B. Homeomorphisms and Embeddings

Definition. A function $h : X \to Y$ between topological spaces is a *homeomorphism* if $h : X \to Y$ is a bijection and both $h : X \to Y$ and $h^{-1} : Y \to X$ are continuous. Hence, if $h : X \to Y$ is a bijection between topological spaces, then the following are equivalent:

- $h : X \to Y$ is a homeomorphism,
- $h : X \to Y$ is continuous and open, and
- $h : X \to Y$ is continuous and closed.

If there is a homeomorphism from X to Y, then we say that X is *homeomorphic* to Y and we write $X \cong Y$. Homeomorphism is the fundamental equivalence relation of topology.

Example II.5. $\mathbb{R}$ is homeomorphic to $(-1, 1)$. Indeed a homeomorphism $h : \mathbb{R} \to (-1, 1)$ is defined by the formula

$$h(x) = \frac{x}{|x| + 1}.$$

A simple way to prove that h is a homeomorphism is to exhibit its inverse. To this end, define the function $k : (-1, 1) \to \mathbb{R}$ by the formula

$$k(x) = \frac{x}{1 - |x|}.$$

The continuity of h and k follows from Example II.1.a and Theorem II.3. Also, it is easy to check that $k \circ h = id_{\mathbb{R}}$ and $h \circ k = id_{(-1, 1)}$. So $h : \mathbb{R} \to (-1, 1)$ is a bijection and h and $h^{-1} = k$ are continuous. Thus, $h : \mathbb{R} \to (-1, 1)$ is a homeomorphism.

A continuous bijection need not be a homeomorphism, as the following example illustrates.

Example II.6. A continuous bijection need not be a homeomorphism. Let $\mathbb{N}$ have the discrete topology, let $Y = \{ 0 \} \cup \{ {}^1/_n : n \in \mathbb{N} - \{ 1 \} \}$, and topologize Y by regarding it as a subspace of $\mathbb{R}$. Define $f : \mathbb{N} \to Y$ by $f(1) = 0$ and $f(n) = {}^1/_n$ for $n > 1$. Then f is a continuous bijection which is not a homeomorphism. **Proof:** Since every subset of $\mathbb{N}$ is open, then for each open subset U of Y, $f^{-1}(U)$ is an open subset of $\mathbb{N}$. So f is continuous. $f : \mathbb{N} \to Y$ is clearly a bijection. Since $\{ 1 \}$ is an open subset of $\mathbb{N}$, but $f(\{ 1 \}) = \{ 0 \}$ is not an open subset of Y, then $f : \mathbb{N} \to Y$ is not an open map. Thus, f is not a homeomorphism.

Definition. A *topological property* or a *topological invariant* is a property of a topological space which is preserved by homeomorphisms; i.e., it is a property which is possessed by a space if and only if it is possessed by all homeomorphic spaces.

Broadly speaking, *topology* is the study of topological invariants. In other words, topology is the study of those properties of topological spaces that are preserved by homeomorphism.

The properties of topological spaces that were defined in the previous chapter are all instances of topological properties, including second countable, first countable, separable, metrizable, T_1, Hausdorff, regular and normal. We illustrate how this is proved in one case. The other cases are similar.

Proof that normality is a topological property. Let $h : X \to Y$ be a homeomorphism between topological spaces and assume that X is normal. Let A and B be disjoint closed subsets of Y. Then $h^{-1}(A)$ and $h^{-1}(B)$ are disjoint closed subsets of X. So there are disjoint open subsets U and V of X such that $h^{-1}(A) \subset U$ and $h^{-1}(B) \subset V$. Then h(U) and h(V) are disjoint open subsets of Y and $A \subset h(U)$ and $B \subset h(V)$. This proves Y is normal. ☐

