Additional Problems – 1

I. Topological Spaces

B. Bases

Problem I.1+. Prove that in a second countable space, every basis contains a countable basis.

Problem I.2+. Prove that if (X, $\mathscr{T}$) is a second countable space, then $\mathscr{T} \preceq \mathbb{R}$ (i.e., there is an injective function from $\mathscr{T}$ to $\mathbb{R}$).

Problem I.3+. This problem presents a variation on Example I.8. Recall that $\mathbb{N}$ denotes the set of natural numbers or positive integers and $\mathbb{Q}$ denotes the set of all rational numbers. Then $\mathbb{Q}^{\mathbb{N}}$ denotes the set of all functions from $\mathbb{N}$ to $\mathbb{Q}$. We now define a topology on $\mathbb{Q}^{\mathbb{N}}$. For each $f \in \mathbb{Q}^{\mathbb{N}}$ and each function $\varepsilon : \mathbb{N} \to (0, \infty)$, define

$$N(f,\varepsilon) = \{ g \in \mathbb{Q}^{\mathbb{N}} : | f(n) - g(n) | < \varepsilon(n) \text{ for every } n \in \mathbb{N} \}.$$

a) Prove that $\{ N(f,\varepsilon) : f \in \mathbb{Q}^{\mathbb{N}} \text{ and } \varepsilon \in (0, \infty)^{\mathbb{N}} \}$ is a basis for a topology on $\mathbb{Q}^{\mathbb{N}}$.

A function $f \in \mathbb{Q}^{\mathbb{N}}$ is said to be *eventually zero* if there is an $n \in \mathbb{N}$ such that $f(k) = 0$ for all $k > n$. Let E denote the subspace of $\mathbb{Q}^{\mathbb{N}}$ consisting of all eventually zero elements of $\mathbb{Q}^{\mathbb{N}}$.

b) Prove that E is a countable set.

c) Prove that for every $f \in E$, E is not first countable at f.

d) Prove that E is Hausdorff.

e) Prove the E is regular.

E is, in fact, normal. This follows from a theorem in a later chapter which implies that all countable regular spaces are normal.

C. Linearly Ordered Spaces

Problem I.4+.

a) Is every separable linearly ordered space necessarily first countable?

b) Is every separable linearly ordered space necessarily second countable?

Problem I.5+. a) Problem I.4 asserts that if (X, <) is a complete linearly ordered set, then it has the property that every decreasing sequence $I_1 \supset I_2 \supset I_3 \supset \ldots$ of closed bounded intervals in X has non-empty intersection. Is the converse to this assertion true? In other words, if (X, <) is a linearly ordered set with the property that every decreasing sequence of closed bounded intervals in X has non-empty intersection, then must (X, <) be complete?

b) Prove that if (X, <) is a separable linearly ordered set with the property that every decreasing sequence of closed bounded intervals in X has non-empty intersection, then (X, <) is complete.

If $\mathscr{C}$ is a collection of sets with the property that for all A and B $\in \mathscr{C}$, either $A \subset B$ or $B \subset A$, then we call $\mathscr{C}$ a *nested* collection.

c) Generalize Problem I.4 by proving that if (X, <) is a complete linearly ordered set, then it has the property that every nested collection of non-empty bounded closed intervals in X has non-empty intersection.

d) Is the converse to the assertion in part c) true? In other words, if (X, <) is a linearly ordered set with the property that every nested collection of non-empty bounded closed intervals in X has non-empty intersection, then must (X, <) be complete?

A collection $\mathscr{C}$ of sets is said to have the *finite intersection property* if every non-empty finite subcollection of $\mathscr{C}$ has non-empty intersection. Thus $\mathscr{C}$ has the finite intersection property if and only if whenever $C_1, C_2, \ldots, C_n \in \mathscr{C}$ for some positive integer n, then $C_1 \cap C_2 \cap \cdots \cap C_n \neq \varnothing$.

e) Prove that a linearly ordered set (X, <) is complete if and only if it has the property that every collection of bounded closed intervals in X with the finite intersection property has non-empty intersection.

f) Let (X, <) be a linearly ordered set with the order topology. Prove that if (X, <) is complete, then every collection of bounded closed subsets of X with the finite intersection property has non-empty intersection.

D. Metric Spaces

Problem I.6+. Prove that if $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^n$, then $\rho_2(\mathbf{x},\mathbf{y}) \le \rho_1(\mathbf{x},\mathbf{y}) \le \sqrt{n}\,\rho_2(\mathbf{x},\mathbf{y})$.

