B. Local and Path Connectedness

Definition. A topological space X is *locally connected* if for each point $x \in X$, each neighborhood of x contains a connected neighborhood of x.

Observe that $\mathbb{R}^n$ is locally connected for each $n \geq 1$. Indeed, each point of $\mathbb{R}^n$ has arbitrarily small neighborhoods that are products of open intervals. Open intervals are connected by Theorem IV.4. Hence, products of open intervals are connected by Theorem IV.11.

Theorem VI.13. A topological space X is locally connected if and only if every component of every open subset of X is open.

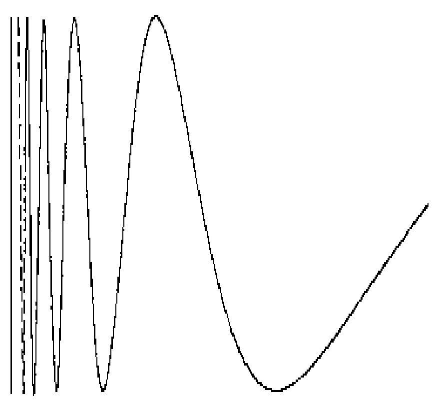
Proof. First assume that X is locally connected. Let C be a component of an open subset U of X. Let $x \in C$. Since X is locally connected and $x \in U$, then there is a connected neighborhood V of x such that $V \subset U$. Observe that $C \cup V$ is connected by Theorem IV.10 because each pair of points of $C \cup V$ is joined by a chain of connected subsets of $C \cup V$ of length one or two. Since C is the maximal connected subset of U, then $C \cup V = C$. Hence, $V \subset C$. Thus, C contains a neighborhood of x. This proves C is an open set.

Now assume every component of every open subset of X is open. Let U be a neighborhood of a point $x \in X$. Let C be the component of U that contains x. (C exists by Theorem IV.12.) By hypothesis, C is an open set. Thus, C is a connected neighborhood of x and $C \subset U$. This proves X is locally connected. ◻

Observe that if C is a component of a locally connected space X, then $\{ C, X - C \}$ is a separation of X. Indeed, since the components of X are open sets, then $X - C$ is a union of open sets. Hence, C and $X - C$ are open sets.

Example IV.1. The *topologist's sine wave* is the subspace of $\mathbb{R}^2$:

$$S = (\{ 0 \} \times [-1, 1]) \cup \{ (x,\sin(1/x)) : 0 < x \leq 1/\pi \}.$$



We now prove that the topologist's sine wave $\mathcal{S}$ is connected but not locally connected.

A simple way to prove that $\mathcal{S}$ is connected is to define the map g : (0,1/π] → $\mathbb{R}^2$ by g(x) = (x,sin(1/x)). (0,1/π] is connected by Theorem IV.4. Since g is continuous, then g((0,1/π]) is connected by Theorem IV.6. Since $\mathcal{S}$ = cl(g((0,1/π])), then $\mathcal{S}$ is connected by Theorem IV.3.

To prove that $\mathcal{S}$ is not locally connected, we will show that no neighborhood of (0,0) in $\mathcal{S}$ which is a subset of ([0, 1/π) × (–1, 1)) ∩ $\mathcal{S}$ is connected. Let U be such a neighborhood. Since $\left\{\left(\frac{1}{2n\pi},0\right)\right\}$ is a sequence in $\mathcal{S}$ that converges to (0,0), then there is an n ≥ 1 such that $\left(\frac{1}{2n\pi},0\right) \in U$. Let L be the vertical line determined by the equation $x = \frac{1}{\left(2n+\frac{1}{2}\right)\pi}$. L intersects $\mathcal{S}$ in the single point $\left(\frac{1}{\left(2n+\frac{1}{2}\right)\pi},1\right)$, and this point doesn't belong to U because U ⊂ [0, 1/π) × (–1, 1). Hence, L ∩ $\mathcal{S}$ = ∅. Therefore, if we let V = $\left\{ (x,y) \in U : x < \frac{1}{\left(2n+\frac{1}{2}\right)\pi} \right\}$ and we let W = $\left\{ (x,y) \in U : x > \frac{1}{\left(2n+\frac{1}{2}\right)\pi} \right\}$, then (0,0) ∈ V, $\left(\frac{1}{2n\pi},0\right) \in W$ and { V, W } is a separation of U.

