Math 751 **Introductory Topology** **Fall 2010**

Additional Problems - 11

II. Continuity

A. Continuous Functions

Problem II.1+. Prove that if f, g : X → Y are maps and Y is Hausdorff, then $\{ x \in X : f(x) = g(x) \}$ is a closed subset of X.

Problem II.2+. a) Let $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ be functions. Define the function $(f_1, f_2) : X \to Y_1 \times Y_2$ by $(f_1, f_2)(x) = (f_1(x), f_2(x))$. Prove that f_1 and f_2 are continuous if and only if (f_1, f_2) is continuous.

b) Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be functions. Define the function $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ by $(f_1 \times f_2)(x_1,x_2) = (f_1(x_1), f_2(x_2))$. Prove that f_1 and f_2 are continuous if and only if $f_1 \times f_2$ is continuous.

Problem II.3+. We regard a function f : X → Y as a subset of X × Y by identifying f with its "graph" $\{ (x,f(x)) \in X \times Y : x \in X \}$.

a) Prove that if f : X → Y is a map and Y is a Hausdorff space, the f is a closed subset of X × Y.

b) Find a T_1 space X such that id_X is not a closed subset of X × X.

c) Find a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ such that f is a closed subset of $\mathbb{R}^2$.

Problem II.4+. Let f : X → Y be an onto map between topological spaces. Either prove or provide a counterexample to each of the following assertions.

a) If X is second countable, then so is Y.

b) If X is first countable, then so is Y.

c) If X is separable, then so is Y.

d) If X is metrizable, then so is Y.

e) If X is T_1, then so is Y.

f) If X is Hausdorff, then so is Y.

g) If X is regular, then so is Y.

h) If X is normal, then so is Y.

Problem II.5+. Let f : X → Y be an open onto map between topological spaces. Either prove or provide a counterexample to each of assertions **a)** through **h)** in Problem II.4+.

Hint. Let Y denote the three-point space $\{ 0, {}^1/_2, 1 \}$ with the topology $\{ \varnothing \} \cup \{ U \subset Y : {}^1/_2 \in U \}$, and consider the function $f : [0, 1] \to Y$ satisfying f(0) = 0, f(1) = 1 and $f(x) = {}^1/_2$ for $0 < x < 1$.

Problem II.6+. Let f : X → Y be an closed onto map between topological spaces. Either prove or provide a counterexample to each of assertions **a)** through **h)** in Problem II.4+.

Hint. Consider the following two functions.

Recall $\mathbb{N} = \{ 1, 2, 3, \cdots \}$. Let X_1 denote the subspace $(\mathbb{R} \times \{ 0 \}) \cup \{ (x, {}^1/_y) : x, y \in \mathbb{N} \}$ of $\mathbb{R}^2$, let Y_1 denote the space defined in Example I.8, and define the function $f_1 : X_1 \to Y_1$ by $f_1(x, {}^1/_y) = (x, y)$ and $f_1(\mathbb{R} \times \{ 0 \}) = \{ \infty \}$.

Let X_2 be a Hausdorff regular space which is not normal. (Such a space occurs among the examples in Chapter I.) Let A and B be disjoint closed subsets of X_2 which don't have disjoint neighborhoods. Let α and β be distinct points not in X_2. Set $Y_2 = (X_2 - (A \cup B)) \cup \{ \alpha, \beta \}$. Define $f_2 : X_2 \to Y_2$ by $f_2(x) = x$ for $x \in X_2 - (A \cup B)$, $f_2(A) = \alpha$ and $f_2(B) = \beta$. Set $\mathscr{T} = \{ U \subset Y_2 : f_2^{-1}(U)$ is an open subset of $X_2 \}$. Verify that $\mathscr{T}$ is a topology on Y_2.

Problem II.7+. Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone increasing function (i.e, $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in \mathbb{R}$). Prove that the set of points of $\mathbb{R}$ at which f is discontinuous is countable.