Proof that metrizability is a topological property. Let $h : X \to Y$ be a homeomorphism between topological spaces and assume that X is metrizable. Let ρ be a metric on X that induces the given topology. Define the function $\sigma : Y \times Y \to [\,0, \infty\,)$ by $\sigma(y,y') = \rho(h^{-1}(y),h^{-1}(y'))$. Since ρ is a metric on X and $h : X \to Y$ is a bijection, it is easy to verify that σ is a metric on Y. (Prove this assertion.) It remains to prove that σ induces the given topology on Y. In other words, we must prove that the collection $\mathscr{B}_\sigma = \{\, N_\sigma(y,\varepsilon) : y \in Y \text{ and } \varepsilon > 0 \,\}$ of all ε-neighborhoods in Y with respect to σ is a basis for the given topology on Y. First, for $y \in Y$ and $\varepsilon > 0$, observe that $h(N_\rho(h^{-1}(y),\varepsilon)) = N_\sigma(y,\varepsilon)$. Indeed,

$$y' \in N_\sigma(y,\varepsilon) \;\Leftrightarrow\; \sigma(y,y') < \varepsilon \;\Leftrightarrow\; \rho(h^{-1}(y),h^{-1}(y')) < \varepsilon \;\Leftrightarrow$$
$$h^{-1}(y') \in N_\rho(h^{-1}(y),\varepsilon) \;\Leftrightarrow\; y' \in h(N_\rho(h^{-1}(y),\varepsilon)).$$

Since ρ induces the given topology on X, then $N_\rho(h^{-1}(y),\varepsilon)$ is an open subset of X. Since h is an open map and $h(N_\rho(h^{-1}(y),\varepsilon)) = N_\sigma(y,\varepsilon)$, then $N_\sigma(y,\varepsilon)$ is an open subset of Y. Thus, $\mathscr{B}_\sigma$ is a subset of the given topology on Y. Suppose y is an element of an open subset U of Y. Since h is continuous, then $h^{-1}(U)$ is an open subset of X and $h^{-1}(y) \in h^{-1}(U)$. Since ρ induces the given topology on X, then Theorem I.10 implies there is an $\varepsilon > 0$ such that $N_\rho(h^{-1}(y),\varepsilon) \subset h^{-1}(U)$. Hence, $h(N_\rho(h^{-1}(y),\varepsilon)) \subset h(h^{-1}(U))$. Since $h(N_\rho(h^{-1}(y),\varepsilon)) = N_\sigma(y,\varepsilon)$ and $h(h^{-1}(U)) = U$, then it follows that $y \in N_\sigma(y,\varepsilon) \subset U$. This completes the proof that $\mathscr{B}_\sigma$ is a basis for the given topology on Y. Consequently, the metric σ induces the given topology on Y. We conclude that Y is metrizable. ☐

Exercise. a) Prove that the other properties listed above are topological invariants.

b) Prove that if X and Y are topological spaces, then $X \times Y \cong Y \times X$.

c) Prove that if $X_1, X_2, \ldots, X_n$ are topological spaces and $1 \leq k < n$, then $(X_1 \times \ldots \times X_k) \times (X_{k+1} \times \cdots \times X_n) \cong X_1 \times X_2 \times \cdots \times X_n$.

(Statements b) and c) play a role in certain proofs about product spaces which proceed by induction on the number of factors.)

Definition. Recall that the *Euclidean norm* (or *2-norm*) on $\mathbb{R}^n$ is defined by the formula

$$\| \mathbf{x} \|_2 = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$$

for $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. We define the following subspaces of $\mathbb{R}^n$:

$$\mathbb{B}^n = \{ \mathbf{x} \in \mathbb{R}^n : \| \mathbf{x} \|_2 \leq 1 \},$$

$$\mathbb{S}^{n-1} = \{ \mathbf{x} \in \mathbb{R}^n : \| \mathbf{x} \|_2 = 1 \},$$

$$\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times [0, \infty).$$

Any space that is homeomorphic to $\mathbb{B}^n$ is called an *n-ball* or an *n-disk.* Any space that is homeomorphic to $\mathbb{B}^1 = [-1, 1]$ (or, equivalently, to $[0, 1]$) is also called an *arc;* and any space that is homeomorphic to $\mathbb{B}^2$ is often called simply a *disk.* Any space that is homeomorphic to $\mathbb{S}^n$ is called an *n-sphere.* Any space that is homeomorphic to $\mathbb{S}^1$ is also called a *simple closed curve.* Any space that is homeomorphic to $\mathbb{R}^n_+$ is called an *n-dimensional half-space.*

Problem II.5. Let $(V, \| \ \|)$ be a normed vector space. Prove that $\{ x \in V : \| x \| < 1 \}$ is homeomorphic to V.