Definition. A map from [0, 1] to a space X is called a *path* in X. (The image of such a map may also be called a *path* in X.) If f : [0, 1] → X is a path in the space X such that f(0) = x and f(1) = y, then the path f is said to *join* the points x and y. If every pair of points of a space X is joined by a path in X, then X is said to be *path connected.*

Theorem IV.14. Every path connected space is connected.

Proof. Let x and y be points of a path connected space X. Then there is a path f : [0, 1] → X that joins x to y. Since [0, 1] is connected by Theorem IV.4, then Theorem IV.6 implies that f([0, 1]) is a connected set that contains x and y. We have proved that every pair of points of X is joined by a chain of connected subsets of X of length 1. Hence, X is connected by Theorem IV.10. □

Problem IV.3. Prove that the topologist's sine wave $\mathcal{S}$ is not path connected.

Definition. A topological space X is *locally path connected* if for each point x ∈ X, each neighborhood of x contains a path connected neighborhood of x.

Since a path connected neighborhood of a point is connected by Theorem IV.14, then every locally path connected space is locally connected.

Theorem IV.15. If a topological space is connected and locally path connected, then it is path connected.

Proof. Let X be a topological space which is connected and locally path connected. Let $\mathscr{U}$ be the set of all path connected open subsets of X. Since X is locally path connected, then $\mathscr{U}$ is an open cover of X.

Let x and y $\in$ X. Since X is connected, then Theorem IV.10 implies there is a chain $U_1, U_2, \ldots, U_n$ of elements of $\mathscr{U}$ that joins x to y. Then for $1 \leq i < n$, we can choose a point $z_i \in U_i \cap U_{i+1}$. Let $z_0 = x$ and $z_n = y$. For $1 \leq i \leq n$, since z_{i-1} and $z_i \in U_i$ and since U_i is path connected, then there is a path $f_i : [0, 1] \to U_i$ joining z_{i-1} to z_i. Now define a map $g : [0, 1] \to X$ by

$$g(t) = f_i(nt - (i - 1))$$

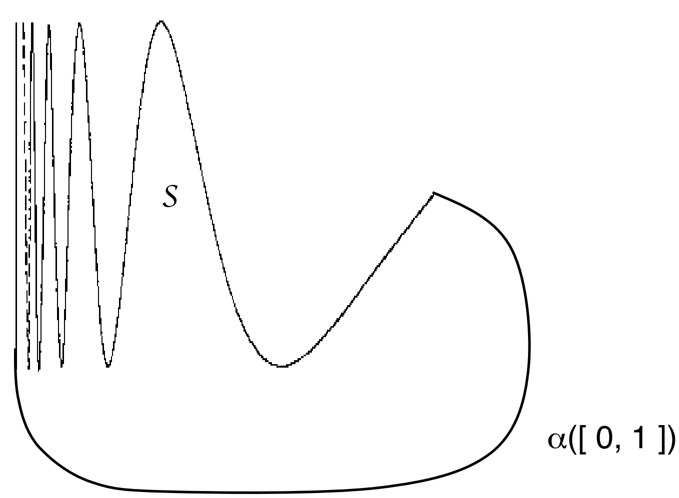
for $t \in [(i-1)/n, i/n]$ for $1 \leq i \leq n$. g is well defined because $f_i(1) = z_i = f_{i+1}(0)$ for $1 \leq i < n$. Since $g(0) = f_1(0) = z_0 = x$ and $g(1) = f_n(1) = z_n = y$, then g is a path in X joining x to y. This proves X is path connected. □

The following example illustrates that a path connected space need not be locally path connected.

