IV. Connectedness and Disconnectedness

A. Fundamental Properties of Connectedness

Definition. A *separation* of a topological space is an unordered pair { U, V } of non-empty disjoint open subsets of X such that U ∪ V = X. A space is *connected* if it has no separation.

Observe that every one-point space is connected.

The following result shows that in the definition of "separation", the condition that U and V be open subsets of X can be replaced by either the condition that U and V be closed subsets of X, or the condition that (cl(U)) ∩ V = ∅ = U ∩ (cl(V)).

Lemma IV.1. If U and V are disjoint subsets of a topological space X such that U ∪ V = X, the the following statements are equivalent.

a) U and V are open subsets of X.

b) U and V are closed subsets of X.

c) (cl(U)) ∩ V = ∅ = U ∩ (cl(V)).

Proof. Since X – U = V and X – V = U, then U and V are open if and only if U and V are closed. Hence, a) and b) are equivalent. If U and V are closed, then (cl(U)) ∩ V = U ∩ V = ∅ and U ∩ (cl(V)) = U ∩ V = ∅. So b) implies c). If (cl(U)) ∩ V = ∅ = U ∩ (cl(V)), then V = X – cl(U) and U = X – cl(V). So U and V are open. Thus, c) implies a). □

Theorem IV.2. If { U, V } is a separation of a topological space X and C is a connected subset of X, then either C ⊂ U or C ⊂ V.

Proof. If C ∩ U and C ∩ V are both non-empty, then { C ∩ U, C ∩ V } is a separation of C. Since C is connected, it has no separation. Hence, either C ∩ U = ∅ or C ∩ V = ∅. Thus, either C ⊂ V or C ⊂ U. □

Theorem IV.3. If A is a connected subset of a topological space X and A ⊂ B ⊂ cl(A), then B is connected.

Proof. Suppose B has a separation { C, D }. Theorem IV.2 implies either A ⊂ C or A ⊂ D. Assume without loss of generality that A ⊂ C. Since C is a relatively closed subset of B, then there is a closed subset E of X such that C = B ∩ E. Since A ⊂ C, then A ⊂ E. Since E is a closed subset of X, then cl(A) ⊂ E. Since B ⊂ cl(A), then B ⊂ E. Consequently, B = B ∩ E. So B = C. Hence, D = B – C = ∅. But D, being an element of a separation of B, must be non-empty. We have reached a contradiction. We conclude that B is connected. □

Recall that in a linearly ordered set, an *interval* is a set of any of the following nine types:

- *open intervals:* (x,y), (x,∞), $(-\infty,y)$ and $(-\infty,\infty)$,
- *closed intervals:* $[x,y]$, $[x,\infty)$ and $(-\infty,y]$, and
- *half open intervals:* $[x,y)$ and $(x,y]$,

for $-\infty < x < y < \infty$.

Theorem IV.4. A subset of $\mathbb{R}$ is connected if and only if it is an interval.

Proof. First we prove that every open interval is connected. Suppose an open interval J has a separation $\{ U, V \}$. Then U and V are open subsets of $\mathbb{R}$ as well as of J. Choose $x \in U$ and $y \in V$. Assume without loss of generality that $x < y$. Set $W = \{ u \in U : u < y \}$. Then $x \in W$. Set $z = \sup(W)$. (z exists because y is an upper bound of W and $\mathbb{R}$ is a complete linearly ordered set.) Then $x \leq z \leq y$. Since J is an interval and $x, y \in J$, then $z \in J$. Hence, either $z \in U$ or $z \in V$.

Case 1: $z \in U$. Since U is an open subset of $\mathbb{R}$, there is an $\varepsilon > 0$ such that $[z, z + \varepsilon] \subset U$. Since $z \leq y$ and $y \notin U$, then $z + \varepsilon < y$. So $z + \varepsilon \in W$. This contradicts the fact that z is an upper bound of W.