Thus, $\mathbb{B}^n - \mathbb{S}^{n-1}$ is homeomorphic to $\mathbb{R}^n$.

Next we note an interesting topological phenomenon: Cartesian factors are not topologically unique. In other words, there are topological spaces X, Y and Z such that $X \times Z \cong Y \times Z$ but $X \ncong Z$. Perhaps the simplest illustration of this phenomenon is presented in part a) of the next problem.

Problem II.6. a) Prove that $\mathbb{R} \times [0, \infty)$ is homeomorphic to $[0, \infty) \times [0, \infty)$.

b) Prove that $\mathbb{R}^n \times [0, \infty)^k$ is homeomorphic to $[0, \infty)^{n+k}$.

Of course, for Problem II.6.a to be an illustration of the non-uniqueness of Cartesian factors, one must prove that $\mathbb{R} \not\cong [\,0, \infty\,)$. The proof of this fact is slightly beyond our current level of understanding, but will become easily understandable based on work in a later chapter. However, we easily can (and do) explain the idea of this proof at this point. The piece of information from a subsequent chapter that is not yet available to us is that the space $(\,0, \infty\,)$ is *connected;* in other words, $(\,0, \infty\,)$ can't be expressed as the union of two non-empty disjoint open subsets. If we assume that $(\,0, \infty\,)$ is connected, then the rest of the argument that $\mathbb{R} \not\cong [\,0, \infty\,)$ is straightforward.

Proof that $\mathbb{R} \not\cong [\,0, \infty\,)$. Assume that $\mathbb{R} \cong [\,0, \infty\,)$. Then there is a homeomorphism $h : [\,0, \infty\,) \to \mathbb{R}$. Let $z = h(0)$. Then

$$h(\,(\,0, \infty\,)\,) \;=\; h(\,[\,0, \infty\,) - \{\,0\,\}\,) \;=\; h(\,[\,0, \infty\,)\,) - \{\,h(0)\,\}$$
$$=\; \mathbb{R} - \{\,z\,\} \;=\; (\,-\infty, z\,) \cup (\,z, \infty\,).$$

Thus, $(\,0, \infty\,)$ is the union of the two non-empty disjoint open sets $h^{-1}(\,(\,-\infty, z\,)\,)$ and $h^{-1}(\,(z, \infty\,)\,)$. This contradicts the connectedness of $(\,0, \infty\,)$. We conclude that $\mathbb{R} \not\cong [\,0, \infty\,)$. □

At the current time, there is an fundamental unsolved topological problem related to the non-uniqueness of Cartesian factors. For each $n \geq 3$, there are many known examples of spaces X with the property that $X \times \mathbb{R} \cong \mathbb{R}^{n+1}$ but $X \not\cong \mathbb{R}^n$. (For $n = 1$ or 2, $X \times \mathbb{R} \cong \mathbb{R}^{n+1}$ implies $X \cong \mathbb{R}^n$.) Thus, in high dimensional situations, an $\mathbb{R}$-factor usually can't be cancelled. However, there are some high dimensional situations in which an $\mathbb{R}$-factor can be cancelled. For instance, it is known that for any topological space X: if $X \times \mathbb{R}^3 \cong \mathbb{R}^{n+3}$, then $X \times \mathbb{R}^2 \cong \mathbb{R}^{n+2}$. (It follows by induction that for any $k \geq 3$, if $X \times \mathbb{R}^k \cong \mathbb{R}^{n+k}$, then $X \times \mathbb{R}^2 \cong \mathbb{R}^{n+2}$.) Thus, an $\mathbb{R}$-factor can be cancelled if there are at least two other $\mathbb{R}$-factors present. What remains unknown at this time is whether an $\mathbb{R}$-factor can be cancelled if there is just one other $\mathbb{R}$-factor present. In other words:

Unsolved Problem. For every topological space X, does $X \times \mathbb{R}^2 \cong \mathbb{R}^{n+2}$ imply $X \times \mathbb{R} \cong \mathbb{R}^{n+1}$?

Problem II.7. Prove that if $\mathbf{p} \in \mathbb{S}^n$, then $\mathbb{R}^n$ is homeomorphic to $\mathbb{S}^n - \{\,\mathbf{p}\,\}$.

