C. Continuous Functions on Normal Spaces

One of the surprising properties of normal spaces is that they possess lots of continuous functions. In fact, normality can be characterized in terms of the exisence of sufficiently many continuous functions. Our first result – Urysohn's Lemma - establishes a direct link between normality and the existence of continuous functions. Urysohn's Lemma is an important and valuable tool. Furthermore, its proof is a piece of topological magic which appears to fabricate a continuous function from thin air.

Theorem II.12: Urysohn's Lemma. If A and B are disjoint closed subsets of a normal space X, then there is a map $f : X \to [\,0, 1\,]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Remark. In the special case that X is a metric space, the proof of Urysohn's Lemma is much simpler than in the general case because the metric can be used to construct the function f. Moreover, in the metric case, a version of Urysohn's Lemma can be proved that is apparently stronger than Theorem II.12. Indeed, when A and B are disjoint closed subsets of a **metric** space X, then there is a map $f : X \to [\,0, 1\,]$ satisfying $f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$. (The problem of constructing such a map was assigned in Problem II.2.c.) The conditions $f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$ imply the conclusion of Theorem II.12; indeed, $f(A) = f(f^{-1}(\{0\})) \subset \{0\}$ and $f(B) = f(f^{-1}(\{1\})) \subset \{1\}$. In fact, the conditions $f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$ are strictly stronger than the conclusion of Theorem II.12, and these conditions can't necessarily be achieved in the general case that X is a normal space. (See Problem II.15.)

Motivation for the proof of Urysohn's Lemma. Suppose $f : X \to [\,0, 1\,]$ is a continuous function. Let D be any dense subset of $[\,0, 1\,]$. For each $t \in D$, let $C_t = f^{-1}([\,0, t\,])$. Then $\{ C_t : t \in D \}$ is a collection of closed subsets of X with the property:

$$\text{If } s \text{ and } t \in D \text{ and } s < t, \text{ then } C_s \subset \text{int}(C_t). \qquad \ldots \textbf{(i)}$$

Proof of (i): If s and $t \in D$ and $s < t$, then

$$C_s = f^{-1}([\,0, s\,]) \subset f^{-1}([\,0, t\,)) \subset f^{-1}([\,0, t\,]) = C_t.$$

Since $f^{-1}([\,0, t\,))$is an open subset of X, then it follows that $C_s \subset \text{int}(C_t)$. □

The function $f : X \to [\,0, 1\,]$ determines the collection of closed sets $\{ C_t : t \in D \}$. Conversely, the collection of closed sets $\{ C_t : t \in D \}$ can be used to reconstruct the function $f : X \to [0,1]$. Indeed, f is determined from $\{ C_t : t \in D \}$ by the formula

$$f(x) = \inf\,(\{ t \in D : x \in C_t \} \cup \{1\}) \qquad \ldots \textbf{(ii)}$$

Proof of (ii): For each $x \in X$, let $D(x) = \{ t \in D : x \in C_t \} \cup \{1\}$, and define the function $g : X \to [\,0, 1\,]$ by $g(x) = \inf D(x)$. We must prove that $f = g$. Assume $f(x) \neq g(x)$ for some $x \in X$. Then either $f(x) < g(x)$ or $g(x) < f(x)$.

- If f(x) < g(x), then there is an $s \in D$ such that $f(x) < s < g(x)$. In this case: $f(x) < s \;\Rightarrow\; x \in f^{-1}([\,0, s\,]) = C_s \;\Rightarrow\; s \in D(x) \;\Rightarrow\; g(x) \le s$, a contradiction.

- If g(x) < f(x), then there is an $s \in D$ such that $g(x) < s < f(x)$. In this case, $s < f(x) \;\Rightarrow\; x \notin f^{-1}([\,0, r\,]) = C_r$ for all $r \le s \;\Rightarrow\; [\,0, s\,] \cap D(x) = \varnothing \;\Rightarrow\; D(x) \subset (s,1] \;\Rightarrow\; g(x) \ge s$, a contradiction.

We conclude that f = g. Therefore, f is determined by the formula **(i)**. □

The fact that the function f can be reconstructed from the collection of closed sets $\{ C_t : t \in D \}$ suggests the following question. Given a collection of closed sets $\{ C_t : t \in D \}$ satisfying the condition **(i)** but which may not originally arise from a continuous function, might it never the less be possible to construct a continuous function f from the sets $\{ C_t : t \in D \}$ by the formula **(ii)**? The answer is "yes", and this fact is the key idea in the proof of Urysohn's Lemma.