Case 2: $z \in V$. Since V is an open subset of $\mathbb{R}$, there is an $\varepsilon > 0$ such that $[z - \varepsilon, z] \subset V$. Since $W \subset U$, $U \cap V = \varnothing$, and z is an upper bound of W, then $u \in W$ implies $u < z - \varepsilon$. So $z - \varepsilon$ is an upper bound of W. This contradicts the fact that z is the least upper bound of W.

Since we have reached a contradiction in both cases, we must conclude that J is connected.

Now let K be any interval in $\mathbb{R}$. Let $J = \text{int}(K)$. Then J is an open interval and $J \subset K \subset \text{cl}(J)$. Since J is connected, then Theorem IV.3 implies that K is connected. Thus, every interval in $\mathbb{R}$ is connected.

We now prove that every connected subset of $\mathbb{R}$ is an interval. Let J be a connected subset of $\mathbb{R}$. If $J = \varnothing$, then J is the interval $(0,0) = \{ x \in \mathbb{R} : 0 < x < 0 \}$. Now assume $J \neq \varnothing$. Regard $\mathbb{R}$ as a subset of the linearly ordered space $[-\infty,\infty]$, where $-\infty < x < \infty$ for every $x \in \mathbb{R}$. Let $a = \inf(J)$ and $b = \text{sub}(J)$. Then $-\infty \leq a \leq b \leq \infty$, and $[a,b] \cap \mathbb{R}$ is an interval containing J. We assert that $(a,b) \subset J$. To see this let $z \in (a,b)$. Then z is neither a lower bound nor an upper bound of J. So there are elements x and y of J such that $a \leq x < z < y \leq b$. Consequently $J \cap (-\infty,z)$ and $J \cap (z,\infty)$ are non-empty. So if $z \notin$

J, then { $J \cap (-\infty,z)$, $J \cap (z,\infty)$ } is a separation of J. As J is connected, we must conclude that $z \in J$. This proves our assertion: $(a,b) \subset J$. Thus, $(a,b) \subset J \subset [a,b] \cap \mathbb{R}$. Since $[a,b] \cap \mathbb{R} = (a,b) \cup (\{a,b\} \cap \mathbb{R})$, then J is one of the following four sets: (a,b), $(a,b] \cap \mathbb{R}$, $[a,b) \cap \mathbb{R}$ or $[a,b] \cap \mathbb{R}$. Since each of these four sets is an interval, we conclude that J is an interval in $\mathbb{R}$. □

Recall that a linearly ordered set X is *densely ordered* if between any two distinct points of X, there is a third point of X.

Theorem IV.5. A linearly ordered space is connected if and only if it is complete and densely ordered.

Problem IV.1. Prove Theorem IV.5.

Observe that the well-ordered spaces Ω and Ω^+ (Examples I.10 and I.11) are not connected because they are not densely ordered. However, the lexicographically ordered square $[\,0, 1]^2$ (Example I.9) is a connected linearly ordered set because it is complete and densely ordered.

Theorem IV.6. Maps preserve connectedness. In other words, if $f : X \to Y$ is a map from a connected space X to a topological space Y, then f(X) is connected.

Proof. We prove the contrapositive of this assertion. Assume f(X) is not connected. Then there is a separation { U, V } of f(X). Since U and V are non-empty and disjoint and $U \cup V = f(X)$, then it follows that $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty and disjoint and $f^{-1}(U) \cup f^{-1}(V) = X$. Since U and V are relatively open subsets of f(X), then there are open subsets U´ and V´ of Y such that $U = U' \cap f(X)$ and $V = V' \cap f(X)$. Hence, $f^{-1}(U')$ and $f^{-1}(V')$ are open subsets of X. It is also obvious that $f^{-1}(U') = f^{-1}(U)$ and $f^{-1}(V') = f^{-1}(V)$. We conclude that { $f^{-1}(U)$, $f^{-1}(V)$ } is a separation of X. Thus, X is not connected. □

Corollary IV.7: The Intermediate Value Theorem. Suppose $f : X \to \mathbb{R}$ is a map from a connected space X into $\mathbb{R}$. If $a < b < c$ and a and $c \in f(X)$, then $b \in f(X)$.