An approach to the solution of this problem is outlined in an Additional Problem.

Definition. A *topological characterization* of a space X is a list of properties of X together with the assertion that if a space satisfies the properties on the list, then it must be homeomorphic to X.

Exercise. Let X be a space with the discrete discrete topology. Prove the following topological characterization of X. A space Y is homeomorphic to X if and only if Y has the discrete topology and X ≈ Y (i.e., there is a bijection from X to Y).

Topological characterizations of the simplest spaces – balls, spheres and Euclidean spaces – are regarded as some of the most central and important results of topology. This is true not only because such characterizations are the first problems that occur to researchers in the area, but because such results serve as tools in the efforts to characterize more complicated objects. For n = 1 and 2, there are simple and satisfying topological characterizations of $\mathbb{B}^n$, $\mathbb{S}^n$ and $\mathbb{R}^n$. (Later we will acquire the concepts needed to state these results and prove some of them.) In dimension n = 3, we encounter a topological characterization result which until recently was the best known and arguably the most fundamental unsolved problem in topology: the **Poincaré conjecture**. The Poincaré conjecture is a topological characterization of $\mathbb{S}^3$. (We don't yet have the terminology to formulate this characterization.) It was settled affirmatively in 2002 by Grigori Perelman. (Perelman, who apparently disapproves of the effect that prizes have on the mathematical community, refused to accept a Fields Medal and the $1,000,000 Millenium prize offered by the Clay Foundation for this work!) The affirmative resolution of the Poincaré conjecture immediately led to proofs of topological characterizations of $\mathbb{B}^3$ and $\mathbb{R}^3$.

One of the paradoxes of modern topology is that although the conjectured characterizations of 3-dimensional balls, spheres and Euclidean spaces were proved only recently, the high dimensional analogues of these conjectures were settled and, in fact, well understood much earlier. In the early 1960's work of J. Milnor, S. Smale, J. Stallings and E. C. Zeeman answered many of the fundamental questions in dimensions ≥ 5. In particular, they established a high-dimensional analogue of the Poincaré conjecture, thereby characterizing $\mathbb{S}^n$ for n ≥ 5. In the early 1980's, M. Freedman and S. Donaldson made breakthroughs which provide answers to a number of characterization conjectures in dimension 4. Freedman proved a 4-dimensional analogue of the Poincaré conjecture and thus characterized $\mathbb{S}^4$. However, many unsolved problems remain in dimension 4. The methods that worked to resolve these characterization conjectures in dimension 4 were inspired by techniques that worked in dimensions 5 and above. However, the methods used in dimension 3 are completely different from the higher dimensional approaches. Dimension 3 is too "cramped" for the strategies that succeed in the "roominess" of higher dimensions.

The following three problems ask for proofs of topological characterizations.

Definition. A subset C of a vector space V is *convex* if whenever C contains two points v and w, it also contains the straight line segment $\{ (1 - t)v + t w : 0 \le t \le 1 \}$ joining v to w.

Problem II.8. Let $(V, \| \ \|)$ be a normed vector space, and let $B = \{ x \in V : \| x \| \le 1 \}$. Prove that if X is a closed bounded convex subset of V with non-empty interior, then there is a homeomorphism $h : V \to V$ such that $h(B) = X$.

The result of Problem II.8 is extended in two Additional Problems.

The next two problems propose topological characterizations of $\mathbb{Q}$ and $\mathbb{R}$ regarded as linearly ordered spaces up to *order preserving* homeomorphisms. (Recall that $\mathbb{Q}$ is the set of rational numbers regarded as a subspace of $\mathbb{R}$.)

Definition. A linearly ordered set X is *densely ordered* if between any two distinct points of X there is a third point of X. A function $f : X \to Y$ between linearly ordered sets is *order preserving* if for any two points x and $x' \in X$, $x < x' \Rightarrow f(x) < f(x')$.

Problem II.9. Prove that if X is a countable densely ordered linearly ordered space with no least element and no greatest element, then there is an order preserving homeomorphism from X to $\mathbb{Q}$.

The result of Problem II.9 is generalized in an Additional Problem. The result of Problem II.9 can be used in the solution of the next problem.