Example IV.2. The *Warsaw circle* is the subspace

$$S \cup \alpha([0, 1])$$

of $\mathbb{R}^2$, where S is the topologist's sine wave and $\alpha : [0, 1] \to \mathbb{R}^2$ is a embedding such that $\alpha(0) = (0,-1)$, $\alpha(1) = (1/\pi, 0)$ and $\alpha((0, 1)) \cap ([0, 1/\pi] \times [-1, 1]) = \varnothing$.



We observe that the Warsaw circle is not locally connected for the same reason that the topologist's sine wave S is not locally connected. Hence, the Warsaw circle is not locally path connected. However, the Warsaw circle is path connected. For instance, any point of the "limit segment" $\{ 0 \} \times [-1, 1]$) can be joined to any point of the "tail" $\{ (x,\sin(1/x)) : 0 < x \leq 1/\pi \}$ via a path that includes $\alpha([0, 1])$.

The following theorem is a substantial improvement of Theorem IV.15 in the context of locally compact metric spaces.

Theorem IV.16. Every connected, locally connected, locally compact metric space is path connected.

Proof. Let X be a connected, locally connected, locally compact metric space. Let $x, y \in X$. We will construct a path in X joining x to y.

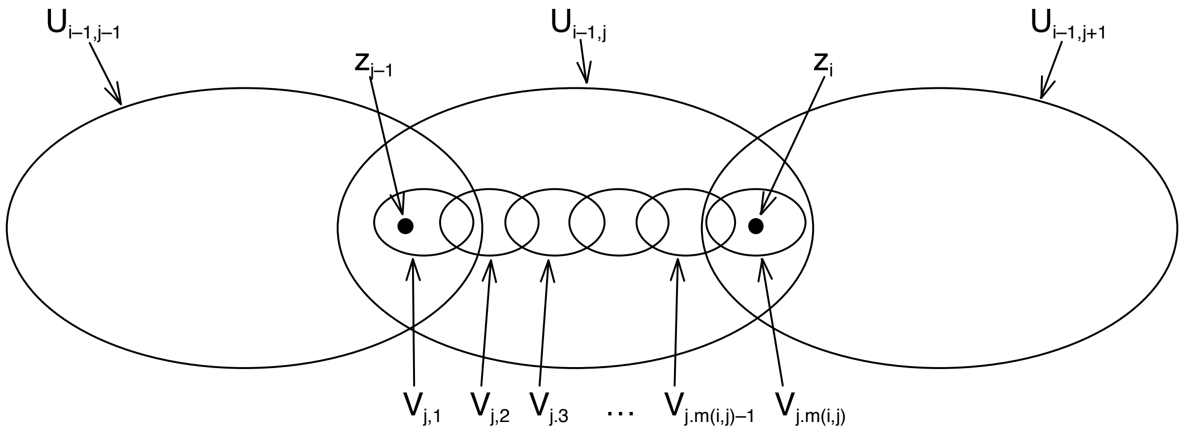
This construction is somewhat involved. The first step is to construct, for each $i \in \mathbb{N}$, a chain $U_{i,1}, U_{i,2}, \dots , U_{i,n(i)}$ joining x to y with the following properties.

a) For each $i \in \mathbb{N}$, for $1 \leq j \leq n(i)$, $U_{i,j}$ is a connected open subset of X of diameter $< 1/i$ with compact closure.

b) For each $i \geq 2$, there is a sequence of positive integers $m(i,1), m(i,2), \dots , m(i,n(i-1))$ such that $m(i,1) + m(i,2) + \dots + m(i,n(i-1)) = n(i)$ and so that for $1 \leq j \leq n(i-1)$, $cl(U_{i,k}) \subset U_{i-1,j}$ whenever $m(i,1) + \dots + m(i,j-1) < k \leq m(i,1) + \dots + m(i,j-1) + m(i,j)$.

To begin the construction of these chains, observe that since X is a locally connected, locally compact metric space, there is an open cover $\mathscr{U}_1$ of X by connected open sets of diameter < 1 with compact closures. Since X is connected, then Theorem IV.10 implies there is a chain $U_{1,1}, U_{1,2}, \dots , U_{1,n(1)}$ of elements of $\mathscr{U}_1$ joining x to y.

