B. Various Forms of Compactness

In addition to the formulation of compactness presented above in terms of open covers, there are other useful versions. We will introduce several alternative notions of compactness and explore conditions under which they are equivalent.

Recall that a point x of a topological space X is a *limit point* or *accumulation point* of a subset A of X if every neighborhood of x intersects $A - \{ x \}$.

Definition. A topological space X is *limit point compact* or *accumulation point compact* if every infinite subset of X has a limit point in X.

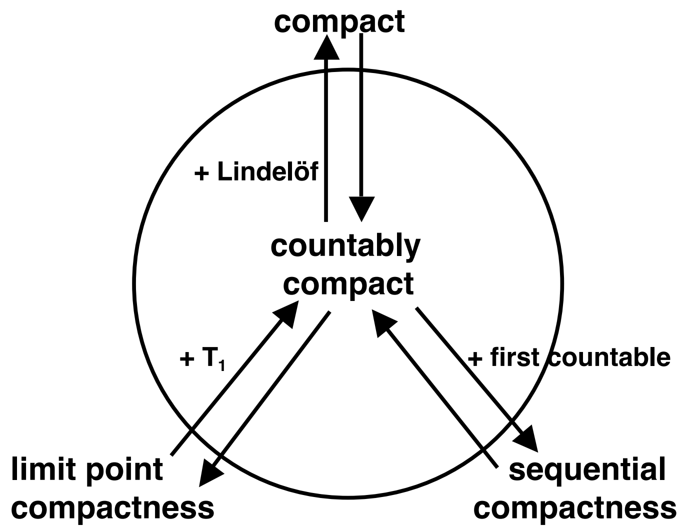
Definition. Let $x : \mathbb{N} \to X$ and $y : \mathbb{N} \to X$ be sequences in a topological space X. y is a *subsequence* of x if there is a strictly increasing function $n : \mathbb{N} \to \mathbb{N}$ such that $y = x \circ n$. In other words, $\{ y_i \}$ is a subsequence of $\{ x_i \}$ is there are positive integers $n(1) < n(2) < n(3) < \ldots$ such that $y_i = x_{n(i)}$ for $i \in \mathbb{N}$. Recall that the sequence $y : \mathbb{N} \to X$ *converges* to a point $z \in X$ if for every neighborhood U of z in X, there is a positive integer n such that $y(i) \in U$ for every $i \geq n$.

Definition. A topological space X is *sequentially compact* if every sequence in X has a converging subsequence.

Definition. A topological space X is *countably compact* if every countable open cover of X has a finite subcover.

Definition. A topological space X is *Lindelöf* if every open cover of X has a countable subcover.

The relations between the various forms of compactness just defined are illustrated by the following figure.



The Great Wheel of Compactness

We now prove the relations indicated in the figure. The first assertion is obvious.

Theorem III.20. Every compact space is countably compact. □

Theorem III.21. Every countably compact Lindelöf space is compact.

Exercise. Prove Theorem III.21.

Theorem III.22. Every limit point compact T_1 space is countably compact.

Proof. Let X be T_1 space which is not countably compact. We will prove that X is not limit point compact.

X has a countable open cover $\{ U_n : n \in \mathbb{N} \}$ which has no finite subcover. Consequently, for each $n \in \mathbb{N}$, the set $X - \left(\bigcup_{i=1}^{n} U_i \right)$ is non-empty, and we can choose a point a_n from this set. Let $A = \{ a_n : n \in \mathbb{N} \}$.

We assert that A is an infinite set. Indeed, if A were a finite set, then A would be covered by some finite subset of the open cover $\{ U_n : n \in \mathbb{N} \}$ of X. Then there would be an $n \in \mathbb{N}$ such that $A \subset \bigcup_{i=1}^{n} U_i$. But this is impossible because $a_n \in A$ and $a_n \notin \bigcup_{i=1}^{n} U_i$.