Problem II.8+. Prove that every continuous function $f : \Omega \to \mathbb{R}$ is eventually constant (i.e., there is an $x \in \Omega$ such that $f(y) = f(x)$ for every $y \in [x,\infty)$).

B. Homeomorphisms and Embeddings

Problem II.9+. Let f : X → Y be a function. Identify f with its graph in X × Y as in Problem II.3+.

a) Prove that f : X → Y is continuous if and only if the function h : X → f defined by h(x) = (x, f(x)) is a homeomorphism.

b) Suppose Y is a normed vector space. Prove that f : X → Y is continuous if and only if there is a homeomorphism F : X × Y → X × Y such that F(x,0) = (x, f(x)).

Problem II.10+. In this problem, we introduce two special types of homeomorphisms of $\mathbb{R}^n$: reflections in planes and inversions in spheres. Reflections and inversions have great geometric significance: reflections in planes generate the isometry group of $\mathbb{R}^n$ with the Euclidean metric (the standard model of Euclidean geometry), and inversions in spheres that preserve $\mathbb{S}^{n-1}$ generate the isometry group of the Poincaré ball model of hyperbolic geometry. (An *isometry* of a metric space (X, ρ)

is a bijection $f : X \to X$ which *preserves distance* in the sense that $\rho(f(x),f(x')) = \rho(x,x')$ for all $x, x' \in X$.) Furthermore, inversions yield a very efficient solution to Problem II.7.

To define reflection and inversion, first recall that the standard inner product on $\mathbb{R}^n$ is defined by

$$\langle \mathbf{x},\mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i ,$$

and that the Euclidean norm on $\mathbb{R}^n$ is defined by

$$\| \mathbf{x} \| = (\langle \mathbf{x},\mathbf{x} \rangle)^{1/2},$$

for $\mathbf{x} = (x_1, x_2, \ldots , x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots , y_n) \in \mathbb{R}^n$.

Let $\mathbf{c}$, $\mathbf{u} \in \mathbb{R}^n$ such that $\| \mathbf{u} \| = 1$. Define the *hyperplane through* ***c*** *orthogonal to* ***u*** to be the set

$$P(\mathbf{c},\mathbf{u}) = \{ \mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x} - \mathbf{c},\mathbf{u} \rangle = 0 \}.$$

Define the *half-spaces determined by P(**c**,**u**)* to be the sets

$$\{ \mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x} - \mathbf{c},\mathbf{u} \rangle > 0 \} \text{ and } \{ \mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x} - \mathbf{c},\mathbf{u} \rangle < 0 \}.$$

Define the *reflection in P(**c**,**u**)* to be the map $R_{\mathbf{c},\mathbf{u}} : \mathbb{R}^n \to \mathbb{R}^n$ determined by the formula

$$R_{\mathbf{c},\mathbf{u}}(\mathbf{x}) = \mathbf{x} - 2\langle \mathbf{x} - \mathbf{c},\mathbf{u} \rangle \mathbf{u}$$

for $\mathbf{x} \in \mathbb{R}^n$.

Let $\mathbf{c} \in \mathbb{R}^n$ and let $r > 0$. Define the *sphere centered at* ***c*** *of radius r* to be the set

$$S(\mathbf{c},r) = \{ \mathbf{x} \in \mathbb{R}^n : \| \mathbf{x} - \mathbf{c} \| = r \}.$$

Define the *ball centered at* ***c*** *of radius r* to be the set

$$B(\mathbf{c},r) = \{ \mathbf{x} \in \mathbb{R}^n : \| \mathbf{x} - \mathbf{c} \| \leq r \}.$$

Define the *interior* and *exterior of B(**c**,r)* to be the sets

$$\{ \mathbf{x} \in \mathbb{R}^n : \| \mathbf{x} - \mathbf{c} \| < r \} \text{ and } \{ \mathbf{x} \in \mathbb{R}^n : \| \mathbf{x} - \mathbf{c} \| > r \},$$

respectively. Define the *inversion in S(**c**,r)* to be the map $I_{\mathbf{c},r} : \mathbb{R}^n - \{ \mathbf{c} \} \to \mathbb{R}^n - \{ \mathbf{c} \}$ determined by the formula

$$I_{\mathbf{c},r}(\mathbf{x}) = \left(\frac{r^2}{\| \mathbf{x} - \mathbf{c} \|^2} \right)(\mathbf{x} - \mathbf{c}) + \mathbf{c}$$

for $\mathbf{x} \in \mathbb{R}^n - \{ \mathbf{c} \}$.