We continue the construction of the chains inductively. Let $i \in \mathbb{N}$ such that $i \geq 2$ and assume that we have already constructed the chain $U_{i-1,1}, U_{i-1,2}, \dots , U_{i-1,n(i-1)}$ joining x to y and satisfying conditions **a)** and **b)**. To construct the i^{th} chain, let $z_0 = x$, let $z_{n(i-1)} = y$ and choose $z_j \in U_{i-1,j} \cap U_{i-1,j+1}$ for $0 < j < n(i-1)$. For $1 \leq j \leq n(i-1)$, since $U_{i-1,j}$ is a locally connected, locally compact metric space, there is an open cover $\mathscr{V}_j$ of $U_{i-1,j}$ by connected open sets of diameter $< 1/i$ whose closures are compact subsets of $U_{i-1,j}$. For $1 \leq j \leq n(i-1)$, since $U_{i-1,j}$ is connected, then Theorem IV.10 implies there is a chain $V_{j,1}, V_{j,2}, \dots , V_{j,m(i,j)}$ of elements of $\mathscr{V}_j$ joining z_{j-1} to z_j. Let $n(i) = m(i,1) + m(i,2) + \dots + m(i,n(i-1))$. We "concatenate" the chains $V_{j,1}, V_{j,2}, \dots , V_{j,m(i,j)}$ for $1 \leq j \leq n(i-1)$ to obtain a chain $U_{i,1}, U_{i,2}, \dots , U_{i,n(i)}$. Specifically, for $1 \leq r \leq n(i)$, there is a j such that $1 \leq j \leq n(i-1)$ and $m(i,1) + \dots + m(i,j-1) < r \leq m(i,1) + \dots + m(i,j-1) + m(i,j)$; then let $k = r - (m(i,1) + \dots + m(i,j-1))$ and let $U_{i,r} = V_{j,k}$. It follows that the chain $U_{i,1}, U_{i,2}, \dots , U_{i,n(i)}$ satisfies conditions **a)** and **b)**.



The second step of the proof is to construct for each $i \in \mathbb{N}$, a partition $0 = w_{i,0} < w_{i,1} < \ldots < w_{i,n(i)} = 1$ of the closed interval [0, 1] so that if $i \geq 2$, $1 \leq j \leq n(i-1)$, $r = m(i,1) + m(i,2) + \ldots + m(i,j-1)$ and $s = r + m(i,j)$, then

$$w_{i-1,j-1} = w_{i,r} < w_{i,r+1} < \ldots < w_{i,s} = w_{i-1,j}.$$

Clearly, it is possible to construct such a sequence of partitions of [0, 1].

Observe that the following relationship holds between the intervals $[\, w_{i,j-1}, w_{i,j} \,]$ and the chain elements $U_{i,j}$. For $i \geq 2$, $1 \leq j \leq n(i-1)$ and $1 \leq k \leq n(i)$:

$$[\, w_{i,k-1}, w_{i,k} \,] \subset [\, w_{i-1,j-1}, w_{i-1,j} \,]$$

implies

$$m(i,1) + m(i,2) + \ldots + m(i,j-1) < k \leq m(i,1) + m(i,2) + \ldots + m(i,j-1) + m(i,j)$$

which implies

$$cl(U_{i,k}) \subset U_{i-1,j}.$$

The third and final step of the proof is to construct a map $f : [\, 0, 1 \,] \to X$ such that $f(0) = x$ and $f(1) = y$. Let $t \in [\, 0, 1 \,]$. For each $i \in \mathbb{N}$, choose $j(t,i) \in \{ 1, 2, \ldots , n(i) \}$ so that we obtain a sequence of nested intervals

c) $$[\, w_{1,j(t,1)-1}, w_{1,j(t,1)} \,] \supset [\, w_{2,j(t,2)-1}, w_{2,j(t,2)} \,] \supset [\, w_{3,j(t,3)-1}, w_{3,j(t,3)} \,] \supset \ldots$$