Next we assert that the set A has no limit points in X. For consider a point $x \in X$. We will find a neighborhood V of x in X that is disjoint from $A - \{ x \}$. Since $\{ U_n : n \in \mathbb{N} \}$ covers X, then $x \in U_n$ for some $n \in \mathbb{N}$. Let $F = A \cap (U_n - \{ x \})$. Then $A \cap U_n \subset F \cup \{ x \}$. Since $a_i \notin U_n$ for $i \geq n$, then $F \subset \{ a_i : 1 \leq i < n \}$. Thus, F is a finite set. Since X is a T_1 space, then F is a closed set. Clearly, $x \notin F$. Let $V = U_n - F$. Then V is a neighborhood of x in X. Furthermore,

$$V \cap (A - \{x\}) = (U_n - F) \cap (A - \{x\}) = (U_n \cap A) - (F \cup \{x\}) = \varnothing.$$

It follows that x is not a limit point of A. We conclude that A has no limit points.

Since X contains an infinite set with no limit points, then X is not limit point compact. □

Theorem III.23. Every countably compact space is limit point compact.

Proof. Let X be a space which is not limit point compact. We will prove that X is not countably compact.

Since X is not limit point compact, then there is an infinite subset A of X that has no limit points. We assert that every subset of A is a closed subset of X. For suppose $B \subset A$. Let $x \in X - B$. Since A has no limit points, then x has a neighborhood U which is disjoint from $A - \{x\}$. Since $B \subset A - \{x\}$, then $U \cap B = \varnothing$. Thus, every point of $X - B$ has a neighborhood which is disjoint from B. This proves B is a closed subset of X.

Now choose an infinite sequence $a_1, a_2, a_3, \ldots$ of distinct points of A. For each positive integer $n \in \mathbb{N}$, define $U_n = X - \{a_i : i \geq n\}$. For each $n \in \mathbb{N}$, since $\{a_i : i \geq n\}$ is a subset of A, then it is a closed subset of X. Hence, each U_n is an open subset of X. Since $U_1 = X - \{a_i : i \geq 1\}$ and $a_n \in U_{n+1}$ for each $n \in \mathbb{N}$, then $\{U_n : n \in \mathbb{N}\}$ covers X. However, since $a_n \notin U_i$ for $1 \leq i \leq n$, then no finite subset of $\{U_n : n \in \mathbb{N}\}$ covers X.

Thus, $\{U_n : n \in \mathbb{N}\}$ is a countable open cover of X that has no finite subcover. This proves X is not countably compact. □

Theorem III.24. Every sequentially compact space is countably compact.

Problem III.4. Prove Theorem III.24.

Theorem III.25. Every countably compact first countable space is sequentially compact.

Proof. Let X be a first countable space which is not sequentially compact. We will prove that X is not countably compact.

Since X is not sequentially compact, it contains a sequence of points $\{x_n\}$ that has no converging subsequence. For each positive integer $n \in \mathbb{N}$, set $T_n = \{x_i : i \geq n\}$ and define $U_n = X - \mathrm{cl}(T_n)$. Each U_n is clearly an open subset of X. We will prove that $\{U_n : n \in \mathbb{N}\}$ covers X but no finite subset of $\{U_n : n \in \mathbb{N}\}$ covers X.

Since $\bigcup_{n=1}^{\infty} U_n = X - \left(\bigcap_{n=1}^{\infty} \mathrm{cl}(T_n) \right)$, then to prove that $\{ U_n : n \in \mathbb{N} \}$ covers X, it suffices to prove that $\bigcap_{n=1}^{\infty} \mathrm{cl}(T_n) = \varnothing$. To this end, assume $\bigcap_{n=1}^{\infty} \mathrm{cl}(T_n) \neq \varnothing$ and let $y \in \bigcap_{n=1}^{\infty} \mathrm{cl}(T_n)$. We will derive a contradiction by showing that some subsequence of $\{ x_n \}$ converges to y.