Two hyperplanes $P(\mathbf{c},\mathbf{u})$ and $P(\mathbf{d},\mathbf{v})$ are *orthogonal* if $\langle \mathbf{u},\mathbf{v} \rangle = 0$. A hyperplane $P(\mathbf{c},\mathbf{u})$ and a sphere $S(\mathbf{d},r)$ are *orthogonal* if $\mathbf{d} \in P(\mathbf{c},\mathbf{u})$. Two spheres $S(\mathbf{c},r)$ and $S(\mathbf{d},s)$ are *orthogonal* if $S(\mathbf{c},r) \cap S(\mathbf{d},s) \neq \varnothing$ and $\langle \mathbf{x} - \mathbf{c}, \mathbf{x} - \mathbf{d} \rangle = 0$ for every $\mathbf{x} \in S(\mathbf{c},r) \cap S(\mathbf{d},s)$.

Let $\mathbf{c}$, $\mathbf{u} \in \mathbb{R}^n$ such that $\| \mathbf{u} \| = 1$.

a) Prove that $R_{\mathbf{c},\mathbf{u}} \circ R_{\mathbf{c},\mathbf{u}} = \mathrm{id}_{\mathbb{R}^n}$, and conclude that $R_{\mathbf{c},\mathbf{u}} : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphisms.

b) Prove that $\{ \mathbf{x} \in \mathbb{R}^n : R_{\mathbf{c},\mathbf{u}}(\mathbf{x}) = \mathbf{x} \} = P(\mathbf{c},\mathbf{u})$.

c) Prove that $R_{\mathbf{c},\mathbf{u}}$ maps:
i) hyperplanes to hyperplanes,
ii) half-spaces to half-spaces,
iii) spheres of radius r to spheres of radius r, and
iv) balls of radius r to balls of radius r.

d) Prove that
i) $R_{\mathbf{c},\mathbf{u}}$ maps a hyperplane to itself if and only if the hyperplane either coincides with $P(\mathbf{c},\mathbf{u})$ or is orthogonal to $P(\mathbf{c},\mathbf{u})$, and
ii) $R_{\mathbf{c},\mathbf{u}}$ maps a sphere to itself if and only if the sphere is orthogonal to $P(\mathbf{c},\mathbf{u})$.

Let $\mathbf{c} \in \mathbb{R}^n$ and let $r > 0$.

e) Prove that $I_{\mathbf{c},r} \circ I_{\mathbf{c},r} = \mathrm{id}_{\mathbb{R}^n - \{ \mathbf{c} \}}$, and conclude that $I_{\mathbf{c},r} : \mathbb{R}^n - \{ c \} \to \mathbb{R}^n - \{ c \}$ is a homeomorphism.

f) Prove that $\{ \mathbf{x} \in \mathbb{R}^n - \{ c \} : I_{\mathbf{c},r}(\mathbf{x}) = \mathbf{x} \} = S(\mathbf{c},r)$.

g) Prove that $I_{\mathbf{c},r}$ maps:
i) a hyperplane containing $\mathbf{c}$ to itself,
ii) a halfspace containing $\mathbf{c}$ in its boundary to itself,
iii) a hyperplane missing $\mathbf{c}$ to a sphere containing $\mathbf{c}$ and vice versa,
iv) a half-space missing $\mathbf{c}$ to the interior of a ball containing $\mathbf{c}$ in its boundary and vice versa,
v) a sphere missing $\mathbf{c}$ to a sphere missing $\mathbf{c}$,
vi) a ball missing $\mathbf{c}$ to a ball missing $\mathbf{c}$, and
vii) the interior of a ball containing $\mathbf{c}$ in its interior to the exterior of a ball containing $\mathbf{c}$ in its interior and vice versa.

