Additional Problems - 23

III. Compactness

A. Fundamental Properties

Problem III.1+. Define the *fish skeleton metric* ρ_{fish} on $\mathbb{R}^2$ by

$$\rho_{\text{fish}}(\,(x,y),\,(x',y')\,) = |\,y\,| + |\,x - x'\,| + |\,y'\,|.$$

In the metric space ($\mathbb{R}^2$, ρ_{fish}), is a closed bounded subset necessarily compact?

Problem III.2+. A cover of a topological space is *minimal* if no proper subcollection covers the space. Prove that a topological space is compact if and only if every open cover of the space has a minimal subcover.

Problem III.3+. a) Prove that a compact Hausdorff space X is first countable if and only if $X - \{ x \}$ is the union of countably many closed sets for each $x \in X$.

b) Prove that every countable compact Hausdorff space is second countable. (Thus, there is no compact space of the type constructed in Example I.8.)

Problem III.4+. a) Is a first countable compact Hausdorff space necessarily separable?

b) Is a first countable separable compact Hausdorff space necessarily second countable?

Problem III.5+. Let X and Y be topological spaces and let $\pi : X \times Y \to X$ denote the projection to first coordinate: $\pi(x,y) = x$.

a) Prove that if Y is compact, then $\pi : X \times Y \to X$ is a closed map.

b) Let $f : X \to Y$ be a function, and identify f with its graph in $X \times Y$ as in Problem II.3+. Prove the following converse to Problem II.3+.a: if f is a closed subset of $X \times Y$ and Y is compact, then f is continuous.

Problem III.6+. Let (X,ρ) be a metric space.

a) Recall that $\text{diam}(X) = \sup \{ \rho(x,y) : x, y \in X \}$. Prove that if X is compact, then there are points $x, y \in X$ such that $\text{diam}(X) = \rho(x,y)$.

b) For $A, B \subset X$, set $\rho(A,B) = \inf \{ \rho(x,y) : x \in A \text{ and } y \in B \}$. Prove that if A and B are compact subsets of X, then there are points $x \in A$ and $y \in B$ such that $\rho(A,B) = \rho(x,y)$.

Problems III.7+ through III.14+ involve the notions of local and σ- compactness.

Problem III.7+. a) Prove that in a locally compact space, every compact subset is contained in the interior of a compact closed subset.

b) Prove that every locally compact Hausdorff space is regular.

Problem III.8+. a) Prove that every open subset of a locally compact Hausdorff space is locally compact.

Surprisingly, **a)** has a sort of converse.

b) Prove that if X is a locally compact dense subspace of a Hausdorff space Y, then X is an open subset of Y.

Problem III.9+. a) Prove that every locally compact Lindelöf space is σ-compact. Conclude that every locally compact second countable space is σ-compact, and that every locally compact separable metric space is σ-compact.

b) Prove that every σ-compact space is Lindelöf.

c) Is every σ-compact space locally compact?

Problem III.10+. a) Prove that if X is a metric space in which every closed bounded subset is compact, then X is locally compact and separable.

b) Show that the converse of **a)** is false: find a locally compact separable metric space in which closed bounded sets are not necessarily compact.

Problem III.11+. Despite part **b)** of Problem III.10+, there is a sort of converse to part **a)**. Indeed, here we outline a proof of the following proposition. If X is a locally compact separable metrizable space, then there is a metric on X which induces the given topology and with respect to which every closed bounded subset of X is compact. Complete the proof of this proposition by verifying assertions **i)** through **vi)** below.

Assume (X,ρ) is a locally compact separable metric space. First we construct a map $f : X \to [\,0, \infty\,)$ such that $f^{-1}([\,0, t\,])$ is compact for every $t \in [\,0, \infty\,)$.

i) There is a sequence $C_1, C_2, C_3, \ldots$ of compact subsets of X that cover X such that $C_n \subset \mathrm{int}(C_{n+1})$ for each $n \in \mathbb{N}$.

ii) For each $n \in \mathbb{N}$, there is a map $f_n : X \to [\,0, 1\,]$ such that $f_n(C_n) = \{\,0\,\}$ and $f_n(X - C_{n+1}) = \{\,1\,\}$.