Since X is first countable, there is a countable basis $\{ B_n : n \in \mathbb{N} \}$ for X at y. We can assume $B_1 \supset B_2 \supset B_3 \supset \ldots$. (If not, replace each B_k by $B_1 \cap B_2 \cap \ldots \cap B_k$.)

We will now inductively construct a sequence $n(1) < n(2) < n(3) < \ldots$ of positive integers such that $x_{n(k)} \in B_{n(k)}$ for each $k \in \mathbb{N}$. To begin: since $y \in \mathrm{cl}(T_1)$, then $B_1 \cap T_1 \neq \varnothing$. Hence, there is an $n(1) \geq 1$ such that $x_{n(1)} \in B_1$. Suppose $k \geq 1$ and we have already found $n(1) < n(2) < \ldots < n(k)$ such that $x_{n(i)} \in B_{n(i)}$ for $1 \leq i \leq k$. Since $y \in \mathrm{cl}(T_{n(k)+1})$, then $B_{n(k)+1} \cap T_{n(k)+1} \neq \varnothing$. Hence, there is an $n(k+1) \geq n(k)+1$ such that $x_{n(k+1)} \in B_{n(k)+1}$. This completes our inductive construction of the sequence $n(1) < n(2) < n(3) < \ldots$ with the property that $x_{n(k)} \in B_{n(k)}$ for each $k \in \mathbb{N}$.

We assert that the subsequence $\{ x_{n(k)} \}$ of $\{ x_n \}$ converges to y. To prove this, let V be a neighborhood of y in X. Then there is a $j \in \mathbb{N}$ such that $B_j \subset V$. Since $1 \leq n(1) < n(2) < n(3) < \ldots$, then $n(k) \geq k$ for each $k \in \mathbb{N}$. Hence, for $k \in \mathbb{N}$: $k \geq j \Rightarrow x_{n(k)} \in B_{n(k)} \subset B_k \subset B_j \subset V$. This proves our assertion: $\{ x_{n(k)} \}$ is a subsequence of $\{ x_n \}$ that converges to y.

Since $\{ x_n \}$ was chosen to be a sequence in X that has no converging subsequences, then we have reached a contradiction. Hence, our original assumption that $\bigcap_{n=1}^{\infty} \mathrm{cl}(T_n) \neq \varnothing$ must be false. Therefore, $\bigcap_{n=1}^{\infty} \mathrm{cl}(T_n) = \varnothing$. Hence, $\bigcup_{n=1}^{\infty} U_n = X$. Thus, $\{ U_n : n \in \mathbb{N} \}$ covers X.

For each $n \in \mathbb{N}$, for $1 \leq i \leq n$, since $x_n \in T_i$, then $x_n \notin U_i$. Hence, $x_n \notin \bigcup_{i=1}^{n} U_i$. Therefore, no finite subset of $\{ U_n : n \in \mathbb{N} \}$ covers X.

Thus $\{ U_n : n \in \mathbb{N} \}$ is a countable open cover of X that has no finite subcover. We conclude that X is not countably compact. □

Problem III.5. Recall that Ω (Example I.10) is an uncountable well-ordered space in which every element has countably many predecessors. Prove that Ω is sequentially compact but not compact. Then observe that Theorems III.21, III.23 and III.24 imply that Ω is countably compact and limit point compact but not Lindelöf.