each of which contains t. Specifically, if $t = 0$, choose $j(t,i) = 1$ for all $i \in \mathbb{N}$. On the other hand, if $0 < t \leq 1$, then for each $i \in \mathbb{N}$, choose $j(t,i)$ to be the unique element of $\{ 1, 2, \ldots , n(i) \}$ so that the interval $(\, w_{i,j(t,i)-1}, w_{i,j(t,i)} \,]$ contains t for each $i \in \mathbb{N}$. It is not hard to see that this prescription insures that the intervals satisfy the nesting condition **c)**. As we observed earlier, this nesting condition implies a corresponding nesting of chain elements:

$$cl(U_{1,j(t,1)}) \supset cl(U_{2,j(t,2)}) \supset cl(U_{3,j(t,3)}) \supset \ldots$$

Thus, $\{ cl(U_{i,j(t,i)}) \}$ is a nested sequence of compact subsets of X. Therefore, Theorem III.10 implies that $\bigcap_{i=1}^{\infty} cl(U_{i,j(t,i)}) \neq \varnothing$. Since $diam(cl(U_{k,j(t,k)})) < 1/k$ and $cl(U_{k,j(t,k)}) \supset \bigcap_{i=1}^{\infty} cl(U_{i,j(t,i)})$ for each $k \geq 1$, then $diam\left(\bigcap_{i=1}^{\infty} cl(U_{i,j(t,i)})\right) = 0$. Hence, $\bigcap_{i=1}^{\infty} cl(U_{i,j(t,i)})$ is a one-point set. Call this point f(t). Thus, $\bigcap_{i=1}^{\infty} cl(U_{i,j(t,i)}) = \{ f(t) \}$. We have now defined the function $f : [\,0,1\,] \rightarrow X$.

Since $j(0,i) = 1$ for each $i \in \mathbb{N}$, then $\bigcap_{i=1}^{\infty} cl(U_{i,1}) = \{ f(0) \}$. Since $U_{i,1}$ contains the point x for each $i \in \mathbb{N}$, then $x \in \bigcap_{i=1}^{\infty} cl(U_{i,1})$. Hence, $f(0) = x$.

Clearly, $j(1,i) = n(i)$ for each $i \in \mathbb{N}$. Therefore, $\bigcap_{i=1}^{\infty} cl(U_{i,n(i)}) = \{ f(1) \}$. Since $U_{i,n(i)}$ contains the point y for each $i \in \mathbb{N}$, then $y \in \bigcap_{i=1}^{\infty} cl(U_{i,n(i)})$. Hence, $f(1) = y$.

Finally, we begin the proof of the continuity of f by observing that the definition of f implies that f has the following two properties:

- For $i \in \mathbb{N}$, since $j(t,i) = 1$ for every $t \in [\, w_{i,0}, w_{i,1} \,]$, then f maps the entire interval $[\, w_{i,0}, w_{i,1} \,]$ into the set $cl(U_{i,1})$.
- For $i \in \mathbb{N}$ and $2 \leq j \leq n(i)$, since $j(t,i) = j$ for every $t \in (\, w_{i,j-1}, w_{i,j} \,]$, then f maps the entire interval $(\, w_{i,j-1}, w_{i,j} \,]$ into the set $cl(U_{i,j})$.

Consequently:

- for $i \in \mathbb{N}$ and $1 \leq j \leq n(i) - 1$, f maps $(\, w_{i,j-1}, w_{i,j+1} \,]$ into the set $cl(U_{i,j}) \cup cl(U_{i,j+1})$.

For $i \in \mathbb{N}$ and $1 \leq j \leq n(i)$, we know $diam(cl(U_{i,j})) < 1/i$. Furthermore, for $i \in \mathbb{N}$ and $1 \leq j \leq n(i) - 1$, since $U_{i,j} \cap U_{i,j+1} \neq \varnothing$, then $diam(\, cl(U_{i,j}) \cup cl(U_{i,j+1}) \,) < 2/i$. Thus, we conclude that for $i \in \mathbb{N}$:

- $diam(\, f(\, [\, w_{i,0}, w_{i,1} \,] \,) \,) < 1/i$, $diam(\, f(\, [\, w_{i,n(i)-1}, w_{i,n(i)} \,] \,) \,) < 1/i$ and

  $diam(\, f(\, (\, w_{i,j-1}, w_{i,j+1} \,] \,) \,) < 2/i$ for $1 \leq j \leq n(i) - 1$.