Proof. Theorem IV.6 implies that f(X) is a connected subset of $\mathbb{R}$. Therefore, f(X) is an interval by Theorem IV.4. Consequently, if $a < b < c$ and a and $c \in f(X)$, then $b \in f(X)$, because all intervals have this property. □

Definition. Let X be a topological space. A point $x \in X$ is a *fixed point* of a map $f : X \to X$ if $f(x) = x$. The space X has the *fixed point property* if every map from X to itself has a fixed point.

Corollary IV.8. [0, 1] has the fixed point property.

Proof. Let f : [0, 1] → [0, 1] be a map. We must prove that f has a fixed point. If either f(0) = 0 or f(1) = 1, then either 0 or 1 is a fixed point of f, and we're done. So assume $f(0) \neq 0$ and $f(1) \neq 1$. Then $f(0) > 0$ and $f(1) < 1$.

A map $g : [0, 1] \to \mathbb{R}$ is defined by the formula $g(x) = f(x) - x$ for all $x \in [0, 1]$. Then $g(0) = f(0) - 0 > 0$ and $g(1) = f(1) - 1 < 0$. In this situation, the Intermediate Value Theorem (Corollary IV.7) implies there is an $x \in [0, 1]$ such that $g(x) = 0$. Hence, $f(x) - x = 0$. So $f(x) = x$. Thus, x is a fixed point of f. □

We state without proof a generalization of Corollary IV.8 which is one of the fundamental topological properties of Euclidean space.

The Brouwer Fixed Point Theorem. For every positive integer n, the n-cube $[0, 1]^n$ has the fixed point property.

Recall that $\mathbb{S}^n = \{ \mathbf{x} \in \mathbb{R}^{\mathbf{n+1}} : \| \mathbf{x} \| = 1 \}$. Two points $\mathbf{x}$ and $\mathbf{y} \in \mathbb{S}^n$ are called *antipodal points* and are said to be *diametrically opposed* if $\mathbf{y} = -\mathbf{x}$.

Corollary IV.9. For every map $f : \mathbb{S}^1 \to \mathbb{R}$, there is an $\mathbf{x} \in \mathbb{S}^1$ such that $f(\mathbf{x}) = f(-\mathbf{x})$.

An entertaining application of Corollary IV.9 is obtained by identifying $\mathbb{S}^1$ with the Earth's equator and letting the map f record the temperature at each point of the equator at the same instant. We must make the modest assumption that temperature varies continuously with position along the Earth's equator. Then Corollary IV.9 implies that at every instant there are two diametrically opposed points on the Earth's equator with the same temperature.

Proof of Corollary IV.9. First we observe that $\mathbb{S}^1$ is connected. This follows from Theorem IV.6 because $\mathbb{S}^1$ is the continuous image of the connected space $\mathbb{R}$ under the map $\theta \mapsto (\cos(\theta), \sin(\theta))$. Next define the map $g : \mathbb{S}^1 \to \mathbb{R}$ by $g(\mathbf{x}) = f(\mathbf{x}) - f(-\mathbf{x})$. Observe that for each $\mathbf{x} \in \mathbb{S}^1$, $g(-\mathbf{x}) = f(-\mathbf{x}) - f(-(-\mathbf{x})) = -(f(\mathbf{x}) - f(-\mathbf{x})) = -g(\mathbf{x})$. Fix a point $\mathbf{x}_0 \in \mathbb{S}^1$. Since $g(-\mathbf{x}_0) = -g(\mathbf{x}_0)$, then either $g(\mathbf{x}_0) = 0$, $g(-\mathbf{x}_0) < 0 < g(\mathbf{x}_0)$, or $g(\mathbf{x}_0) < 0 < g(-\mathbf{x}_0)$. In either of the latter two cases, the Intermediate Value Theorem implies there is a point $\mathbf{x}_1 \in \mathbb{S}^1$ such that $g(\mathbf{x}_1) = 0$. Hence, either $g(\mathbf{x}_0) = 0$ or $g(\mathbf{x}_1) = 0$. Thus, either $f(\mathbf{x}_0) = f(-\mathbf{x}_0)$ or $f(\mathbf{x}_1) = f(-\mathbf{x}_1)$. □

We state without proof a generalization of Corollary IV.9 which, like the Brouwer Fixed Point Theorem, is a fundamental topological properties of Euclidean space.