Define the function $f : X \to [\,0, \infty\,)$ by $f(x) = \sum_{n=1}^{\infty} f_n(x)$.

iii) Then f is well-defined and continuous.

iv) For every $t \in [\,0, \infty\,)$, $f^{-1}([\,0, t\,])$ is compact.

Define the metric σ on $X \times [0, \infty)$ by $\sigma((x,r),(y,s)) = \max\{\rho(x,y), | s - t |\}$. Then σ induces the product topology on $X \times [0, \infty)$ by Theorem I.32. Identify f with its graph in $X \times [0, \infty)$. Then a homeomorphism $h : X \to f$ is defined by $h(x) = (x,f(x))$. (See Problem II.9+.a.)

v) A metric ρ' on X which induces the given topology is defined by the formula $\rho'(x,y) = \sigma(h(x),h(y))$.

vi) With respect to ρ', every closed bounded subset of X is compact.

Problems III.12+ and III.14+ involve the notion of a compactification of a space.

Problem III.12+. A *one-point compactification* of a space X is a compactification (Y, e) of X such that $Y - e(X)$ contains exactly one point.

a) Suppose that (Y, e) is a one-point compactification of a space X and $Y - e(X) = \{\infty\}$. Prove that a subset U of Y is a neighborhood of ∞ in Y if and only if there is a compact subset C of X such that $U = Y - e(C)$.

b) Prove that a space X has a one-point compactification if and only if X is a non-compact locally compact Hausdorff space.

c) Prove that a one-point compactification of a locally compact Hausdorff space X is unique in the following sense. If (Y_1, e_1) and (Y_2, e_2) are both one-point compactifications of X, then there is a homeomorphism $h : Y_1 \to Y_2$ such that $h \circ e_1 = e_2$.

This fact justifies our speaking of *the* one-point compactification of a non-compact locally compact Hausdorff space.

d) Prove that **i)** $\mathbb{S}^n$ is the one-point compactification of $\mathbb{R}^n$ and that **ii)** Ω^+ is the one-point compactification of Ω. (Ω and Ω^+ refer to Examples I.10 and I.11.)

e) Prove that a non-compact locally compact Hausdorff space is second countable if and only if its one-point compactification is second countable.

f) Let X be a non-compact locally compact Hausdorff space. Let (Y, e) be a compactification of X. Prove that a compactification (Y, e) of X is a one-point compactification of X if and only if it has the following property: for every other compactification (Y´, e´) of X, there is a unique map $f : Y' \to Y$ such that $f \circ e' = e$.
(Hence, this property characterizes the one-point compactification among all possible compactifications of X.)

Problem III.13+. A topological space X is said to be *compactly generated* if for every $U \subset X$, U is an open subset of X if and only if $U \cap C$ is a (relatively) open subset of C for every compact subset C of X. A compactly generated space is also called a *k-space.*

a) Prove that every locally compact space is compactly generated.

b) Prove that every first countable space is compactly generated.

Problem III.14+. A map $f : X \to Y$ is *proper* if f is a closed map and for each $y \in Y$, $f^{-1}(\{ y \})$ is a compact subset of X.

a) Prove that every map from a compact space to a Hausdorff space is proper.

b) Prove that if $f : X \to Y$ is a proper map, then for each compact subset C of Y, $f^{-1}(C)$ is a compact subset of X.

c) Let (X,ρ) be a metric space, let $x_0 \in X$, and define the function $f : X \to [\, 0, \infty\,)$ by $f(x) = \rho(x_0,x)$. Prove that f is a proper map if and only if every closed bounded subset of X is compact.

d) Suppose that Y is a compactly generated Hausdorff space. Prove that a map $f : X \to Y$ is proper if and only if $f^{-1}(C)$ is a compact subset of X for every compact subset C of Y.