Problem I.7+. On the set $[\,0, 1\,]^{\mathbb{N}}$ of all functions from $\mathbb{N}$ to $[\,0, 1\,]$, we define three metrics:

i) $\sigma_1(\mathbf{x},\mathbf{y}) = \sum_{i \in \mathbb{N}} 2^{-i} \,|\, \mathbf{x}(i) - \mathbf{y}(i) \,|$.

ii) $\sigma_2(\mathbf{x},\mathbf{y}) = \left(\sum_{i \in \mathbb{N}} 2^{-i} \,(\, \mathbf{x}(i) - \mathbf{y}(i) \,)^2 \right)^{1/2}$. (Here $\mathbf{x}$, $\mathbf{y} \in [\,0, 1\,]^{\mathbb{N}}$.)

iii) $\sigma_\infty(\mathbf{x},\mathbf{y}) = \sup \{\, 2^{-i} \,|\, \mathbf{x}(i) - \mathbf{y}(i) \,|\, : i \in \mathbb{N} \,\}$.

a) Verify that these three formulas define metrics on $[\,0, 1\,]^{\mathbb{N}}$. (Observe that the factors 2^{-i} occuring in these formulas prevent these metrics from arising from norms in the obvious way that the taxicab, Euclidean and supremum metrics on $\mathbb{R}^n$ do.)

b) Prove that these three metrics on $[\,0, 1\,]^{\mathbb{N}}$ are equivalent.
The space $[\,0, 1\,]^{\mathbb{N}}$ with the topology induced by any one of these three equivalent metrics is called the *Hilbert cube.*

c) Is the Hilbert cube separable?

Problem I.8+. Does every norm on $\mathbb{R}^n$ induce the standard topology?

Problem I.9+. This problem outlines a way to decide whether $\mathbb{R}_{bad}$ is metrizable without appealing to Theorem I.13. To begin assume there is a metric ρ on $\mathbb{R}_{bad}$ which induces the closed-open interval topology; and for $x \in \mathbb{R}_{bad}$ and $\varepsilon > 0$, let $N(x,\varepsilon)$ denote the ε-neighborhood of x determined by the metric ρ. For m, n $\in \mathbb{N}$, define

$$S_{m,n} = \{\, x \in \mathbb{R}_{bad} : [x,x+(1/m)) \subset N(x,1/n) \subset [x,\infty) \,\}.$$

Now complete steps a), b) and c).

a) Prove that the collection of sets $\{\, S_{m,n} : m, n \in \mathbb{N} \,\}$ covers $\mathbb{R}$; i.e., every element of $\mathbb{R}$ belongs to $S_{m,n}$ for some m, n $\in \mathbb{N}$.

b) Prove there are elements m, n $\in \mathbb{N}$ such that the set $S_{m,n}$ is uncountable.

b) Assume $S_{m,n}$ is uncountable. Prove that there are elements x, y $\in S_{m.n}$ such that $x < y < x+(1/m)$.

c) Derive a contradiction.

Problem I.10+. There is a way to prove that Ω is not metrizable using properties of compactness introduced in a later chapter. (Ω can be shown to be sequentially compact but not compact. On the other hand, all sequentially compact metric spaces are compact.) This problem outlines a proof that Ω is not metrizable which does not rely on compactness properties.

a) Prove that if A is an uncountable subset of Ω, then there is an $x \in \Omega$ such that $(-\infty, x) \neq \varnothing$ and $(y, x) \cap A \neq \varnothing$ for every $y \in (-\infty, x)$.

b) Assume there is a metric ρ on Ω which induces the order topology; and for $x \in \Omega$ and $\varepsilon > 0$, let $N(x,\varepsilon)$ denote the ε-neighborhood of x determined by the metric ρ. For each $n \in \mathbb{N}$, define $S_n = \{ x \in \Omega : N(x,1/n) \subset (-\infty, x] \}$. Prove there is an $n \in \mathbb{N}$ such that S_n is uncountable. Then use part a) of this problem to obtain a contradiction.

Problem I.11+. Let (X, ρ) be a metric space. For $\varepsilon > 0$, a subset S of X is ε *-separated* if $\rho(x,y) \geq \varepsilon$ for all distinct points $x, y \in S$.

a) Prove that a metric space is separable if and only if for every $\varepsilon > 0$, every ε-separated subset of X is countable.

b) Prove that a metric space is separable if and only if every pairwise disjoint collection of non-empty open subsets of X is countable.

Remark. Additional Problem I.11+. b) becomes false if the "metric" hypothesis is removed. Indeed, Additional Problem I.17+ outlines the construction of a non-separable space X in which every pairwise disjoint collection of open subsets of X is countable. Of course, this space is not metrizable.

Problem I.12+. Let X be an infinite set and let B(X) denote the metric space described in Example I.15.

a) Prove there is a pairwise disjoint collection $\mathscr{U}$ of non-empty open subsets of B(X) such that $\mathscr{U} \approx \mathscr{P}(X)$.

b) Assume $\mathbb{N} \times X \approx X$. (This is not hard to prove for $X = \mathbb{N}$ or $\mathbb{R}$ and can be proved in general if X well-ordered. If we assume Zermelo's Well Ordering Principle, then it follows, of course, that X is well-ordered). Prove $B(X) \approx \mathscr{P}(X)$. Conclude that if D is any dense subset of B(X), then $D \approx B(X)$.