The Borsuk-Ulam Theorem. For every positive integer n, for every map $f : \mathbb{S}^n \to \mathbb{R}^n$, there is an $\mathbf{x} \in \mathbb{S}^n$ such that $f(\mathbf{x}) = f(-\mathbf{x})$.

Observe that the Borsuk-Ulam Theorem has the following entertaining application: at every instant there are two diametrically opposed points on the Earth's surface with the same temperature and humidity (assuming that temperature and humidity vary continuously with position along the Earth's surface).

Definition. If n is a positive integer and $A_1, A_2, \dots, A_n$ are sets such that $A_i \cap A_{i+1} \neq \varnothing$ for $1 \le i < n$, then we call $A_1, A_2, \dots, A_n$ a *chain of sets of length n.* If x and y are points and $A_1, A_2, \dots, A_n$ is a chain of sets such that $x \in A_1$ and $y \in A_n$, then we say that $A_1, A_2, \dots, A_n$ *joins* x to y. If U and V are sets and $A_1, A_2, \dots, A_n$ is a chain of sets that joins a point of U to a point of V, then we say that $A_1, A_2, \dots, A_n$ *joins* U to V.

Theorem IV.10. Let X be a topological space. The the following three statements are equivalent.

a) X is connected.

b) For every open cover $\mathscr{U}$ of X, every pair of points in X is joined by a chain of elements of $\mathscr{U}$.

c) Every pair of non-empty open subsets of X is joined by a chain of connected subsets of X.

Proof. a) implies b). Assume X is connected. Let $\mathscr{U}$ be an open cover of X. Let $x \in X$. Set $C = \{ y \in X : x$ is joined to y by a chain of elements of $\mathscr{U} \}$.

First we prove that C is an open set. Suppose $y \in C$. Then there is a chain $U_1, U_2, \dots, U_n$ of elements of $\mathscr{U}$ that joins x to y. Thus, $y \in U_n$. Clearly, $U_1, U_2, \dots, U_n$ chains x to every element of U_n. Hence, $U_n \subset C$. Hence, C contains a neighborhood of y. This proves C is open.

Second we prove $X - C$ is an open set. Suppose $y \in X - C$. $y \in V$ for some $V \in \mathscr{U}$. We claim $V \subset X - C$. For if not, then there is a point $z \in C \cap V$. It follows that there is a chain $U_1, U_2, \dots, U_n$ of elements of $\mathscr{U}$ joining x to z. Then $z \in U_n \cap V$. Therefore, $U_1, U_2, \dots, U_n, V$ is a chain of elements of $\mathscr{U}$ joining x to y. This implies $y \in C$, contradicting our hypothesis that $y \in X - C$. We conclude as claimed that $V \subset X - C$. Hence, $X - C$ contains a neighborhood of y. This proves $X - C$ is open.

If $X - C \neq \varnothing$, then $\{ C, X - C \}$ is a separation of X. Since X is connected, we conclude that $X - C = \varnothing$. Therefore, $X = C$. Consequently, x is joined to every element of X by a chain of elements of $\mathscr{U}$.

b) implies a). We prove the contrapositive. Assume that X is not connected. Then there is a separation { U, V } of X. Then { U, V } is an open cover of X. Furthermore, since U and V are non-empty and disjoint, then there is no chain of elements of { U, V } that joins a point of U to a point of V.

a) implies c). Assume that X is connected. Then, clearly, X is a chain of connected sets of length one that joins every pair of non-empty open subsets of X.