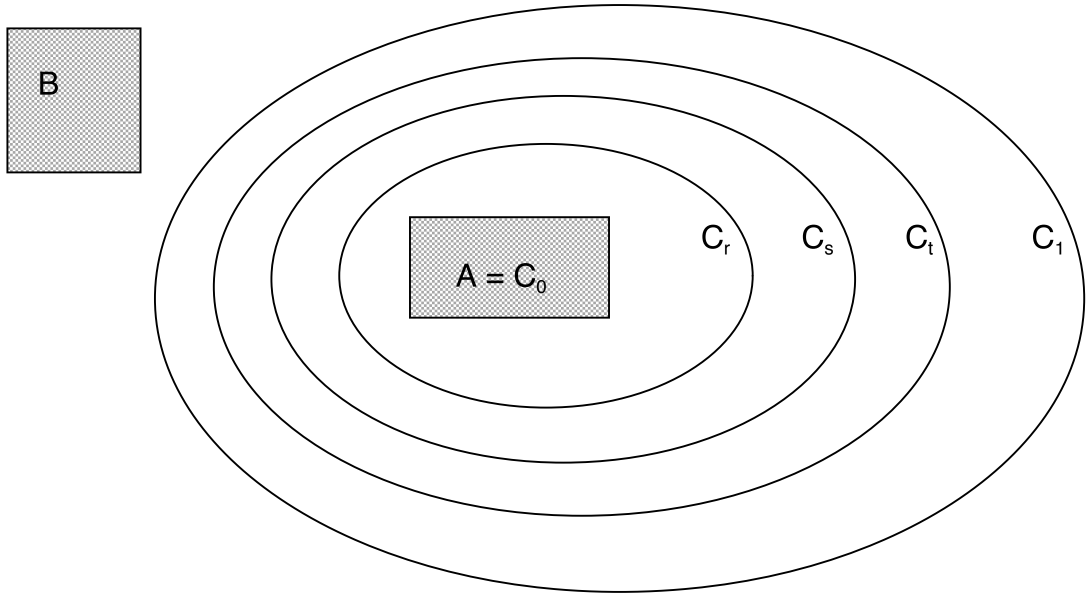
Proof of Urysohn's Lemma. Let A and B be disjoint closed subsets of a normal space X. Let D be a countable dense subset of [0, 1] that contains the points 0 and 1.

Step 1. We exploit the normality of X to construct a collection $\{ C_t : t \in D \}$ of closed subset of X such that

$$A \subset C_0,\ C_1 \subset X - B, \text{ and } C_s \subset \text{int}(C_t) \text{ whenever } s \text{ and } t \in D \text{ and } s < t. \;.\;.\;.\; \textbf{(iii)}$$

We will construct the sets $\{ C_t : t \in D \}$ by induction. First we enumerate D as D = $\{ t_1, t_2, t_3, \cdots \}$ where $t_1 = 0$, $t_2 = 1$ and $t_i \ne t_j$ for $i \ne j$.

We begin the inductive construction of the C_t's by setting $C_0 = A$. For the second step in our inductive construction of the C_t's: since X is normal and since the closed set C_0 is contained in the open set X – B, then Theorem I.23.c implies there is a closed subset C_1 of X such that $C_0 \subset \text{int}(C_1) \subset C_1 \subset X - B$. Now let $n \ge 2$, and assume we have constructed closed sets $C_{t_1}, C_{t_2}, \cdots , C_{t_n}$ so that if i and j are integers between 1 and n and $t_i < t_j$, then $C_{t_i} \subset \text{int}(C_{t_j})$. Now consider t_{n+1} (the $(n+1)^{st}$ element of D). There are integers i and j between 1 and n such that $t_i < t_{n+1} < t_j$, and no element of $\{ t_1, t_2, \cdots , t_n \}$ lies strictly between t_i and t_j. Since X is normal and since the closed set C_{t_i} is contained in the open set $\text{int}(C_{t_j})$, then Theorem I.23.c implies there is a closed set $C_{t_{n+1}}$ such that $C_{t_i} \subset \text{int}(C_{t_{n+1}}) \subset C_{t_{n+1}} \subset \text{int}(C_{t_j})$. Given this choice of $C_{t_{n+1}}$, it is now easily verified that the closed sets $C_{t_1}, C_{t_2}, \cdots , C_{t_n}$ satisfy $C_{t_i} \subset \text{int}(C_{t_j})$ whenever i and j are integers between 1 and n+1 such that $t_i < t_j$. It follows by induction that we can choose an infinite sequence of closed sets $C_{t_1}, C_{t_2}, C_{t_3}, \cdots$ such that $C_{t_i} \subset \text{int}(C_{t_j})$ whenever i and j are positive integers such that $t_i < t_j$. This completes Step 1.



$0 < r < s < t < 1$

Step 2. For each $x \in X$, let $D(x) = \{ t \in D : x \in C_t \} \cup \{1\}$. Now we define the function $f : X \to [0, 1]$ by

$$f(x) = \inf D(x) \qquad \ldots \textbf{(iv)}$$

We must prove that $f(A) = \{ 0 \}$, $f(B) = \{ 1 \}$ and that f is continous.

To prove $f(A) = \{ 0 \}$, assume $x \in A = C_0$. Then $0 \in D(x) \subset [0,1]$. Hence, $f(x) = 0$.

To prove $f(B) = \{ 1 \}$, assume $x \in B$. Since $C_1 \subset X - B$, then $x \notin C_1$. Hence, $t \in D \Rightarrow t \le 1 \Rightarrow C_t \subset C_1 \Rightarrow x \notin C_t$. Therefore, $D(x) = \{ 1 \}$. Hence, $f(x) = 1$.

Next we make two observations that will help us to prove the continuity of f.

First: $x \in C_t \Rightarrow t \in D(x) \Rightarrow f(x) \le t$.