h) Prove that:
i) $I_{\mathbf{c},r}$ maps a hyperplane to itself if and only if the hyperplane is orthogonal to $S(\mathbf{c},r)$, and
ii) $I_{\mathbf{c},r}$ maps a sphere to itself if and only if the sphere either coincides with $S(\mathbf{c},r)$ or is orthogonal to $S(\mathbf{c},r)$.

i) Let $\mathbf{n} = (0, 0, \ldots, 0, 1) \in \mathbb{S}^{n-1}$. Prove there is an $r > 0$ such that $I_{\mathbf{n},r}$ maps:

i) $\mathbb{R}^{n-1} \times \{0\}$ onto $\mathbb{S}^{n-1} - \{\mathbf{n}\}$, $I_{\mathbf{n},r}$,

ii) the half-space $\{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x},\mathbf{n} \rangle > 0\}$ onto the interior of the ball $\mathbb{B}^n = B(\mathbf{0},1)$, and

iii) the half-space $\{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x},\mathbf{n} \rangle < 0\}$ onto the exterior of the ball $\mathbb{B}^n = B(\mathbf{0},1)$.

The following six problems develop elementary properties of balls and spheres.

Problem II.9+. *Invariance of domain* is a fundamental topological property of $\mathbb{R}^n$ which asserts that if U and V are homeomorphic subsets of $\mathbb{R}^n$ and U is an open subset of $\mathbb{R}^n$, then so is V. A proof of invariance of domain requires techniques not developed in these notes. Assume invariance of domain, and prove the following two propositions.

a) If $h : \mathbb{B}^n \to \mathbb{B}^n$ is a homeomorphism, then $h(\mathbb{B}^n - \mathbb{S}^{n-1}) = \mathbb{B}^n - \mathbb{S}^{n-1}$ and $h(\mathbb{S}^{n-1}) = \mathbb{S}^{n-1}$.

b) If C is an n–ball and $g : \mathbb{B}^n \to C$ and $h : \mathbb{B}^n \to C$ are homeomorphisms, then $g(\mathbb{B}^n - \mathbb{S}^{n-1}) = h(\mathbb{B}^n - \mathbb{S}^{n-1})$ and $g(\mathbb{S}^{n-1}) = h(\mathbb{S}^{n-1})$.

Definition. If C is an n–ball and $h : \mathbb{B}^n \to C$ is a homeomorphism, then we call the subset $h(\mathbb{S}^{n-1})$ the *boundary* of C and denote it by ∂C, and we call the subset $h(\mathbb{B}^n - \mathbb{S}^{n-1}) = C - \partial C$ the *interior* of C and denote it by Int(C). The result of Problem II.9+.b shows that the boundary and interior of C are independent of the homeomorphism h.

Problem II.10+. Prove that if C and D are n–balls, $h : \partial C \to \partial D$ is a homeomorphism, $x \in \mathrm{Int}(C)$ and $y \in \mathrm{Int}(D)$, then there is a homeomorphism $H : C \to D$ such that $H|\partial C = h$ and $H(x) = y$. Furthermore, prove that if E is an n–ball in int(C) that contains x and U is a neighborhood of y in D, then the homeomorphism H can be chosen so that $H(E) \subset U$.

Problem II.11+. Prove that if a space X is the union of two closed subsets C and D that are n–balls such that $C \cap D = \partial C = \partial D$, then X is an n–sphere.

Definition. Let C be an n–ball. If E is an (n–1)–ball in ∂C with the property that there is an (n–1)–ball F in ∂C such that $E \cup F = \partial C$ and $E \cap F = \partial E = \partial F$, then we call E a *face* of C, and we call E and F *complementary faces* of C. We remark that for $n \geq 4$, every n–ball C has the property that its boundary ∂C contains an (n–1)–ball E which is not a face of C because $\mathrm{cl}(\partial C - E)$ is not an (n–1)–ball.