To finish the proof of the continuity of f, let $t \in [\, 0, 1 \,]$ and let $\varepsilon > 0$. Choose $i \in \mathbb{N}$ such that $2/i < \varepsilon$. Then one of the intervals $[\, 0, w_{i,1} \,)$, $(\, w_{i,n(i)-1}, 1 \,]$ or $(\, w_{i,j-1}, w_{i,j+1} \,)$ where $1 \leq j \leq n(i) - 1$ is a neighborhood of t in $[\, 0, 1 \,]$ and f maps each of these intervals to a subset of X of diameter $< 2/i < \varepsilon$. Thus, f is continuous. □

Definition. An embedding of [0, 1] in a space X is called an *arc* in X. (The image of such an embedding may also be called a *arc* in X.) If $f : [\, 0, 1 \,] \rightarrow X$ is an arc in the space X such that $f(0) = x$ and $f(1) = y$, then the arc f is said to *join* the points x and y. If every pair of distinct points of a space X is joined by an arc in X, then X is said to be *arc connected.*

Theorem IV.17. Every connected, locally connected, locally compact metric space is arc connected.

Problem IV.4. Prove Theorem IV.17 by solving the problems stated in a) through d) below.

Definition. Suppose $A_1, A_2, \ldots, A_n$ is a chain of sets of length n. We call $A_1, A_2, \ldots, A_n$ a *simple* chain if $A_i \cap A_j = \varnothing$ whenever $| i - j | > 1$.

a) Prove that if X is a connected space, then for every open cover $\mathscr{U}$ of X, every pair of points in X is joined by a simple chain of elements of $\mathscr{U}$.

We now outline how to modify the proof of Theorem IV.16 to change the map f constructed there into an embedding, thereby proving Theorem IV.17.

b) Show that in the first step of the proof of Theorem IV.16 , each chain $U_{i,1}, U_{i,2}, \ldots, U_{i,n(i)}$ can be constructed so that it is a *simple* chain and so that $n(1) \geq 2$ and so that for each $i \geq 2$ and $1 \leq j \leq n(i-1)$, $m(i,j) \geq 2$.

c) Show that in the second step of the proof of Theorem IV.16, for each $i \in \mathbb{N}$, the partition $0 = w_{i,0} < w_{i,1} < \ldots < w_{i,n(i)} = 1$ can be chosen so that $w_{i,j} - w_{i,j-1} \leq 2^{-i}$ for $1 \leq j \leq n(i)$.

d) Show that, once the changes described in parts b) and c) of this problem are made, then the map $f : [\,0,1\,] \to X$ constructed in the third step of the proof of Theorem IV.16 must be an embedding.

There are several interesting examples which show that the hypotheses in Theorems IV.16 and IV.17 can't be omitted. Since the topologists sine wave is a connected compact metric space which is not locally connected and not path connected, then the "local connectedness" hypothesis in Theorem IV.16 can't be omitted.

Problem IV.5. Prove that $[\,0, 1\,]^2$ with the lexicographic order topology is a connected, locally connected, compact Hausdorff space that is not metrizable and not path connected. Thus, the "metric" hypothesis in Theorem IV.16, the can't be weakened to "compact Hausdorff".

It turns out that in Theorem IV.16, the "locally compact metric" hypothesis can be replaced by "complete metric" with essentially the same proof. However, if "locally compact metric" is simply reduced to "metric" (not assuming "*complete* metric"), then Theorem IV.16 becomes false. We illustrate this in an Additional Problem.

In the Additional Problems, we outline a proof that every path connected Hausdorff space is arc connected. Observe that this result, combined with Theorem IV.16, yield a very quick alternative proof of Theorem IV.17. The following problem illustrates why the "Hausdorff" hypothesis is needed for this result.

Problem IV.6. Prove that [0, 1] with the finite complement topology is a compact non-Hausdorff space that is path connected but not arc connected.