Second: $x \notin C_t \Rightarrow x \notin C_s$ for all $s \in D$ such that $s \le t$ (because $s \le t$ implies $C_s \subset C_t$) $\Rightarrow D(x) \cap [0, t] = \varnothing \Rightarrow D(x) \subset (t, 1] \Rightarrow f(x) \ge t$.

To summarize: $x \in C_t \Rightarrow f(x) \le t$, and $x \notin C_t \Rightarrow f(x) \ge t$. . . . **(v)**

Passing to contrapositives, we obtain:

$$f(x) > t \Rightarrow x \notin C_t, \text{ and } f(x) < t \Rightarrow x \in C_t. \qquad \ldots \textbf{(vi)}$$

To prove the continuity of f, let $x \in X$ and let U be a neighborhood of f(x) in [0, 1]. We consider three cases: $0 < f(x) < 1$, $f(x) = 0$ and $f(x) = 1$.

Case 1: $0 < f(x) < 1$. Since D is a dense subset of [0, 1], then there are elements r, s and t of D such that $r < f(x) < s < t$ and $[r, t] \subset U$. Then **(vi)** implies $x \in C_s \subset \text{int}(C_t)$ and $x \notin C_r$. Hence, $\text{int}(C_t) - C_r$ is a neighborhood of x in X. Furthermore, **(v)** implies: $y \in \text{int}(C_t) - C_r \Rightarrow f(y) \le t$ and $f(y) \ge r \Rightarrow f(y) \in [r, t] \subset U$. Hence, $f(\text{int}(C_t) - C_r) \subset U$.

Case 2: $f(x) = 0$. Since D is a dense subset of [0, 1], there are elements s and t of D such that $f(x) = 0 < s < t$ and $[0, t] \subset U$. As before, **(vi)** implies $x \in C_s \subset \text{int}(C_t)$. Thus, $\text{int}(C_t)$ is a neighborhood of x in X. Also, as before, **(v)** implies: $y \in \text{int}(C_t) \Rightarrow f(y) \le t \Rightarrow f(y) \in [0, t] \subset U$. Therefore, $f(\text{int}(C_t)) \subset U$.

Case 3: $f(x) = 1$. Since D is a dense subset of [0, 1], there is an element r of D such that $r < f(x) = 1$ and $[r, 1] \subset U$. As before, **(vi)** implies $x \notin C_r$. Hence, $X - C_r$ is a neighborhood of x in X. Also, as before, **(v)** implies: $y \in X - C_r \Rightarrow f(y) \ge r \Rightarrow f(y) \in [r, 1] \subset U$. Thus, $f(X - C_r) \subset U$.

Therefore, in each case, there is a neighborhood V of x in X such that $f(V) \subset U$. We conclude that $f : X \to [0, 1]$ is continuous. □

We remarked earlier that normality can be characterized in terms of the existence of sufficiently many continuous functions. Urysohn's Lemma yields such a characterization.

Corollary II.13. A topological space X is normal if and only if for any two disjoint closed subsets A and B of X, there is a map $f : X \to [0, 1]$ such that $f(A) = \{ 0 \}$ and $f(B) = \{ 1 \}$.

Proof. Urysohn's Lemma clearly implies the forward direction of this result.

To prove the converse direction assume that the space X has the property that for any two disjoint closed subsets A and B of X, there is a map $f : X \to [0, 1]$ such that $f(A) = \{ 0 \}$ and $f(B) = \{ 1 \}$. To prove X is normal, let A and B be disjoint closed subsets of X. Then there is a map $f : X \to [0, 1]$ such that $f(A) = \{ 0 \}$ and $f(B) = \{ 1 \}$. Hence, $A \subset f^{-1}(\{ 0 \}) \subset f^{-1}([0, {}^1/_2))$, $B \subset f^{-1}(\{ 1 \}) \subset f^{-1}(({}^1/_2, 1])$ and $f^{-1}([0, {}^1/_2)) \cap f^{-1}(({}^1/_2, 1]) = f^{-1}([0, {}^1/_2) \cap ({}^1/_2, 1]) = f^{-1}(\varnothing) = \varnothing$. Thus, $f^{-1}([0, {}^1/_2))$ and $f^{-1}(({}^1/_2, 1])$ are disjoint neighborhoods of A and B, respectively. It follows that X is normal. □

The next problem illustrates that the strong form of Urysohn's Lemma that holds for metric spaces is not, in general, valid in normal spaces.

Problem II.15. Recall the space $\Omega^+ = \Omega \cup \{ \omega^+ \}$ which was defined in Example I.11. Ω^+ is normal and in section I.F, we discussed several approaches to a proof that Ω^+ is normal. Prove there is no continuous function $f : \Omega^+ \to [0, 1]$ such that $f^{-1}(\{0\}) = \{ \omega^+ \}$. It follows that if x is any point of Ω, then there is no continuous function $f : \Omega^+ \to [0, 1]$ such that $f^{-1}(\{0\}) = \{ \omega^+ \}$ and $f^{-1}(\{1\}) = \{ x \}$, even though $\{ \omega^+ \}$ and $\{ x \}$ are disjoint closed subsets of Ω^+. Thus, the strong form of Urysohn's Lemma that holds for metric spaces is not valid in the normal space Ω^+.