e) Let $f : X \to Y$ be a map between non-compact locally compact Hausdorff spaces. Let (X^*, e_X) and (Y^*, e_Y) be one-point compactifications of X and Y, respectively; and let $\{ \infty_X \} = X^* - e_X(X)$ and $\{ \infty_Y \} = Y^* - e_Y(Y)$. Define the function $f^* : X^* \to Y^*$ by $f^* \mid e_X(X) = e_Y \circ f \circ (e_X^{-1})$ and $f^*(\infty_X) = \infty_Y$. Prove that $f^* : X^* \to Y^*$ is continuous if and only if $f : X \to Y$ is a proper map.

f) Suppose that X is a dense subset of a compact Hausdorff space X^*, Y is a subset of a Hausdorff space Y^*, and $f^* : X^* \to Y^*$ is a map such that $f^*(X) \subset Y$. Define the map $f : X \to Y$ by $f = f^* \mid X$. Prove that $f : X \to Y$ is a proper map if and only if $f^*(X^* - X) \subset Y^* - Y$.

B. Various Forms of Compactness

Problem III.15+. If $\mathscr{U}$ is a cover of X, then a *proper subcover* of $\mathscr{U}$ is a proper subset of $\mathscr{U}$ that also covers X. Prove that a T_1 space is countably compact if and only every infinite open cover has a proper subcover.

Problem III.16+. Let X be a set, let $\{ 0, 1 \}^X$ denote the set of all functions from X to $\{ 0, 1 \}$, and assign $\{ 0, 1 \}^X$ a topology in the same way that $\{ 0, 1 \}^\Sigma$ is given a topology in Problem III.6. We will sketch a proof that the space $\{ 0, 1 \}^X$ is compact. It will follow that the space $\{ 0, 1 \}^\Sigma$ of Problem III.6 is compact. As we remarked in

Problem III.6, the Tychonoff Theorem (proved in Chapter 5) is the general principle which implies the compactness of $\{ 0, 1 \}^X$. The proof we outline here is simply a specialization to $\{ 0, 1 \}^X$ of a proof of this general principle. Complete the proof sketched below by verifying assertions **i)** through **vii)**.

Our proof will rely (as does the proof of the Tychonoff Theorem in Chapter V) on the Well Ordering Principle. Recall that this principle, which is equivalent to the Axiom of Choice, says that every set can be well ordered. We will also require the following elementary proposition about well ordered sets.

The Principle of Transfinite Induction. Let $(X, <)$ be a well ordered set. Let $P(x)$ be a meaningful proposition about the elements x of X. Suppose that $P(x)$ satisfies the following condition: for each $x \in X$, the truth of $P(y)$ for every $y \in (-\infty,x)$ implies the truth $P(x)$. Then $P(x)$ is true for every $x \in X$.

Proof. Assume that for each $x \in X$, the truth of $P(y)$ for each $y \in (-\infty,x)$ implies the truth of $P(x)$, and assume that $P(x)$ is false for some $x \in X$. We will obtain a contradiction. Since $\{ x \in X : P(x) \text{ is false} \}$ is a non-empty subset of X, and X is well ordered, then this subset has a least element x_0. Then $P(y)$ is true for every $y \in (-\infty,x_0)$ but $P(x_0)$ is false. We have reached a contradiction. □

We now begin our outline of the proof that $\{ 0, 1 \}^X$ is compact.

Let X be a set, and let $\{ 0, 1 \}^X$ denote the set of all functions from X to $\{ 0, 1 \}$. For every $f \in \{ 0, 1 \}^X$ and every finite subset A of X, set

$$N(f,A) = \{ g \in \{ 0, 1 \}^X : g|A = f|A \}.$$

Set

$$\mathscr{B} = \{ N(f,A) : f \in \{ 0, 1 \}^X \text{ and } A \text{ is a finite subset of } X \}.$$

i) Then $\mathscr{B}$ is a basis for a topology on $\{ 0, 1 \}^X$.

We endow $\{ 0, 1 \}^X$ with this topology.

We will prove that $\{ 0, 1 \}^X$ is compact by contradiction. Assume $\{ 0, 1 \}^X$ is not compact.

ii) Then there is a subset $\mathscr{N}$ of $\mathscr{B}$ such that $\mathscr{N}$ covers $\{ 0, 1 \}^X$ and no finite subset of $\mathscr{N}$ covers $\{ 0, 1 \}^X$.