c) implies a). We prove the contrapositive. Assume that X is not connected. Then there is a separation { U, V } of X. Suppose that $C_1, C_2, \ldots, C_n$ is a chain of connected subsets of X such that $C_1 \cap U \neq \varnothing$. Theorem IV.2 implies that for each i between 1 and n, either $C_i \subset U$ or $C_i \subset V$. We claim that $C_i \subset U$ for $1 \leq i \leq n$. Since C_1 intersects U, then $C_1 \not\subset V$; hence, $C_1 \subset U$. Proceeding inductively, let $1 \leq i < n$ and assume $C_i \subset U$. Since $C_i \cap C_{i+1} \neq \varnothing$, then C_{i+1} intersects U; hence, $C_{i+1} \not\subset V$. Consequently $C_{i+1} \subset U$. Our claim follows by induction. Therefore, $C_1 \cup C_2 \cup \ldots \cup C_n \subset U$. Since $U \cap V = \varnothing$, then it follows that no chain of connected subsets of X joins U to V. ◻

Theorem IV.11. If $X_1, X_2, \ldots, X_n$ are connected topological spaces, then the Cartesian product $X_1 \times X_2 \times \ldots \times X_n$ (with the product topology) is connected.

Problem IV.2. Prove Theorem IV.11.

Hint. Use Theorem II.9 to prove that any two points of $X_1 \times X_2 \times \ldots \times X_n$ are joined by a chain of connected subsets of $X_1 \times X_2 \times \ldots \times X_n$.

Definition. C is a *component* of a topological space X if C is a connected subset of X such that the only connected subset of X that contains C is C itself. Thus, a component of a space is simply a maximal connected subset.

Theorem IV.12. The collection $\mathscr{C}$ of all components of a topological space X partitions X into closed subsets. In other words, $\mathscr{C}$ covers X and distinct elements of $\mathscr{C}$ are disjoint closed sets.

Proof. First we prove that $\mathscr{C}$ covers X. In other words, we prove that every point of X belongs to a component of X. Let $x \in X$. Let D be the union of all the connected subsets of X that contain x. Since { x } is a connected set that contains x, then $\{ x \} \subset D$. So $x \in D$. We will prove that $D \in \mathscr{C}$.

To prove that D is connected, suppose y and $z \in D$. Then there are connected subsets E and F of X both of which contain x such that $y \in E$ and $z \in F$. Furthermore, since D is the union of all connected subsets of X that contain x, then E and $F \subset D$.

Thus E,F is a chain of connected subsets of D (of length 2) joining y to z. This proves that every pair of points of D is joined by a chain of connected subsets of D. Therefore, Theorem IV.10 implies that D is connected.

To prove D is a maximal connected set, suppose D is contained in a connected subset G of X. Since $x \in D$, then $x \in G$. Since D is be the union of all the connected subsets of X that contain x, then it follows that $G \subset D$. Thus, G = D, proving that D is a maximal connected subset of X.

It follows that $D \in \mathscr{C}$. Hence, $\mathscr{C}$ covers X.

To prove that distinct elements of $\mathscr{C}$ are disjoint, assume that C and D are elements of $\mathscr{C}$ that intersect. Since C intersects D, then C, D is a length 2 chain of connected sets that joins any two points of $C \cup D$. Hence, $C \cup D$ is connected by Theorem IV.10. Since $C \cup D$ contains both C and D, and C and D are maximal connected subsets of X, then $C = C \cup D = D$. This proves that any two intersecting elements of $\mathscr{C}$ are equal. It follows that distinct elements of $\mathscr{C}$ are disjoint.

To prove that the elements of $\mathscr{C}$ are closed sets, let $C \in \mathscr{C}$. Then C is connected. Hence, cl(C) is connected by Theorem IV.3. Since $C \subset \text{cl}(C)$ and C is a maximal connected subset of X, then C = cl(C). Hence, C is a closed set. □