Problem II.12+. Suppose that C and D are n–balls, E is a face of C and F is a face of D. Prove that any homeomorphism from E to F extends to a homeomorphism from C to D.

Problem II.13+. Prove that if a space X is the union of two closed subsets C and D that are n–balls such that $C \cap D = F$ is a face of both C and D, then X is an n–ball and $\partial X = (\partial C - \mathrm{Int}(F)) \cup (\partial D - \mathrm{Int}(F))$.

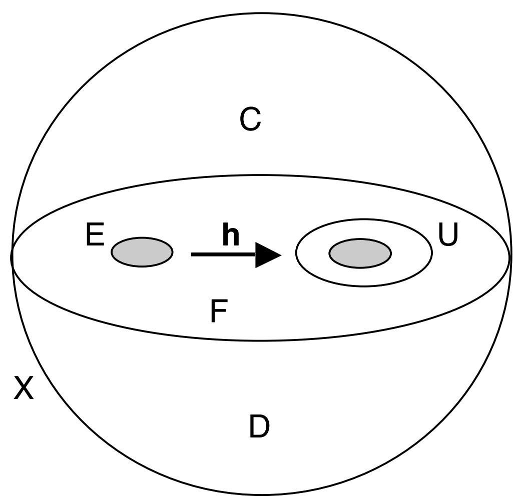
Problem II.14+. Let C be an n–ball.

a) Prove that $C \times [\,0, 1\,]$ is an (n+1)–ball such that $\partial(C \times [\,0, 1\,]) =$ $(\,C \times \{\,0, 1\,\}) \cup (\,(\partial C) \times [\,0, 1\,]\,)$, and $C \times \{\,0\,\}$ and $C \times \{\,1\,\}$ are faces of $C \times [\,0, 1\,]$.

b) Let $f : C \to [\,0, \infty\,)$ be a map such that $f^{-1}(0) = \partial C$, and let $P =$ $\{\,(x,y) \in C \times [\,0, \infty\,) : 0 \leq y \leq f(x)\,\}$. Prove that P is an (n+1)–ball and $C \times \{\,0\,\}$ is a face of P.

The following problem contributes to a rigorous foundation for the solutions of Problems II.12 through II.15. The homeomorphisms sought in those problems can be constructed as compositions of simpler homeomorphisms some of which are of the type of produced in this problem.

Problem II.15+. Suppose that the space X is the union of two closed subsets C and D that are n–balls and $C \cap D = F$ is a face of both C and D. (Then X is an n–ball.) Also suppose that E is an (n–1)–ball in int(F) and U is a non-empty relatively open subset of F. Prove there is a homeomorphism $h : X \to X$ such that $h(C) = C$, $h(D) = D$ (and, therefore, $h(F) = F$), $h(D) \subset U$ and $h = \mathrm{id}$ on ∂X.



Problem II.16+. Any topological space that is homeomorphic to $\mathbb{S}^1 \times \mathbb{B}^2$ is called a *solid torus.* Prove that a topological space X is a 3-sphere if and only if X is the union of two closes subsets V and W that are solid tori and there are homeomorphisms $g : \mathbb{S}^1 \times \mathbb{B}^2 \to V$ and $h : \mathbb{S}^1 \times \mathbb{B}^2 \to W$ such that $V \cap W = g(\mathbb{S}^1 \times \mathbb{S}^1) = h(\mathbb{S}^1 \times \mathbb{S}^1)$ and $g(x,y) = h(y,x)$ for $(x,y) \in \mathbb{S}^1 \times \mathbb{S}^1$.

The next two problems generalize the result of Problem II.8.