Hint: Assume there is a continuous function $f : \Omega^+ \to [0, 1]$ such that $f^{-1}(\{0\}) = \{ \omega^+ \}$. Prove that $\{ \omega^+ \}$ equals the intersection of the countable collection of open sets $\{ f^{-1}([0, {}^1/_n)) : n \in \mathbb{N} \}$. Contradict this statement by proving that $\{ \omega^+ \}$ can't be expressed as the intersection of a countable collection of open sets.

As we remarked above, Urysohn's Lemma is a valuable tool with many important consequences. For instance, it plays a crucial role in the proof of Urysohn's Metrization Theorem which states that every second countable regular T_1 space is metrizable. Urysohn's Metrization Theorem will be proved in a later chapter. In this chapter, we prove the following important consequence of Urysohn's Lemma.

Theorem II.14: The Tietze Extension Theorem. If X is a normal space, then every map from a closed subset of X to [0, 1] extends to a map from X to [0, 1].

In other words, every normal space X has the following property. If $f : A \to [0, 1]$ is a map from a closed subset A of X to [0, 1], then there is a map $g : X \to [0, 1]$ such that $g \mid A = f$. In our proof, the map $g : X \to [0, 1]$ will be obtained as the limit of a sequence of "approximate extensions" of $f : A \to [0,1]$. The construction of these approximate extensions is the goal of the following two lemmas.

Lemma II.15: The First Approximate Extension Lemma. Let X be a normal space, and let $f : A \to [a, b]$ be a map from a closed subset A of X to a closed interval [a, b] in $\mathbb{R}$. Then for every $\delta > 0$, there is a map $g : X \to [a, b]$ such that $| f(x) - g(x) | \le \delta$ for every $x \in A$.

Proof. Let $\delta > 0$. Choose $n \in \mathbb{N}$ so that $\frac{b-a}{n} \le \delta$. For $0 \le i \le n$, let $c_i = a + i\left(\frac{b-a}{n}\right)$. Then $a = c_0 < c_1 < \ldots < c_n = b$ and $c_i - c_{i-1} = \frac{b-a}{n} \le \delta$ for $1 \le i \le n$.

For $1 \le i \le n$, $f^{-1}([a, c_{i-1}])$ and $f^{-1}([c_i, b])$ are disjoint closed subsets of A. Hence, they are disjoint closed subsets of X by Theorem I.26.g. Therefore, Urysohn's Lemma provides a map $\lambda_i : X \to [0, 1]$ such that $\lambda_i(f^{-1}([a, c_{i-1}])) = \{0\}$ and $\lambda_i(f^{-1}([c_i, b])) = \{1\}$.

Now define the function $g : X \to \mathbb{R}$ by $g(x) = a + \left(\frac{b-a}{n}\right)\sum_{i=1}^{n} \lambda_i(x)$ for $x \in X$.

Since g is a sum of constant multiples of continuous functions, then the continuity of g follows from Theorem II.3.

Since $\frac{b-a}{n} > 0$ and $0 \le \lambda_i(x) \le 1$ for $x \in X$, then

$$a \le g(x) \le a + \left(\frac{b-a}{n}\right)n = b$$

for each $x \in X$. Hence, $g(X) \subset [\, a, b \,]$.

To prove that $|\, f(x) - g(x) \,| \le \delta$ for every $x \in A$, let $x \in A$. Since $f(x) \in [\, a, b \,]$ and $a = c_0 < c_1 < \ldots < c_n = b$, then there is an integer k between 1 and n such that $f(x) \in [\, c_{k-1}, c_k \,]$. Observe that $1 \le i \le k-1 \Rightarrow c_i \le c_{k-1} \le f(x) \Rightarrow x \in f^{-1}([\, c_i, b \,]) \Rightarrow \lambda_i(x) = 1$, and that $k+1 \le i \le n \Rightarrow f(x) \le c_k \le c_{i-1} \Rightarrow x \in f^{-1}([\, a, c_{i-1} \,]) \Rightarrow \lambda_i(x) = 0$. Hence,

$$g(x) = a + \left(\frac{b-a}{n}\right)\left(\left(\sum_{i=1}^{k-1} 1\right) + \lambda_k(x) + \left(\sum_{i=k+1}^{n} 0\right) \right) =$$

$$a + \left(\frac{b-a}{n}\right)(k-1) + \left(\frac{b-a}{n}\right)\lambda_i(x) = c_{k-1} + \left(\frac{b-a}{n}\right)\lambda_i(x).$$

Therefore, $c_{k-1} \le g(x) \le c_{k-1} + \left(\frac{b-a}{n}\right)1 = c_k$. So $g(x) \in [\, c_{k-1}, c_k \,]$. Since both f(x) and g(x) lie in $[\, c_{k-1}, c_k \,]$, then $|\, f(x) - g(x) \,| \le c_k - c_{k-1} = \frac{b-a}{n} \le \delta$. □