E. Closure and Convergence Properties

Problem I.13+. Prove that if A is an uncountable subset of a second countable space X, then there is a closed subset B of X such that $A \cap B$ is uncountable and for every $x \in B$, every neighborhood of x intersects $A \cap B$ in an uncountable set.

Problem I.14+. Let X be a topological space. The *closure operator* A $\mapsto$ cl(A) and the *complement operator* A $\mapsto$ (X – A) are two functions from the collection $\mathscr{P}$(X) of all subsets of X to itself. They are known as the *Kuratowski operators* on $\mathscr{P}$(X).

a) Given a subset A of X, find the maximum number of distinct subsets of X that can theoretically be formed from A by repeated application of the Kuratowski operators.

b) Give an example of a subset A of a topological space X for which this theoretical maximum is achieved.

F. Separation Properties

Problem I.15+. Is every linearly ordered space **a)** T_1, **b)** Hausdorff, **c)** regular, **d)** normal?

Problem I.16+. a) Prove that if X is a second countable T_1 space, then X $\preceq \mathbb{R}$.

b) Prove that if X is a separable Hausdorff space, then X $\preceq \mathscr{P}(\mathbb{R})$.

Remark. The estimate in Additional Problem I.16+. b) can't be improved. In fact, Additional Problem I.17+. e) provides an example of a separable Hausdorff space X such that X $\approx \mathscr{P}(\mathbb{R})$.

Problem I.17+. a) Prove that if X is a separable regular space, then X has a basis $\mathscr{B}$ such that $\mathscr{B} \preceq \mathbb{R}$.

b) Here we outline a construction which (when combined with Additional Problem I.18+) shows that the "regular" hypothesis in part a) of this problem can't be replaced by "Hausdorff". Suppose (X, $\mathscr{T}$) is a separable Hausdorff space in which every non-empty open subset is uncountable, and suppose D is a countable dense subset of X. Set $\mathscr{B}_* = \{ \{ x \} \cup (U \cap D) : x \in U \in \mathscr{T}\}$.

i) Prove $\mathscr{B}_*$ is a basis for a topology on X, and let $\mathscr{T}_*$ denote this topology.

Prove that (X, $\mathscr{T}_*$) is **ii)** Hausdorff but not regular, and **iii)** separable.

iv) Prove that if $\mathscr{B}_{**}$ is any basis for $\mathscr{T}_*$, then $\mathscr{B}_{**} \succeq$ X.

Now suppose that (X, $\mathscr{T}$) is a separable Hausdorff space satisfying X $\approx \mathscr{P}(\mathbb{R})$ in which every non-empty open subset is uncountable. (Such a space is constructed in Additional Problem I.18+. See the remark following Additional Problem I.18+.) Then (X, $\mathscr{T}_*$) is a separable Hausdorff space with the property that if $\mathscr{B}_{**}$ is any basis for $\mathscr{T}_*$, then $\mathscr{B}_{**} \succeq \mathscr{P}(\mathbb{R})$. We conclude that the "regular" hypothesis in part a) of this problem can't be replaced by "Hausdorff".

Remark. Again suppose that (X, $\mathscr{T}$)is a separable Hausdorff space satisfying X $\approx \mathscr{P}(\mathbb{R})$. Since $\mathscr{T} \subset \mathscr{P}(X)$, it is possible in principle that $\mathscr{T} \approx \mathscr{P}(X) \approx \mathscr{P}(\mathscr{P}(\mathbb{R}))$ and, further, that if $\mathscr{B}$ is any basis for $\mathscr{T}$, then $\mathscr{B} \approx \mathscr{P}(X) \approx \mathscr{P}(\mathscr{P}(\mathbb{R}))$. Then $\mathscr{T}$ would in some sense be a "largest possible topology" on a separable Hausdorff space. The construction just carried out in Additional Problem I.17+. b) does not provide such a topology because the basis $\mathscr{B}_*$ constructed there satisfies $\mathscr{B}_* \approx X \times \mathscr{P}(D) \approx \mathscr{P}(\mathbb{R})$. Additional Problem I.19+ fills this gap by outlining the construction of a separable Hausdorff space with a "largest possible topology".

Problem I.18+. Let X be a set. Let $\{ 0, 1 \}^X$ denote the set of all functions from X to { 0, 1 }. For every $f \in \{ 0, 1 \}^X$ and every finite subset A of X, set

$$N(f,A) = \{ g \in \{ 0, 1 \}^X : g|A = f|A \}.$$

Set $\mathscr{B} = \{ N(f,A) : f \in \{ 0, 1 \}^X$ and A is a finite subset of X }.

a) Prove $\mathscr{B}$ is a basis for a topology on $\{ 0, 1 \}^X$.