Problem III.6. Let $\Sigma = \{ 0, 1 \}^{\mathbb{N}}$, the set of all functions from $\mathbb{N}$ to $\{ 0, 1 \}$. Thus, Σ may be regarded as the set of all sequences of 0's and 1's. Now consider the set $\{ 0, 1 \}^{\Sigma}$, the set of all functions from Σ to $\{ 0, 1 \}$. ($\{ 0, 1 \}^{\Sigma}$ may be regarded as the set of all "Σ-indexed sequences" of 0's and 1's.) For every $f \in \{ 0, 1 \}^{\Sigma}$ and every finite subset A of Σ, let

$$N(f,A) = \{ g \in \{ 0, 1 \}^{\Sigma} : g|A = f|A \}.$$

a) Prove that $\{ N(f,A) : f \in \{ 0, 1 \}^{\Sigma}$ and A is a finite subset of $\Sigma \}$ is a basis for a topology on $\{ 0, 1 \}^{\Sigma}$.

Endow $\{ 0, 1 \}^{\Sigma}$ with this topology. In a subsequent chapter, we will see that $\{ 0, 1 \}^{\Sigma}$ can be regarded as an infinite Cartesian product of "Σ-many" copies of the two-point space $\{ 0, 1 \}$. Furthermore, we will learn that topology on $\{ 0, 1 \}^{\Sigma}$ that we have just described is known as the *product topology*. In the infinite Cartesian product $\{ 0, 1 \}^{\Sigma}$, each factor $\{ 0, 1 \}$ has the discrete topology. $\{ 0, 1 \}$ with the discrete topology is a compact space. Thus, $\{ 0, 1 \}^{\Sigma}$ is a Cartesian product of compact spaces with the product topology. One of the fundamental theorems about infinite Cartesian products, the *Tychonoff Theorem*, tells us that the Cartesian product of compact spaces with the product topology is compact. (Also see Additional Problem III.16+.) It follows that $\{ 0, 1 \}^{\Sigma}$ with the topology described above is compact. Since, $\{ 0, 1 \}^{\Sigma}$ is compact, it is countably compact and limit point compact by Theorems III.20 and III.23.

b) Prove that $\{ 0, 1 \}^{\Sigma}$ with the topology described above is not sequentially compact.

Hint. First, prove that if a sequence $\{ f_n \}$ in $\{ 0, 1 \}^{\Sigma}$ converges to an element g of $\{ 0, 1 \}^{\Sigma}$, then for each $\sigma \in \Sigma$, there is an $n \in \mathbb{N}$ such that $f_i(\sigma) = g(\sigma)$ for every $i \geq n$. Second, define a sequence $\{ f_n \}$ in $\{ 0, 1 \}^{\Sigma}$ as follows. For each $n \in \mathbb{N}$, define $f_n \in \{ 0, 1 \}^{\Sigma}$ by the formula $f_n(\sigma) = \sigma(n)$ for each $\sigma \in \Sigma$. Now prove that no subsequence of $\{ f_n \}$ converges by the following argument. Let $\{ f_{n_i} \}$ be a subsequence of $\{ f_n \}$. Define an element σ of Σ such that the value of $\sigma(n_i)$ oscillates between 0 and 1 as i increases. Then argue that this behavior prevents the sequence $\{ f_{n_i} \}$ from converging to any element of $\{ 0, 1 \}^{\Sigma}$.

Problem III.7. Regard Ω and $\{ 0, 1 \}^{\Sigma}$ as disjoint sets, and consider their union $\Omega \cup \{ 0, 1 \}^{\Sigma}$.

a) Prove that the collection of sets

$$\{ U \cup V : U \text{ is an open subset of } \Omega \text{ and } V \text{ is an open subset of } \{ 0, 1 \}^{\Sigma} \}$$

is a topology on $\Omega \cup \{ 0, 1 \}^{\Sigma}$. This topology is called the *disjoint union* topology on $\Omega \cup \{ 0, 1 \}^{\Sigma}$.

Endow $\Omega \cup \{ 0, 1 \}^{\Sigma}$ with the disjoint union topology.

b) Prove that $\Omega \cup \{ 0, 1 \}^{\Sigma}$ with the disjoint union topology is countably compact but not compact and not sequentially compact.

Then observe that $\Omega \cup \{ 0, 1 \}^{\Sigma}$ is limit point compact but not Lindelöf and not first countable by Theorems III.21, III.23 and III.25.