Problem II.10. Prove that if X is a separable densely ordered complete linearly ordered space with no least element and no greatest element, then there is an order preserving homeomorphism from X to $\mathbb{R}$.

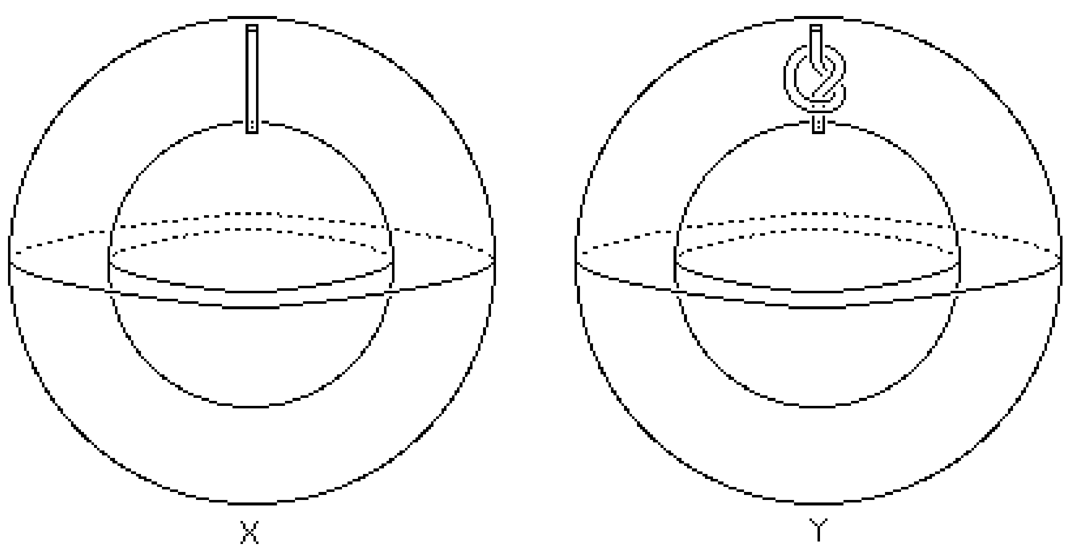
Observe that the result of Problem II.9 implies that there is an order preserving homeomorphism from $\mathbb{Q}$ to $\mathbb{Q} \cap (0, 1)$, where both $\mathbb{Q}$ and $\mathbb{Q} \cap (0, 1)$ are regarded as subspaces of $\mathbb{R}$. However, because an order preserving homeomorphism must preserve least elements and greatest elements, there can be no order preserving homeomorphisms between any two of the spaces $\mathbb{Q} \cap (0, 1)$, $\mathbb{Q} \cap [0, 1)$, $\mathbb{Q} \cap (0, 1]$ and $\mathbb{Q} \cap [0, 1]$. None the less, these spaces might be homeomorphic via non-order preserving homeomorphisms.

Problem II.11. Are any of the four spaces $\mathbb{Q} \cap (0, 1)$, $\mathbb{Q} \cap [0, 1)$, $\mathbb{Q} \cap (0, 1]$ and $\mathbb{Q} \cap [0, 1]$ (regarded as subspaces of $\mathbb{R}$) homeomorphic?

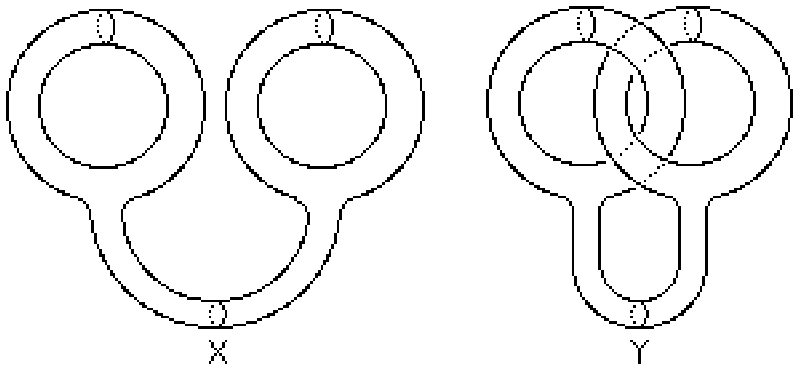
The following four problems are exercises in visualizing homeomorphisms of $\mathbb{R}^3$ that carry one subspace to another. They may initially appear to be impossible. In fact, the homeomorphisms sought in these problems perform rather complicated motions of $\mathbb{R}^3$. To solve one of these problems, a student should describe a motion of the points of $\mathbb{R}^3$ that will carry one subspace to the other. A convincing way to describe a complicated motion of the points of $\mathbb{R}^3$ that is intended to be a homeomorphism is to express this motion as a composition of simple moves each of which shifts points only within a small set. We describe one such simple move following the statements of these four problems. We call this simple move *scooching a tube.*