Problem II.17+. Let (V, $\|\ \|$) be a normed vector space, and set B = $\{ x \in V : \|x\| \leq 1 \}$.

a) Prove that every non-empty open convex subset of V is homeomorphic to int(B).

b) Prove that every convex subset of V with non-empty interior is homeomorphic to a set Y satisfying $\mathrm{int}(B) \subset Y \subset B$.

Problem II.18+. a) Prove that every non-empty closed bounded convex subset of $\mathbb{R}^n$ that is not a single point is homeomorphic to $\mathbb{B}^k$ for some k such that $1 \leq k \leq n$.

b) Prove that every non-empty convex subset of $\mathbb{R}^n$ that is not a single point is homeomorphic to a set Y satisfying $\mathrm{int}(\mathbb{B}^k) \subset Y \subset \mathbb{B}^k$ for some k such that $1 \leq k \leq n$.

The next problem generalizes Problem II.9.

Problem II.19+. Recall that a point x in a topological space X is *isolated* if { x } is an open subset of X. Prove that every countable metric space with no isolated points is homeomorphic to $\mathbb{Q}$.

Suggestion. For the purpose of proving this result, we do not recommend using $\mathbb{Q}$ as a model of a countable metric space with no isolated points. Rather, we recommend using a space we call $\{ 0, 1 \}^{\mathbb{N}}_{\text{fin}}$ because this space is structured in a way that makes it easier to describe a homeomorphism from this space to an arbitrary countable metric space without isolated points. The structure that makes $\{ 0, 1 \}^{\mathbb{N}}_{\text{fin}}$ a convenient model for countable metric spaces without isolated points is not immediately apparent in $\mathbb{Q}$. Clearly, if we prove that every countable metric space with no isolated points is homeomorphic to $\{ 0, 1 \}^{\mathbb{N}}_{\text{fin}}$, then it will follow that every countable metric space with no isolated points is homeomorphic to $\mathbb{Q}$. (Why?)

To describe $\{ 0, 1 \}^{\mathbb{N}}_{\text{fin}}$, first consider the set $\{ 0, 1 \}^{\mathbb{N}}$ consisting of all functions from $\mathbb{N}$ to { 0, 1 }. (In other words, $\{ 0, 1 \}^{\mathbb{N}}$ is the set of all sequences of 0's and 1's.) Prove that a metric ρ on $\{ 0, 1 \}^{\mathbb{N}}$ is defined by the equation

$$\rho(\mathbf{x},\mathbf{y}) = \sup\left\{ \frac{\mathbf{x}(n) - \mathbf{y}(n)}{2^n} : n \in \mathbb{N} \right\}$$

for $\mathbf{x}$, $\mathbf{y} \in \{0, 1\}^{\mathbb{N}}$. Define $\{0, 1\}^{\mathbb{N}}_{\text{fin}} = \{ \mathbf{x} \in \{0, 1\}^{\mathbb{N}} : \mathbf{x}^{-1}(1)$ is a finite set $\}$. Prove that $\{0, 1\}^{\mathbb{N}}_{\text{fin}}$ is a countable dense subset of $\{0, 1\}^{\mathbb{N}}$. Thus, as a subspace of $\{0, 1\}^{\mathbb{N}}$, $\{0, 1\}^{\mathbb{N}}_{\text{fin}}$ is a countable metric space. Prove that $\{0, 1\}^{\mathbb{N}}_{\text{fin}}$ has no isolated points. Now, to solve Problem II.19+, you must prove that that every countable metric space with no isolated points is homeomorphic to $\{0, 1\}^{\mathbb{N}}_{\text{fin}}$.

C. Continuous Functions on Normal Spaces

Problem II.20+. A space Y is an *absolute retract* if whenever $e : Y \to X$ is an embedding of Y onto a closed subset of a normal space X, then there is a map $r : X \to Y$ such that $r \circ e = \text{id}_Y$.

a) Prove that every absolute extensor is an absolute retract.

The converse of this assertion is also true, and the rest of this problem is devoted to proving it.