Given a map $f : A \to [\, 0, 1 \,]$, the First Approximate Extension Lemma provides a map $g : X \to [\, 0, 1 \,]$ such that $g \,|\, A$ closely approximates f. Repeated use of the First Approximation Lemma would provide us with a sequence of maps $g_n : X \to [\, 0, 1 \,]$ ($n \ge 1$) such that $\{ g_n \,|\, A \}$ converges to f in C(A). This does not complete the proof of the Tietze Extension Theorem because we have not controlled the behavior of the g_n's at points of X – A. So we have no guarantee that $\{ g_n \}$ converges to a function that is defined at points of X – A. The purpose of the Second Approximate Extension Lemma is to produce a sequence of maps from X to [0, 1] whose restrictions to A converge to f while their restrictions to X – A exhibit enough control insure that they converge to a continuous function. Specifically, given a map $f : A \to [\, 0, 1 \,]$, an $\varepsilon > 0$ and a map $g : X \to [\, 0, 1 \,]$ such that $g \,|\, A$ is ε-close to f, then the Second Approximate Extension Lemma provides a map $h : X \to [0,1]$ that simultaneously achieves two objectives: (i) $h|A$ approximates f arbitrarily closely, and (ii) h is ε-close to g. The Second Approximate Extension Lemma is proved by a clever use of the First Approximate Extension Lemma.

Lemma II.16: The Second Approximate Extension Lemma. Let X be a normal space, and let $f : A \to [\,0, 1\,]$ be a map from a closed subset A of X to [0, 1]. Suppose $\varepsilon > 0$ and $g : X \to [\,0, 1\,]$ is a map such that $|\, f(x) - g(x)\,| \leq \varepsilon$ for every $x \in A$. Then for every $\delta > 0$, there is a map $h : X \to [\,0, 1]$ such that $|\, f(x) - h(x)\,| \leq \delta$ for every $x \in A$ and $|\, g(x) - h(x)\,| \leq \varepsilon$ for every $x \in X$.

Proof. Let $\delta > 0$. Since $|\, f(x) - g(x)\,| \leq \varepsilon$ for every $x \in A$, then $f - g \,|\, A$ maps A into $[\,-\varepsilon, \varepsilon\,]$. We apply the First Approximate Extension Lemma to the map $f - g \,|\, A : A \to [\,-\varepsilon, \varepsilon\,]$ to obtain a map $d : X \to [\,-\varepsilon, \varepsilon\,]$ such that $|\,(f(x) - g(x)) - d(x)\,| \leq \delta$ for every $x \in A$. Next define the map $h_0 : X \to \mathbb{R}$ by $h_0(x) = g(x) + d(x)$ for $x \in X$. Then clearly:

- $|\, f(x) - h_0(x)\,| = |\, f(x) - g(x) - d(x)\,| \leq \delta$ for $x \in A$, and
- $|\, g(x) - h_0(x)\,| = |\,-d(x)\,| \leq \varepsilon$ for $x \in X$

The only reason we are not finished with the proof at this point is that $h_0(X)$ may not be a subset of [0, 1]. To remedy this flaw, define the function $h : X \to [\,0, 1\,]$ by

$$h(x) = \begin{cases} 0 & \text{if } x \in h_0^{-1}((-\infty, 0\,]) \\ h_0(x) & \text{if } x \in h_0^{-1}([\,0, 1\,]) \\ 1 & \text{if } x \in h_0^{-1}([\,1, \infty)) \end{cases}.$$

$h : X \to [\,0, 1\,]$ is well defined and continuous by Theorem II.5.b. It remains to show that $|\, f(x) - h(x)\,| \leq \delta$ for every $x \in A$ and $|\, g(x) - h(x)\,| \leq \varepsilon$ for every $x \in X$.

Let $x \in A$. Then $f(x) \in [\,0, 1\,]$. If $x \in h_0^{-1}([\,0, 1\,])$, then $|\, f(x) - h(x)\,| = |\, f(x) - h_0(x)\,| \leq \delta$. If $x \in h_0^{-1}((-\infty, 0\,])$, then $h_0(x) \leq 0 = h(x) \leq f(x)$. So $|\, f(x) - h(x)\,| \leq |\, f(x) - h_0(x)\,| \leq \delta$. If $x \in h_0^{-1}([\,1, \infty))$, then $h_0(x) \geq 1 = h(x) \geq f(x)$. So $|\, f(x) - h(x)\,| \leq |\, f(x) - h_0(x)\,| \leq \delta$. This proves $|\, f(x) - h(x)\,| \leq \delta$ for all $x \in A$.

Let $x \in X$. Then $g(x) \in [\,0, 1\,]$. If $x \in h_0^{-1}([\,0, 1\,])$, then $|\, g(x) - h(x)\,| = |\, g(x) - h_0(x)\,| \leq \varepsilon$. If $x \in h_0^{-1}((-\infty, 0\,])$, then $h_0(x) \leq 0 = h(x) \leq g(x)$. So $|\, g(x) - h(x)\,| \leq |\, g(x) - h_0(x)\,| \leq \varepsilon$. If $x \in h_0^{-1}([\,1, \infty))$, then $h_0(x) \geq 1 = h(x) \geq g(x)$. So $|\, g(x) - h(x)\,| \leq |\, g(x) - h_0(x)\,| \leq \varepsilon$. This proves $|\, g(x) - h(x)\,| \leq \varepsilon$ for all $x \in A$. □

We now turn to the proof of the Tietze Extension Theorem. Given a map $f : A \to [\,0, 1\,]$ from a closed subset A of a normal space X to [0,1], we will use the two approximate extension lemmas to construct a sequence of maps $g_n : X \to [\,0, 1\,]$ $(n \geq 1)$

that converge to a map $g : X \to [0,1]$ such that the sequence $\{ g_n \mid A \}$ converges to f. It will then follow that $g \mid A = f$.