Problem III.8. Let the set $\mathbb{N}$ of positive integers have the discrete topology, and let $\{ 0, 1 \}^*$ denote the two-point set $\{ 0, 1 \}$ endowed with the *indiscrete* topology. Consider the space $\mathbb{N} \times \{ 0, 1 \}^*$ endowed with the product topology. Prove that $\mathbb{N} \times \{ 0, 1 \}^*$ is limit point compact, but not compact, not countably compact and not sequentially compact. Observe that $\mathbb{N} \times \{ 0, 1 \}^*$ is not T_1 by Theorem III.22.

The relevant properties of the spaces Ω, $\{ 0, 1 \}^{\Sigma}$, $\Omega \cup \{ 0, 1 \}^{\Sigma}$ and $\mathbb{N} \times \{ 0, 1 \}^*$ are summarized in the following table. These spaces illustrate that the additional hypotheses of Lindelöf, T_1 and first countable that are attached to certain arrows in the Great Wheel of Compactness can't be omitted.

Space	**compact**	**sequentially compact**	**countably compact**	**limit point compact**	**other properties**
Ω	**no**	**yes**	**yes**	**yes**	**first countable, not Lindelöf**
$\{ 0, 1 \}^{\Sigma}$	**yes**	**no**	**yes**	**yes**	**not first countable, Lindelöf**
$\Omega \cup \{ 0, 1 \}^{\Sigma}$	**no**	**no**	**yes**	**yes**	**not first countable, not Lindelöf**
$\mathbb{N} \times \{ 0, 1 \}^*$	**no**	**no**	**no**	**yes**	**not T_1**

We conclude this section with an examination of the Lindelöf property.

Theorem III.26. Every second countable space is Lindelöf.

Problem III.9. Prove Theorem III.26.

Theorem III.27. Every regular Lindelöf space is normal.

Proof. Let A and B be disjoint closed subsets of a regular Lindelöf space X.

Since X is a regular space, then for each $x \in A$, there is a neighborhood L_x of x in X such that $cl(L_x) \cap B = \emptyset$. Then $\{ L_x : x \in A \} \cup \{ X - A \}$ is an open cover of X. Since X is a Lindelöf space, then there is a countable subset $\{ M_i : i \in \mathbb{N} \}$ of $\{ L_x : x \in A \}$ such that $\{ M_i : i \in \mathbb{N} \} \cup \{ X - A \}$ covers X. Hence, $\{ M_i : i \in \mathbb{N} \}$ is a countable collection of open subsets of X that covers A such that $cl(M_i) \cap B = \emptyset$ for each $i \in \mathbb{N}$. Similarly there is a countable collection $\{ N_i : i \in \mathbb{N} \}$ of open subsets of X that covers B such that $cl(N_i) \cap A = \emptyset$ for each $i \in \mathbb{N}$.

For each $k \in \mathbb{N}$, let $U_k = M_k - \left(\bigcup_{i=1}^{k} cl(N_i) \right)$ and let $V_k = N_k - \left(\bigcup_{i=1}^{k} cl(M_i) \right)$. Then U_k and V_k are open subsets of X. For each $k \in \mathbb{N}$, since $\left(\bigcup_{i=1}^{k} cl(N_i) \right) \cap A = \emptyset$, then $U_k \cap A = M_k \cap A$. Hence, $\{ U_k : k \in \mathbb{N} \}$ covers A. Similarly, $\{ V_k : k \in \mathbb{N} \}$ covers B. We assert that for all $j, k \in \mathbb{N}$, $U_j \cap V_k = \emptyset$. If $j \leq k$, then $U_j \subset M_j \subset \bigcup_{i=1}^{k} cl(M_i)$; so clearly $U_j \cap V_k = \emptyset$. A symmetric argument show that $U_j \cap V_k = \emptyset$ if $j \geq k$.