Suppose A is a closed subset of a normal space X, Y is a normal space and $f : A \to Y$ is a map. Assume X and Y are disjoint sets, and let $X \cup Y$ denote their union. (If X and Y are not disjoint, replace X by $X \times \{0\}$ and Y by $Y \times \{1\}$.) Let $\mathscr{T}_X$ and $\mathscr{T}_Y$ denote the topologies on X and Y, respectively. Let

$$\mathscr{T}_{X \cup Y} = \{ U \subset X \cup Y : U \cap X \in \mathscr{T}_X \text{ and } U \cap Y \in \mathscr{T}_Y \}$$

and observe that $\mathscr{T}_{X \cup Y}$ is a topology on $X \cup Y$ such that the inclusions of X and Y into $X \cup Y$ are embeddings onto subsets of $X \cup Y$ that are both open and closed. We call $\mathscr{T}_{X \cup Y}$ the *disjoint union topology* on $X \cup Y$. Next let $X \cup_f Y$ denote the following collection of subsets of $X \cup Y$:

$$X \cup_f Y = \{ \{x\} : x \in X - A \} \cup \{ \{y\} \cup f^{-1}(\{y\}) : y \in Y \}.$$

Observe that $X \cup_f Y$ is a partition of $X \cup Y$; in other words, distinct elements of $X \cup_f Y$ are disjoint and non-empty and the union of the elements of $X \cup_f Y$ is $X \cup Y$. It follows that there is a unique onto function $q : X \cup Y \to X \cup_f Y$ is defined by the condition that $z \in q(z)$ for every $z \in X \cup Y$. Let

$$\mathscr{T}_f = \{ U \subset X \cup_f Y : q^{-1}(U) \in \mathscr{T}_{X \cup Y} \}$$

and observe that $\mathscr{T}_f$ is a topology on $X \cup_f Y$ such that $q : X \cup Y \to X \cup_f Y$ is continuous and $q \mid Y : Y \to X \cup_f Y$ is an embedding of Y onto a closed subset of $X \cup_f Y$. $X \cup_f Y$ with the topology $\mathscr{T}_f$ is called an *adjunction space,* $\mathscr{T}_f$ is called the *quotient topology* on $X \cup_f Y$, and $q : X \cup Y \to X \cup_f Y$ is called the *quotient map.*

b) Prove that the adjunction space $X \cup_f Y$ is normal.

Use the concept of adjunction space to complete part c) of this problem.

c) Prove that if Y is a normal absolute retract, then Y is an absolute extensor.

Problem II.21+. A space Y is an *absolute neighborhood extensor* (for the class of all normal spaces) if for every normal space X and every map $f : A \to Y$ where A is a closed subset of X, there is a map $g : U \to Y$ where U is a neighborhood of A in X and $g \mid A = f$.

$$
\begin{tikzcd}
X \\
\cup \\
U \arrow[r, dashed, "g"] & Y \\
\cup \\
A \arrow[uur, "f"]
\end{tikzcd}
$$

Clearly, every absolute extensor is an absolute neighborhood extensor.

a) Prove that every open subset of an absolute neighborhood extensor is an absolute neighborhood extensor.

b) Prove that every retract of an absolute neighborhood extensor is an absolute neighborhood extensor.

c) Use parts a) and b) of this problem to prove that $\mathbb{S}^n$ is an absolute neighborhood extensor for every $n \geq 0$.

Problem II.22+. A map of the form $h : X \times [\,0, 1\,] \to Y$ is called a *homotopy.* If $f : X \to Y$ and $g : X \to Y$ are maps and $h : X \times [\,0, 1\,] \to Y$ is a homotopy such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for every $x \in X$, then we say that f and g are *homotopic*. Prove that if Y is an absolute neighborhood extensor, then Y has the following *Borsuk homotopy extension property:* if X is a space such that $X \times [\,0, 1\,]$ is normal, A is a closed subset of X and $f : (\,A \times [\,0, 1\,]\,) \cup (\,X \times \{\,0\,\}\,) \to Y$ is a map, then f extends to a homotopy $g : X \times [\,0, 1\,] \to Y$ (i.e., $g \mid (\,A \times [\,0, 1\,]\,) \cup (\,X \times \{\,0\,\}\,) = f$).