If $Y \subset X$ and $f : Y \to \{ 0, 1 \}$ is a function, define the subset $E(f)$ of $\{ 0, 1 \}^X$ by

$$E(f) = \{ g \in \{ 0, 1 \}^X : g|Y = f \}.$$

At this point we invoke the Well Ordering Principle to obtain a well ordering $<$ of X. Now consider the following proposition.

Proposition A. For each $x \in X$, $a_x \in \{ 0, 1 \}$ can be chosen so that if the function $f_x : (-\infty,x] \to \{ 0, 1 \}$ is defined by $f_x(y) = a_y$ for $y \in (-\infty,x]$, then no finite subset of $\mathscr{N}$ covers $E(f_x)$.

We will prove Proposition A by applying the Principle of Transfinite Induction in the well ordered set $(X, <)$. Accordingly, we assume that $x \in X$ and that for every $y \in (-\infty,x)$, $a_y \in \{ 0, 1 \}$ has been chosen so that if the function $f_y : (-\infty,y] \to \{ 0, 1 \}$ is defined by $f_y(z) = a_z$ for $z \in (-\infty,y]$, then no finite subset of n covers $E(f_y)$. We must prove that it is possible to chose $a_x \in \{ 0, 1 \}$ so that if the function $f_x : (-\infty,x] \to \{0,1\}$ is defined by $f_x(y) = a_y$ for $y \in (-\infty,x]$, then no finite subset of $\mathscr{N}$ covers $E(f_x)$.

We begin by defining the function $g : (-\infty,x) \to \{ 0, 1 \}$ by $g(y) = a_y$ for $y \in (-\infty,x)$.

iii) Then no finite subset of $\mathscr{N}$ covers $E(g)$.

Next define the functions $h_{x,0} : (-\infty,x] \to \{ 0, 1 \}$ and $h_{x,1} : (-\infty,x] \to \{ 0, 1 \}$ by $h_{x,0}(y) = h_{x,1}(y) = g(y) = a_y$ for $y \in (-\infty,x)$, $h_{x,0}(x) = 0$ and $h_{x,1}(x) = 1$.

iv) Then $E(g) = E(h_{x,0}) \cup E(h_{x,1})$.

v) Hence, either no finite subset of $\mathscr{N}$ covers $E(h_{x,0})$ or no finite subset of $\mathscr{N}$ covers $E(h_{x,1})$.

vi) Therefore, $a_x \in \{ 0, 1 \}$ can be chosen so that if the function $f_X : (-\infty,x] \to \{ 0, 1 \}$ is defined by $f_x(y) = a_y$ for $y \in (-\infty,x]$, then no finite subset of $\mathscr{N}$ covers $E(f_x)$.

Proposition A now follows by the Principle of Transfinite Induction.

Next define $f \in \{ 0, 1 \}^X$ by $f(x) = a_x$ for each $x \in X$. Observe that for each $x \in X$, $f \mid (-\infty,x] = f_x$. Since $\mathscr{N}$ covers $\{ 0, 1 \}^X$, then $f \in N$ for some $N \in \mathscr{N}$. Since $N \in \mathscr{N}$, then $N = N(k,A)$ for some $k \in \{ 0, 1 \}^X$ and some finite subset A of X.

vi) There is an $x \in X$ such that $E(f_X) \subset N$.

vii) We have reached a contradiction.

We conclude that $\{ 0, 1 \}^X$ is compact. ◻

Problem III.17+. A sequence $\{ x_n \}$ is *eventually constant* if there is an $n \in \mathbb{N}$ such that $x_i = x_n$ for all $i \geq n$. Consider the space $\{ 0, 1 \}^\Sigma$ from Problem III.6. This problem outlines the construction of a subspace of $\{ 0, 1 \}^\Sigma$ which is compact and Hausdorff with no isolated points, in which every converging sequence is eventually constant. Thus, this subspace contains no infinite compact metrizable subspace. Also observe that this subspace is a compact Hausdorff space which fails to be sequentially compact in a rather spectacular fashion: if $\{ x_n \}$ is any sequence of distinct points in this subspace (i.e., $x_i \neq x_j$ for $i \neq j$), then no subsequence of $\{ x_n \}$ converges.

a) Prove that $\{ 0, 1 \}^{\Sigma}$ is Hausdorff.