Proof of Theorem II.14: The Tietze Extension Theorem. Let X be a normal space and let $f : A \to [\,0, 1\,]$ be a map from a closed subset A of X to $[\,0, 1\,]$. We begin by applying the First Approximate Extension Lemma to obtain a map $g_1 : X \to [\,0, 1\,]$ such that $| f(x) - g_1(x) | \leq 2^{-1}$ for $x \in A$. Next we repeatedly apply the Second Approximate Extension Lemma to obtain a sequence of maps $g_n : X \to [\,0, 1\,]$ $(n \geq 2)$ such that for each $n \geq 1$: $| f(x) - g_n(x) | \leq 2^{-n}$ for each $x \in A$ and $| g_n(x) - g_{n+1}(x) | \leq 2^{-n}$ for each $x \in X$. (If we have already obtained the map $g_n : X \to [\,0, 1\,]$ such that $| f(x) - g_n(x) | \leq 2^{-n}$ for each $x \in A$, then the Second Approximate Extension Lemma provides a map $g_{n1} : X \to [\,0, 1\,]$ such that $| f(x) - g_{n+1}(x) | \leq 2^{-(n+1)}$ for each $x \in A$ and $| g_n(x) - g_{n+1}(x) | \leq 2^{-n}$ for each $x \in X$.)

Our next step is to show that for each $x \in X$, then sequence $\{ g_n(x) \}$ converges to a point in $[\,0, 1\,]$. Let $x \in X$. For each $n \geq 1$, let

$$J_n(x) \;=\; [\, g_n(x) - 2^{-(n-1)},\, g_n(x) + 2^{-(n-1)} \,] \cap [\,0, 1\,].$$

Then for each $n \geq 1$, $J_n(x)$ is a closed interval in $[\,0, 1\,]$. Furthermore, $J_{n+1}(x) \subset J_n(x)$ for each $n \geq 1$. To see this, let $t \in J_{n+1}(x)$. Then $t \in [\,0, 1\,]$ and $| t - g_{n+1}(x) | \leq 2^{-n}$. Hence,

$$| t - g_n(x) | \;\leq\; | t - g_{n+1}(x) | + | g_{n+1}(x) - g_n(x) | \;\leq\; 2^{-n} + 2^{-n} = \; 2^{-(n-1)}.$$

Therefore, $t \in J_n(x)$. It follows that $J_1(x) \supset J_2(x) \supset J_3(x) \supset ....$ Since $(\mathbb{R}, <)$ is a complete linearly ordered set, then $\bigcap_{n \in \mathbb{N}} J_n(x) \neq \varnothing$ (according to Problems I.4). Choose a point in $\bigcap_{n \in \mathbb{N}} J_n(x)$ and call it g(x). For each $n \geq 1$, since $g(x) \in J_n(x)$, then $g(x) \in [\,0, 1\,]$ and $| g_n(x) - g(x) | \leq 2^{-(n-1)}$. Thus, $\{ g_n(x) \}$ converges to the point $g(x) \in [\,0, 1\,]$.

By choosing the point $g(x) \in [\,0, 1\,]$ for each $x \in X$, we have defined a function $g : X \to [\,0, 1\,]$. Furthermore, we have seen that our choice of g(x) satisfies the condition: $| g_n(x) - g(x) | \leq 2^{-(n-1)}$ for each $x \in X$.

To prove $g \mid A = f$, let $x \in A$. Then for each $n \geq 1$,

$$| f(x) - g(x) | \;\leq\; | f(x) - g_n(x) | + | g_n(x) - g(x) | \;\leq\; 2^{-n} + 2^{-(n-1)} \;\leq 2^{-(n-2)}.$$

If follows that $g(x) = f(x)$. This prove $g \mid A = f$.

Finally we must prove that $g : X \to [\,0, 1\,]$ is continuous. Recall that B(X) denotes the set of all bounded functions from X to $\mathbb{R}$. Let σ denote the supremum metric on B(X). (See Example I.15.) Recall that C(X) denotes the set of all bounded continuous functions from X to $[\,0, 1\,]$, and that C(X) is a closed subset of B(X) by Theorem II.7. For each $n \geq 1$, since g_n is continuous and $g_n(X) \subset [\,0, 1\,]$, then $g_n \in C(X)$.