Define $U = \bigcup_{k=1}^{\infty} U_k$ and $V = \bigcup_{k=1}^{\infty} V_k$. It follows that U and V are disjoint neighborhoods of A and B in X. This proves X is normal. ◻

Problem III.10. Recall that the set $\{ [a, b) : a < b \}$ of all closed-open intervals in the set of real numbers $\mathbb{R}$ is a basis for a topology on $\mathbb{R}$, and $\mathbb{R}$ endowed with this topology is denoted $\mathbb{R}_{bad}$. Recall that $\mathbb{R}_{bad}$ is not second countable.

a) Prove that $\mathbb{R}_{bad}$ is Lindelöf.

b) Is $\mathbb{R}_{bad} \times \mathbb{R}_{bad}$ Lindelöf?

c) Is every subspace of a Lindelöf space Lindelöf?

C. Compactness in Metric Spaces

Our first goal in this section is to show that in a metric space, the four forms of compactness – compactness, countable compactness, limit point compactness and sequential compactness – are equivalent. Since metric spaces are T_1 and first countable, then the theorems of the previous section imply that in the metric space setting, the last three of these properties – countable compactness, limit point compactness and sequential compactness – are equivalent and are implied by compactness. (See Theorems III.20, III.22, III.23, III.24 and III.25.) It remains to show that in a metric space, these three properties imply compactness.

It is most convenient to prove that for metric spaces, compactness is implied by sequential compactness. It suffices to prove that every sequentially compact metric space is separable. This is because every separable metric space is second countable, every second countable space is Lindelöf (Theorem III.26), and every sequentially compact Lindelöf space is compact (Theorems III.21 and III.24).

Definition. A metric space (X,ρ) is *totally bounded* if for every $\varepsilon > 0$, there is a finite subset F of X such that $\{ N(x,\varepsilon) : x \in F \}$ covers X. In other words, X is totally bounded if and only if for every $\varepsilon > 0$, there is a finite subset F of X such that every point of X lies within distance ε of some point of F.

Theorem III.28. Every sequentially compact metric space is totally bounded.

Proof. Assume that the metric space (X,ρ) is not totally bounded. We will prove that X is not sequentially compact. Since X is not totally bounded, then there is an $\varepsilon > 0$ such that for every finite subset F of X, $\bigcup_{x \in F} N(x,\varepsilon) \neq X$. Consequently, one can inductively choose a sequence of points $x_1, x_2, x_3, \ldots$ in X such that $x_{n+1} \notin \bigcup_{i=1}^{n} N(x_i,\varepsilon)$ for each $n \in \mathbb{N}$. (Verify!) It follows that $\rho(x_i,x_j) \geq \varepsilon$ for all $i \neq j$. We assert that the sequence $\{ x_n \}$ has no converging subsequences. Indeed, if a subsequence $\{ x_{n(i)} \}$ of $\{x_n \}$ converged to a point y of X, then there would be a positive integer j such that $x_{n(i)} \in N(y,\varepsilon/2)$ for every $i \geq j$. But then for $i > j$, $\rho(x_{n(i)},x_{n(j)}) \leq \rho(x_{n(i)},y) + \rho(y,x_{n(j)}) < \varepsilon/2 + \varepsilon/2 = \varepsilon$, a contradiction. Thus, as claimed, $\{ x_n \}$ has no converging subsequences. It follows that X is not sequentially compact. ☐

Theorem III.29. Every totally bounded metric space is separable.