A description of a homeomorphism that solves one of these problems can consist of words or pictures or both. Clearly, each such motion should be a continuous bijection of $\mathbb{R}^3$ whose inverse is also continuous. However, at this point, the student is not expected to present a completely rigorous proof that the described motions are homeomorphisms. It suffices for the student to present a clear idea of the motion, possibly using a picture. The details of proving that the motions which solve these problems are homeomorphisms can be made easier by developing some technical tools. Some of the Additional Problems explore the development of relevant technical tools.

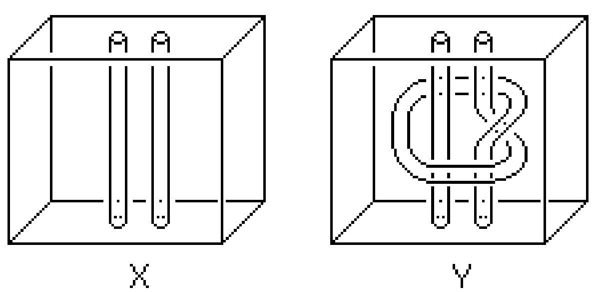
Problem II.12. Let X and Y be the closed bounded subsets of $\mathbb{R}^3$ shown in the following picture. X is the closure of the region between two concentric balls with a hole drilled out. So is Y. In X, the hole is straight. In Y, the hole is knotted. Prove that there is a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(X) = Y$.



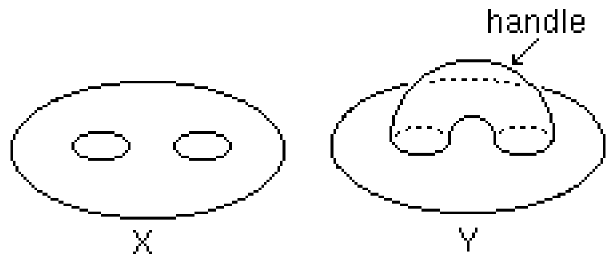
Problem II.13. Let X and Y be the closed bounded subsets of $\mathbb{R}^3$ shown in the following picture. X consists of two rings joined by a tube. So does Y. The rings of X are unlinked, while the rings of Y are linked. Prove that there is a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(X) = Y$.



Problem II.14. Let X and Y be the closed bounded subsets of $\mathbb{R}^3$ shown in the following picture. X and Y are each cubes with two holes drilled out of them. In X, both holes are straight. In Y, one hole is straight and the other is knotted. Prove that there is a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(X) = Y$.

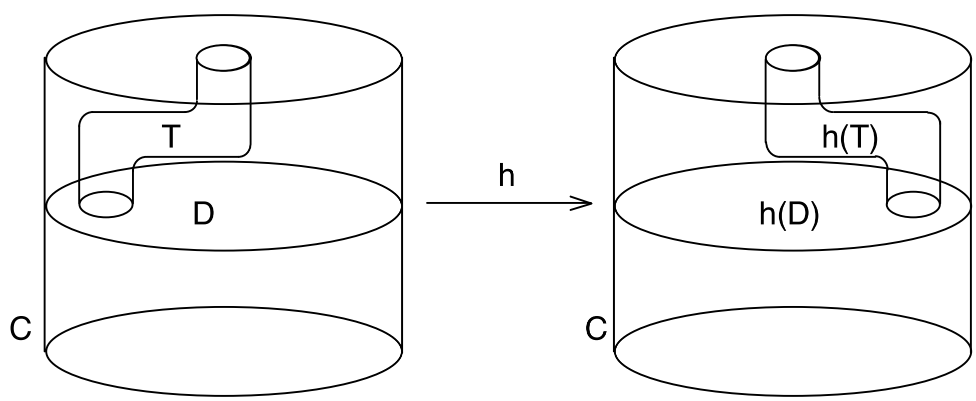


Problem II.15. Let X and Y be the closed bounded subsets of $\mathbb{R}^3$ shown in the following picture. X and Y are 2-dimensional surfaces. X is a disk with two holes. Y is a disk with a "handle". (Y = X ∪ (handle).) Prove X × [0, 1] and Y × [0, 1] can be identified with subsets and $\mathbb{R}^3$, and that there is a homeomorphism h : $\mathbb{R}^3 \to \mathbb{R}^3$ such that h(X × [0,1]) = Y × [0,1].