Since $g(X) \subset [\,0, 1\,]$, then $g \in B(X)$. For $n \geq 1$, since $|\, g_n(x) - g(x) \,| \leq 2^{-(n-1)}$ for each $x \in X$, then $\sigma(g_n,g) \leq 2^{-(n-1)}$. Hence, $\{ g_n \}$ converges to g in B(X). Since $\{ g_n \}$ lies in C(X) and C(X) is a closed subset of B(X), it follows that $g \in C(X)$. Thus, g is continuous. ☐

Remark. The last paragraph of the proof of the Tietze Extension Theorem reveals a characteristically twentieth century approach. The continuity of the function g is proved by appealing to properties of the function spaces B(X) and C(X), rather than working only with properties of the individual functions g and $\{ g_n \}$. This point of view asserts that there is an advantage to considering a topological space comprising the totality of objects (in this case, functions) rather than focusing solely on the individual objects. The argument for this point of view is that there may be easily and clearly formulated "global" properties of a space of objects that become more complex when translated into statements about individual objects of the space. For example, the statement "C(X) is a closed subset of B(X)" is shorter and simpler than the equivalent statement "a bounded function is continuous if it is the uniform limit of a sequence of continuous functions". For another example, the statement "C(X) is a vector space" is briefer than "a linear combination of two continuous functions is continuous". The argument for the global perspective further asserts that the simplicity of the global formulation leads to the discovery of new results and proofs that would be obscured and possibly overlooked by an approach that was restricted to considering only individual objects. There is a lot of evidence supporting the global perspective. Twentieth century topology and analysis are replete with powerful arguments and theorems formulated in terms of *spaces* of functions and other complex objects.

Remark. Surveying the proofs of the two approximate extension lemmas and the Tietze Extension Theorem, we observe that the normality of the domain space X is used in these proofs only to invoke Urysohn's Lemma. In other words, we have proved that Urysohn's Lemma implies the Tietze Extension Theorem. Observe conversely that the Tietze Extension Theorem implies Urysohn's Lemma. For suppose that the Tietze Extension Theorem holds and that A and B are disjoint closed subsets of a normal space X. Then a continuous function $f : A \cup B \rightarrow [\,0, 1\,]$ is defined by specifying that $f(A) = \{ 0 \}$ and $f(B) = \{ 1 \}$. Hence, the Tietze Extension Theorem provides a map $g : X \rightarrow [\,0, 1\,]$ such that $g \,|\, (A \cup B) = f$. Thus, $g(A) = \{ 0 \}$ and $g(B) = \{ 1 \}$. This gives us the conclusion of Urysohn's Lemma.

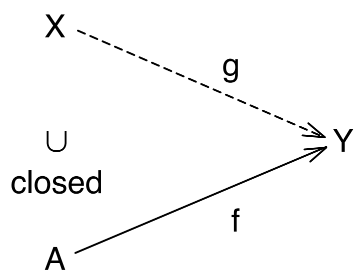
Because the Tietze Extension Theorem implies Urysohn's Lemma (and vice versa), then from Corollary II.13, we get a characterization of normality inspired by the Tietze Extension Theorem.

Corollary II.17. A topological space X is normal if and only every map from a closed subset of X to [0, 1] extends to a map from X to [0, 1].

From one point of view, the Tietze Extension Theorem reveals of a property of normal spaces. However, the Tietze Extension Theorem can also be regarded as revealing a property of the unit interval [0, 1]: maps from closed subsets of a normal spaces into [0, 1] extend to the entire normal space. Spaces which share this property with the unit interval are called "absolute extensors" and have been studied in their own right.

Definition. A space Y is an *absolute extensor* (for the class of all normal spaces) if for every normal space X, every map from a closed subset of X to Y extends to a map from X to Y.

In other words, the space Y is an absolute extensor if and only if for every normal space X and every map $f : A \to Y$ where A is a closed subset of X, there is a map $g : X \to Y$ such that $g | A = f$.



Then the Tietze Extension Theorem can be restated in the form

Corollary II.18. [0, 1] is an absolute extensor.

Remark. Being an absolute extensor is a topological property. In other words, if X is an absolute extensor and X is homeomorphic to Y, then Y is an absolute extensor.

Exercise. Verify the preceding remark.

Theorem II.19. $\mathbb{R}$ is an absolute extensor.

Proof. We will prove that (0, 1) is an absolute extensor. Since $\mathbb{R}$ is homeomorphic to (0, 1), it will then follow that $\mathbb{R}$ is an absolute extensor.

To prove that (0, 1) is an absolute extensor, suppose X is a normal space, A is a closed subset of X and $f : A \to (0, 1)$ is a map. Then f is also a map from A to [0, 1], and we can invoke the Tietze Extension Theorem to obtain a map $g : X \to [0, 1]$ such that $g | A = f$. To finish the proof we must modify g so that its image lies in (0, 1).

Let $B = g^{-1}(\{ 0, 1 \})$. Since $g(A) = f(A) \subset (0, 1)$, then $g(A) \cap g(B) \subset \{ 0, 1 \} \cap (0, 1) = \varnothing$. Hence, $A \cap B = \varnothing$. Thus, A and B are disjoint closed subsets of X. Hence, Urysohn's Lemma provides a map $\lambda : X \to [0, 1]$ such that $\lambda(A) = \{ 0 \}$ and $\lambda(B) = \{ 1 \}$. We now use the map λ to "squeeze" the image of g towards $^1/_2$ without change $g \mid A$.

Define the map $h : X \to \mathbb{R}$ by $h(x) = (1 - \lambda(x))g(x) + \lambda(x)(^1/_2)$. We assert that $h \mid A = f$ and $h(X) \subset (0, 1)$. To prove that $h \mid A = f$, observe that

$$x \in A \;\Rightarrow\; \lambda(x) = 0 \;\Rightarrow\; h(x) = g(x) = f(x).$$

To prove that $h(X) \subset (0, 1)$, let $x \in X$. Since $g(x) \in [0, 1]$, we can break the argument into the following three cases.