Proof. Let (X,ρ) be a totally bounded metric space. Then for each positive integer n, there is a finite subset F_n of X such that $\{ N(x,{}^1/_n) : x \in F_n \}$ covers X. Define D $= \bigcup_{n=1}^{\infty} F_n$. We assert that D is a dense countable subset of X. Clearly D is countable. To see that D is dense in X, let U be a non-empty open subset of X and let $x \in U$. Then there is a positive integer n such that $N(x,{}^1/_n) \subset U$. There is a $y \in F_n$ such that $x \in$

$N(y,{}^1/_n)$. Hence, $\rho(x,y) < {}^1/_n$. So $y \in D \cap N(x,{}^1/_n) \subset D \cap U$. Thus, $D \cap U \neq \emptyset$. This proves D is a dense subset of X. Hence, X is separable. □

Exercise. Is every separable metric space totally bounded?

Theorem III.30. For metric spaces, compactness, countable compactness, limit point compactness and sequential compactness are equivalent.

Proof. We simply repeat the chain of reasoning given at the beginning of this section.

Since metric spaces are T_1 and first countable, then countable compactness, limit point compactness and sequential compactness are equivalent in metric spaces by Theorems III.22 through III.5.

Every compact metric space is sequentially compact by Theorems III.20 and III.25. The converse implication – every sequentially compact metric space is compact – is obtained by combining the following previously proved results.

- Every sequentially compact metric space is totally bounded (Theorem III.28).
- Every totally bounded metric space is separable (Theorem III.29).
- Every separable metric space is second countable (Theorem I.13.)
- Every second countable space is Lindelöf (Theorem III.26).
- Every sequentially compact Lindelöf space is compact (Theorems III.24 and III.11).

We conclude that every sequentially compact metric space is compact. □

Combining Theorems III.28, III.29 and III.30, we have:

Corollary III.31. Every compact metric space is totally bounded and separable.

Definition. A *Lebesgue number* of an open cover $\mathscr{U}$ of a metric space (X,ρ) is a real number $\lambda > 0$ such that every subset of X of diameter $\leq \lambda$ is contained in an element of $\mathscr{U}$.

Exercise. Verify that $\{ ({}^1/_{n+2},{}^1/_n) : n \in \mathbf{N} \}$ is an open cover of $(0, 1)$ that has no Lebesgue number.

Theorem III.32. Every open cover of a compact metric space has a Lebesgue number.

Proof. Let $\mathscr{U}$ be an open cover of a compact metric space (X,ρ). For each $x \in X$, first choose an element U_x of $\mathscr{U}$ such that $x \in U_x$, and then choose $\varepsilon_x > 0$ so that $N(x,\varepsilon_x) \subset U_x$. Since X is compact, there is a finite subset F of X such that $\{ N(x,\varepsilon_x/2) : x \in F \}$ covers X. Let $\lambda = \min \{ \varepsilon_x/2 : x \in F \}$. We assert that λ is a Lebesgue number of $\mathscr{U}$. To prove this assertion, let A be a subset of X of diameter $\leq \lambda$.

Let $y \in A$. Then there is an $x \in F$ such that $y \in N(x,\varepsilon_x/2)$. Hence, for every $z \in A$,

$$\rho(z,x) \le \rho(z,y) + \rho(y,x) \le \mathrm{diam}(A) + \varepsilon_x/2 \le \lambda + \varepsilon_x/2 \le 2(\varepsilon_x/2) = \varepsilon_x.$$

Hence, if $z \in A$, then $z \in N(x,\varepsilon_x)$. This proves $A \subset N(x,\varepsilon_x)$. Since $N(x,\varepsilon_x) \subset U_x$, then $A \subset U_x \in \mathscr{U}$. We conclude that λ is a Lebesgue number of $\mathscr{U}$. □

Definition. A function $f : (X,\rho) \to (Y,\sigma)$ is *uniformly continuous* if for every $\varepsilon > 0$, there is a $\delta > 0$ such that if $x, x' \in X$ and $\rho(x,x') < \delta$, then $\sigma(f(x),f(x')) < \varepsilon$.

Problem III.11. Let $f : (X,\rho) \to (Y,\sigma)$ be a map between metric spaces. Use Lebesgue numbers to prove that if X is compact, then f is uniformly continuous.