The concepts of absolute extensor, absolute retract, absolute neighborhood extensor and the Borsuk homotopy extension property belong to an area of topology called the *theory of retracts* which was developed by the Polish topologist Karol Borsuk in the 1930’s. The class of absolute neighborhood extensors is broader than either the class of topological manifolds or the class polyhedra, but the notion of absolute neighborhood extensor captures some essential topological aspects of manifolds and polyhedra and has played a very useful role in the study of these objects.

A space X is called a *binormal* space if it satisfies the condition that $X \times [\,0, 1\,]$ is normal. The term was coined by the topologist C. H. Dowker in 1951. However, the use of binormality as a hypothesis apparently goes back to Borsuk's study of the homotopy extension property in the 1930's. We know that the product of two normal spaces need not be normal. (Recall $\mathbb{R}_{bad} \times \mathbb{R}_{bad}$.) Hence, it is conceivable that a space X might be normal but not binormal. Indeed, an example of a normal space that is not binormal was created by the renowned topologist M. E. Rudin in 1971. Hence, the hypothesis that X is binormal in Problem II.22+ apparently can't be replaced by the weaker assumption that X is normal.

Problem II.23+. A space X is *contractible* if there is a map $f : X \times [\,0, 1\,] \to X$ such that $f(x,0) = x$ for every $x \in X$ and f maps $X \times \{ 1 \}$ to a one-point subset of X. Thus, X is contractible if and only if id_x is homotopic to a constant map. Observe that $\mathbb{B}^n$ and $\mathbb{R}^n$ are contractible. ($\mathbb{S}^n$ is not contractible, but this is not easy to prove.) Suppose X is a space such that $X \times [\,0, 1\,]$ is normal. Prove that if X is an absolute extensor if and only if X is a contractible absolute neighborhood extensor.

Problem II.24+. A space X is *locally contractible* if for every $x \in X$ and every neighborhood U of x in X, there is a neighborhood V of x in U and a homotopy $f : V \times [\,0, 1\,] \to U$ such that $f(x,0) = x$ for every $x \in V$ and f maps $V \times \{ 1 \}$ to a one-point subset of U. Prove that every absolute neighborhood extensor is locally contractible.

One might be tempted to believe the converse of this result: every locally contractible space is an absolute neighborhood extensor. However, this converse is false. Borsuk provided an example of a locally contractible metric space that is not an absolute neighborhood extensor. One surprising feature of Borsuk's example is that it is infinite dimensional. Furthermore, infinite dimensionality is an essential feature of the example. Every such example must be infinite dimensional. In other words, once an adequate definition of topological dimension is formulated, it can be proved that every finite dimensional locally contractible metric space is an absolute neighborhood extensor.

Problem II.25+. Suppose the space X is the union of two closed subsets Y and Z such that

i) Y, Z and $Y \cap Z$ are absolute neighborhood extensors, and

ii) there is a map $\lambda : X \to [\,0, 1]$ such that $\lambda^{-1}(\,[\,0, {}^1/_2\,]\,) = Y$ and $\lambda^{-1}(\,[\,{}^1/_2, 1\,]\,) = Z$ (and, hence, $\lambda^{-1}(\,\{\,{}^1/_2\,\}\,) = Y \cap Z$).

a) Prove that X is an absolute neighborhood retract.

b) Also prove that if X is a metric space, then condition ii) is automatically satisfied and superfluous.

Since balls and spheres are absolute neighborhood extensors, then this result allows us to construct absolute neighborhood extensors by taking successive unions of balls in which each ball meets the previously added balls in a subset that is a ball or a sphere. Since straight line segments, triangles and 3-simplexes and higher-dimensional simplices are all homeomorphic to balls of various dimensions, then all polyhedra constructed of such pieces are absolute neighborhood retracts.