Case 1: $g(x) \in \{ 0, 1 \}$. In this case, $x \in B$. So $\lambda(x) = 1$. Therefore, $h(x) = {}^1/_2 \in (0, 1)$.

Case 2: $g(x) \in (0, {}^1/_2]$. In this case,

$$g(x) = (1 - \lambda(x))g(x) + \lambda(x)g(x) \le h(x) \le (1 - \lambda(x))(^1/_2) + \lambda(x)(^1/_2) = {}^1/_2.$$

Therefore, $h(x) \in [g(x), {}^1/_2] \subset (0, {}^1/_2] \subset (0, 1)$.

Case 3: $g(x) \in [{}^1/_2, 1)$. In this case,

$${}^1/_2 = (1 - \lambda(x))(^1/_2) + \lambda(x)(^1/_2) \le h(x) \le (1 - \lambda(x))g(x) + \lambda(x)g(x) = g(x).$$

Therefore, $h(x) \in [{}^1/_2, g(x)] \subset [{}^1/_2, 1) \subset (0, 1)$.

We conclude that $h(x) \in (0, 1)$ in all three cases. Thus, h maps X into $(0, 1)$ and $h \mid A = f$. This completes the proof that $(0, 1)$ is an absolute extensor. □

Theorem II.20. $[0, 1)$ and $(0, 1]$ are absolute extensors.

Exercise. Prove Theorem II.20 by modifying the proof of Theorem II.19.

Theorem II.21. If the spaces $X_1, X_2, \ldots, X_n$ are absolute extensors, then their Cartesian product $X_1 \times X_2 \times \ldots \times X_n$ (with the product topology) is also an absolute extensor.

Problem II.16. Prove Theorem II.21.

Corollary II.22. For $n \in \mathbb{R}$, $[0, 1]^n$ and $\mathbb{R}^n$ are absolute extensors.

We now introduce the notion of a *retract.* This concept is very useful in conjunction with the notion of absolute extensor.

Definition. Let X be a subset of a topological space Y. X is a *retract* of Y if X is a closed subset of Y and there is a map $r : Y \to X$ such that $r \mid X = id_X$. In this situation, the map $r : Y \to X$ is called a *retraction* of Y onto X.

The following two theorems link the notions of retract and absolute extensor.

Theorem II.23. Every retract of an absolute extensor is an absolute extensor.

Proof. Suppose Y is an absolute extensor and Z is a retract of Y. Then there is a map $r : Y \to Z$ such that $r \mid Z = id_Z$.

To prove that Z is an absolute extensor, assume A is a closed subset of a normal space X and $f : A \to Z$ is a map. We must extend f to a map from X to Z.

Since f also maps A into Y and Y is an absolute extensor, then there is a map $g : X \to Y$ such that $g \mid A = f$. Then $r \circ g$ maps X into Z. Furthermore, since $f(A) \subset Z$ and $r \mid Z = id_Z$, then $r \circ g \mid A = r \circ f \mid A = id_Z \circ f = f$. So $r \circ g$ extends f. We conclude that Z is an absolute extensor. □

As an application of Theorem II.23, we have:

An Alternative Proof of Theorem II.20. Since $\mathbb{R}$ is an absolute extensor by Theorem II.19, and $(-1, 1)$ and $(0, 2)$ are homeomorphic to $\mathbb{R}$, then $(-1, 1)$ and $(0, 2)$ are absolute extensors. $[0, 1)$ is a retract of $(-1, 1)$, and $(0, 1]$ is a retract of $(0, 2)$. (**Exercise:** Verify this.) Hence, $[0, 1)$ and $(0, 1]$ are absolute retracts by Theorem II.23. □

Problem II.17. Let T be the a topological space homeomorphic to the letter "T". Thus, T is homeomorphic to the subspace $(\{ 0 \} \times [0, 1]) \cup ([-1, 1] \times \{1\})$ of $\mathbb{R}^2$. (T is also homeomorphic to the letter "Y".) Prove T is an absolute extensor.

Hint. Show that T is a retract of an absolute extensor and invoke Theorem II.23.

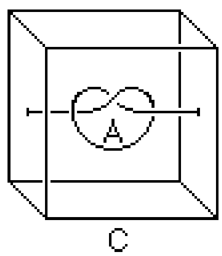
The second theorem linking the notions of absolute extensor and retract is:

Theorem II.24. If X is an absolute extensor and X is a closed subset of a normal space Y, then X is a retract of Y.

Proof. Since X is a closed subset of the normal space Y, and X is an absolute extensor the the map $id_X : X \to X$ extends to a map $r : Y \to X$. Thus, X is a retract of Y. □

Problem II.18. a) If A is a knotted arc inside a cube C ($A \cong [\,0, 1\,]$, $C \cong [\,0, 1\,]^3$) as shown in the following figure, then A is a retract of C.

b) If K is a knotted simple closed curve inside the solid torus T ($K \cong S^1$, $T \cong S^1 \times B^2$) as shown in the following figure, then K is a retract of T.

a)


b)