Problem II.15, like Problem II.6.a, illustrates the non-uniqueness of Cartesian factorization: In Problem II.14, X × [0, 1] ≅ Y × [0, 1], but X ≇ Y. At this point, we can't prove X ≇ Y; however, we can suggest a reason for it: the "boundary" of X is the union of three disjoint simple closed curves, whereas the "boundary" of Y is a single simple closed curve.

We now describe a homeomorphism of $\mathbb{R}^3$ which we call *scooching a tube*. C is a cylindrical 3-ball in $\mathbb{R}^3$. h : $\mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism which shifts only the points of int(C). h(x) = x for every x ∈ $\mathbb{R}^3$ – int(C). Thus, h(C) = C. D is a horizontal disk in C half way between the top and the bottom of C. T is a tube in C with its lower end on the left side of D and its upper end on the top of C. h doesn't move the upper end of T, but h *scooches* the lower end of T from the left side of D to the right side of D. Also h(D) = D. The complicated homeomorphisms which are the solutions to some of the previous problems are the compositions of simple moves like scooching a tube that shift points only within a small set.



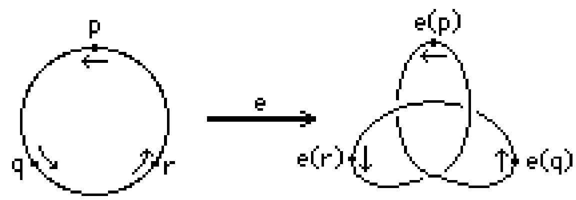
Scooching a tube

Definition. A function $e : X \to Y$ between topological spaces is an *embedding* if $e : X \to e(X)$ is a homeomorphism. (Here, $e(X)$ has the subspace topology it receives as a subset of Y.) Therefore, $e : X \to Y$ is an embedding if and only if e is injective and continuous and $e^{-1} : e(X) \to X$ is continuous. Equivalently, $e : X \to Y$ is an embedding if and only if e is injective and continuous and $e: X \to e(X)$ is open.

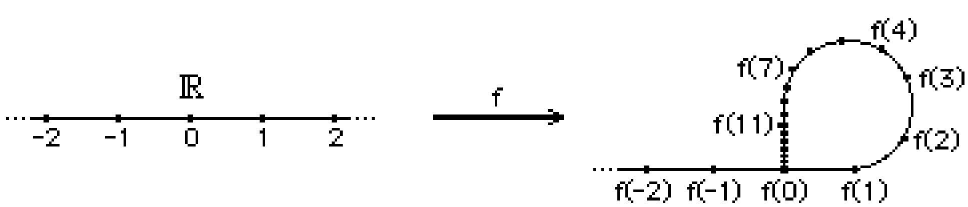
Example II.6 above shows that a continuous bijection need not be a homeomorphism. Consequently, a continuous injection need not be an embedding. This is illustrated further by Examples II.7 and II.9 below.

Example II.7. Assign $\mathbb{N}$ the discrete topology. Define the injective continuous functions e_1, e_2 and $e_3 : \mathbb{N} \to \mathbb{R}$ by $e_1(n) = n$, $e_2(n) = {}^1/_n$, and $e_3(1) = 0$ and $e_3(n) = {}^1/_n$ for $n \geq 2$. Then e_1 and e_2 are embeddings. However, e_3 is not an embedding, because $e_3^{-1} : e_3(\mathbb{N}) \to \mathbb{N}$ is not continuous. (See Example II.6.)