As in the hint for Problem III.6.b, for each $n \in \mathbb{N}$, define $f_n \in \{ 0, 1 \}^{\Sigma}$ by $f_n(\sigma) = \sigma(n)$ for each $\sigma \in \Sigma$. (Recall that $\Sigma = \{ 0, 1 \}^{\mathbb{N}}$.) Set $F = \{ f_n : n \in \mathbb{N} \}$.

b) Prove that every converging sequence in cl(F) is eventually constant.

Hint. Suppose $\{ x_k \}$ is a converging sequence in cl(F) which is not eventually constant. Prove assertions i) through iv) to reach a contradiction.

i) By passing to a subsequence if necessary, we can assume $x_i \neq x_j$ for $i \neq j$.

ii) Again by passing to a subsequence if necessary, we can assume that there is a sequence $U_1, U_2, U_3, \ldots$ of disjoint open subsets of $\{ 0, 1 \}^{\Sigma}$ such that $x_k \in U_k$ for each $k \in \mathbb{N}$.

Set $V = U_1 \cup U_3 \cup U_5 \cup \cdots$. Define $\sigma \in \Sigma$ by stipulating that $\sigma(n) = 1$ if $f_n \in V$ and $\sigma(n) = 0$ if $f_n \notin V$.

iii) Then $x_k(\sigma) = 1$ if k is odd and $x_k(\sigma) = 0$ if k is even.

iv) Hence, $\{ x_k \}$ doesn't converge.

c) Argue that no infinite compact metric space embeds in cl(F).

d) Prove that f_n is an isolated point of the space cl(F) for each $n \in \mathbb{N}$.

e) Conclude that cl(F) – F is a compact Hausdorff space in which every converging sequence is eventually constant.

f) Prove that no point of cl(F) – F is isolated in cl(F) – F.

Hint. Show that an isolated point of cl(F) – F would be the limit of a subsequence of $\{ f_n \}$.

Thus, cl(F) – F is a compact Hausdorff space with no isolated points in which every converging sequence is eventually constant and which, consequently, contains no infinite compact metrizable subspace. Hence, no sequence of distinct points in cl(F) – F has a converging subsequence.

Problem III.18+. a) Is every countable Hausdorff space regular?

b) Is every countable regular space normal?

C. Compact Metric Spaces

Problem III.19+. Let (X,ρ) and (Y,σ) be metric spaces. A function $f : X \to Y$ is *distance preserving* if $\sigma(f(x),f(x')) = \rho(x,x')$ for all $x, x' \in X$. A function from X to Y which is distance preserving and onto is called an *isometry* of X.

a) Prove that every distance preserving function from a compact metric space to itself is an isometry.

b) Suppose that X is a metric space in which every closed bounded set is compact; and suppose that for any two points x and $y \in X$, there is a distance preserving function $f : X \to X$ of X such that $f(x) = y$. Prove that every distance preserving function from X to itself is an isometry.

Problem III.20+. Let (X,ρ) be a metric space and let $f : X \to X$ be a function. A point $x \in X$ is a *fixed point* of f if $f(x) = x$. The function $f : X \to X$ is *non-expansive* if $\rho(f(x),f(y)) < \rho(x,y)$ for all $x, y \in X$.

a) Prove that every non-expansive function is continuous.

b) Prove that if X is a compact metric space, then every non-expansive function from X to itself has a unique fixed point.

c) Find an example which illustrates that if X is non-compact metric space, then a non-expansive function from X to itself need not have a fixed point.

Problem III.21+. Suppose X is a metric space with the property that every open cover has a Lebesgue number.

a) Must X be compact?

b) Suppose that X also has the property that every infinite set of isolated points of X has a limit point in X. Then must X be compact?