap (Z - \{ \mathbf{v} \})$ is a relative neighborhood of $\mathbf{p}_D$ in $Z - \{ \mathbf{v} \}$. Our hypothesis that $Z - \{ \mathbf{v} \}$ is zero dimensional implies that there is a relatively clopen subset U of $Z - \{ \mathbf{v} \}$ such that $\mathbf{p}_D \in U \subset N_D \cap (Z - \{ \mathbf{v} \})$. Let $V = (Z - \{ \mathbf{v} \}) - U$. Since U is a relatively clopen subset of $Z - \{ \mathbf{v} \}$, so is V. Therefore, $\{ U, V \}$ is a separation of $Z - \{ \mathbf{v} \}$. Since $\{ \mathbf{v} \}$ is a relatively closed subset of Z, then $Z - \{ \mathbf{v} \}$ is a relatively open subset of Z. Hence, according to Theorem I.26.f, U and V are relatively open sets in Z. Let $W = N_v \cap Z$. Then W is a relatively open subset of Z such that $\mathbf{v} \in W$. Since $U \subset N_D$ and $W \subset N_v$, then $U \cap W = \emptyset$. It follows that $\{ U, V \cup W \}$ is a separation of Z. We have contradicted the previously proven fact that Z is connected. We must conclude that $Z - \{ \mathbf{v} \}$ is not zero dimensional.

We conclude our discussion of Example IV.4 with the following remark. The construction of this example relies essentially on Zermelo's Well Ordering Principle and, hence, on the Axiom of Choice. There is a way to construct a subset Z´ of $\mathbb{R}^2$ with properties similar to Z without using the Axiom of Choice. Like Z, Z´ is connected and contains a point **v**´ such that $Z' - \{ \mathbf{v}' \}$ is totally disconnected but not zero dimensional. The description of Z´ is relatively straightforward, but the proof that it has the desired properties is more complicated than the proof we have just given and depends on topological ideas we have not yet encountered.

Definition. A continuum is *decomposable* if it is the union of two proper subcontinua. (Recall that a subset S of a set X is called a *proper subset* if $S \neq X$.) A continuum is *indecomposable* if it is not decomposable.

Clearly, [0,1] is decomposable continuum since it is the union of the proper subcontinua [0,1/2] and [1/2,1]. Remarkably, indecomposable continua exist. We now describe one.

Example IV.5. The *Knaster continuum* (also known as the *horseshoe* and the *buckethandle*) is a subcontinuum of $\mathbb{R}^2$ which we now describe. For each integer $n \geq 0$, let $\mathcal{S}_n$ denote the collection

$$\left\{ \left[\frac{i-1}{3^n},\frac{i}{3^n}\right] \times \left[\frac{j-1}{3^n},\frac{j}{3^n}\right] : 1 \leq i \leq 3^n \text{ and } 1 \leq j \leq 3^n \right\}$$

of 9^n squares of side $1/3^n$ covering $[0,1]^2$. For each integer $n \geq 1$, let

$$A_n = \left[\frac{1}{3^n},\frac{2}{3^n}\right] \times \left[\frac{0}{3^n},\frac{1}{3^n}\right].$$

Then $A_n \in \mathcal{S}_n$. We inductively define a nested sequence $K_0 \supset K_1 \supset K_1 \supset \ldots$ of continua in $[0,1]^2$ as follows. Let $K_0 = [0,1]^2$. For each integer $n \geq 1$, assume that K_{n-1} is already defined, let

$$L_n = A_n \cup \left(\bigcup \{ S \in \mathcal{S}_n : S \subset \text{int}(K_{n-1}) \} \right) \text{ and let } K_n = \text{cl}(K_{n-1} - L_n).$$

K_0 K_1 K_2 K_3

The *Knaster continuum* is the subcontinuum $\bigcap_{n=0}^{\infty} K_n$ of $\mathbb{R}^2$.

Theorem IV.25. The Knaster continuum is an indecomposable continuum.

Problem IV.12. Prove Theorem IV.25.