Example II.8. Recall that $\mathbb{S}^1 = \{ x \in \mathbb{R}^2 : \| x \|_2 = 1 \}$ is a circle of radius 1 in $\mathbb{R}^2$. The following picture illustrates an embedding $e : \mathbb{S}^1 \to \mathbb{R}^2$ whose image $e(\mathbb{S}^1)$ is a *trefoil knot* .



Example II.9. There is an injective continuous function $f : \mathbb{R} \to \mathbb{R}^2$ whose image, $f(\mathbb{R})$ is shown in the following picture. f is not an embedding because f^{-1} is not continuous. Indeed, $f(-1, 1)$ is not a relatively open subset of $f(\mathbb{R})$.



The following theorem reveals that we have already encountered embeddings when studying products of topological spaces.

Theorem II.12. Let $X_1, X_2, \ldots, X_n$ be topological spaces, let $X_1 \times X_2 \times \ldots \times X_n$ have the product topology, and choose a point $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in X_1 \times X_2 \times \ldots \times X_n$. Then for $1 \le i \le n$, the i^{th} injection function $e_{\mathbf{a},i} : X_i \to X_1 \times X_2 \times \ldots \times X_n$ is an embedding.

Proof. Let $1 \le i \le n$. Clearly, $e_{\mathbf{a},i} : X_i \to X_1 \times X_2 \times \ldots \times X_n$ is injective. Also $e_{\mathbf{a},i}$ is continuous by Theorem II.8. It remains to show that $e_{\mathbf{a},i} : X_i \to e_{\mathbf{a},i}(X_i)$ is open.

To begin, recall that the ith projection function $\pi_i : X_1 \times X_2 \times \ldots \times X_n \to X_i$ satisfies the equation $\pi_i \circ e_{\mathbf{a},i} = id_{X_i}$ by Theorem I.30.a. Also π_i is continuous by Theorem II.8. Now let V be an open subset of X_i. We must prove that $e_{\mathbf{a},i}(V)$ is a relatively open subset of $e_{\mathbf{a},i}(X_i)$. Since $\pi_i^{-1}(V)$ is an open subset of $X_1 \times X_2 \times \ldots \times X_n$, then it will suffice to prove that $e_{\mathbf{a},i}(V) = \pi_i^{-1}(V) \cap e_{\mathbf{a},i}(X_i)$. To accomplish this, we first prove the inclusion $e_{\mathbf{a},i}(V) \subset \pi_i^{-1}(V) \cap e_{\mathbf{a},i}(X_i)$. Since $\pi_i(e_{\mathbf{a},i}(V)) = V$, then $e_{\mathbf{a},i}(V) \subset \pi_i^{-1}(V)$; also since $V \subset X$, then $e_{\mathbf{a},i}(V) \subset e_{\mathbf{a},i}(X)$. Hence, $e_{\mathbf{a},i}(V) \subset \pi_i^{-1}(V) \cap e_{\mathbf{a},i}(X_i)$. Now we prove the opposite inclusion: $\pi_i^{-1}(V) \cap e_{\mathbf{a},i}(X_i) \subset e_{\mathbf{a},i}(V)$. To this end, assume $y \in \pi_i^{-1}(V) \cap e_{\mathbf{a},i}(X_i)$. Since $y \in e_{\mathbf{a},i}(X_i)$, then there is a point $x \in X_i$ such that $y = e_{\mathbf{a},i}(x)$. Since $y \in \pi_i^{-1}(V)$, then $\pi_i(y) \in V$. Thus, $x = \pi_i(e_{\mathbf{a},i}(x)) = \pi_i(y) \in V$. Therefore, $y = e_{\mathbf{a},i}(x) \in e_{\mathbf{a},i}(V)$. This proves $\pi_i^{-1}(V) \cap e_{\mathbf{a},i}(X_i) \subset e_{\mathbf{a},i}(V)$. We conclude that $e_{\mathbf{a},i}(V) = \pi_i^{-1}(V) \cap e_{\mathbf{a},i}(X_i)$. Since $\pi_i^{-1}(V)$ is an open subset of $X_1 \times X_2 \times \ldots \times X_n$, it follows that $e_{\mathbf{a},i}(V)$ is a relatively open subset of $e_{\mathbf{a},i}(X)$. So $e_{\mathbf{a},i} : X_i \to e_{\mathbf{a},i}(X_i)$ is open. ☐