Endow $\{ 0, 1 \}^X$ with this topology.

b) Prove that $\{ 0, 1 \}^X$ is a regular Hausdorff space.

c) Prove that if X is infinite, then every non-empty open subset of $\{ 0, 1 \}^X$ is uncountable.

d) Prove that the following are equivalent: **i)** X is countable, **ii)** $\{ 0, 1 \}^X$ is second countable, and **iii)** $\{ 0, 1 \}^X$ is first countable.

e) Prove that if X is countable, then $\{ 0, 1 \}^X$ is metrizable.

f) Prove that if $X \preceq \mathbb{R}$, then $\{ 0, 1 \}^X$ is separable.

g) Prove that if $X \succ \mathbb{R}$ and $\mathscr{B}_*$ is any basis for $\{ 0, 1 \}^X$, then $\mathscr{B}_* \succ \mathbb{R}$ and $\{ 0, 1 \}^X$ is not separable.

h) Prove that every pairwise disjoint collection of open subsets of $\{ 0, 1 \}^X$ is countable.

Observe that if $X \succ \mathbb{R}$, then $\{ 0, 1 \}^X$ provides an example of the type mentioned in the remark following Addition Problem I.11+: a non-separable regular Hausdorff space in which every pairwise disjoint collection of open subsets is countable.

Remark. It follows that $\{ 0, 1 \}^{\mathbb{R}}$ is a separable regular Hausdorff space satisfying $\{ 0, 1 \}^{\mathbb{R}} \approx \mathscr{P}(\mathbb{R})$. Hence, that the estimate of Additional Problem I.16+. b) can't be improved. Also this is the sort of space needed in Additional Problem I.17+. b).

Problem I.19+. This problem outlines the rather lengthy construction of a separable Hausdorff space Y with the property that if $\mathscr{B}$ is any basis for Y, then $\mathscr{B} \approx \mathscr{P}(\mathscr{P}(\mathbb{R}))$. Prove the propositions labelled **a)** through **m).**

To begin, let X denote an infinite set. Assume $\mathbb{N} \times X \approx X$. (As we remarked in Problem I.12+. b), this can be proved if $X = \mathbb{N}$ or $\mathbb{R}$ or if X well-ordered. Later we will set $X = \mathbb{R}$.) An *ultrafilter* on X is a collection α of subsets of X satisfying

i) $A, B \in \alpha \Rightarrow A \cap B \in \alpha$,

ii) $A \in \alpha$ and $A \subset B \subset X \Rightarrow B \in \alpha$, and

iii) $\forall\, A \subset X$, exactly one of A and $X - A$ is an element of α.

(It follows that $X \in \alpha$ and $\varnothing \notin \alpha$.)

Let $\mathscr{U}$ denote the set of all ultrafilters on X.

For each $x \in X$, set $\alpha_x = \{ A \subset X : x \in A \}$.

a) Prove that for each $x \in X$, α_x is an ultrafilter on X.

The ultrafilters of the form α_x (where $x \in X$) are called *principal ultrafilters* on X. Let $\mathscr{U}_p$ denote the set of all principal ultrafilters on X.

A collection σ of subsets of X has the *finite intersection property* if for all finite sequences $S_1, S_2, \ldots, S_n$ of elements of σ, $\bigcap_{i=1}^{n} S_i \neq \varnothing$.

b) Prove that if σ is a collection of subsets of X with the finite intersection property, then there is an ultrafilter α on X such that $\sigma \subset \alpha$.

A collection σ of subsets of X is *independent* if for all finite sequences $S_1, S_2, \ldots, S_m, T_1, T_2, \ldots, T_n$ of distinct elements of σ, $\left(\bigcap_{i=1}^{m} S_i \right) \cap \left(\bigcap_{j=1}^{n} (X - T_j) \right) \neq \varnothing$.

c) Prove that if σ is an independent collection of subsets of X, then for each subset τ of σ, there is an ultrafilter α_τ on X such that $\tau \cup \{ X - S : S \in \sigma - \tau \} \subset \alpha_\tau$. Also prove that the function $\tau \mapsto \alpha_\tau : \mathscr{P}(\sigma) \to \mathscr{U}$ is one-to-one. Conclude that $\mathscr{P}(\sigma) \preceq \mathscr{U}$.

d) Prove that there exists an independent collection σ of subsets of X such that $\sigma \approx \mathscr{P}(X)$.

Outline of proof of d): For any set S, let $\mathscr{F}(S)$ denote the collection of all finite subsets of S. Set $Y = \mathscr{F}(X) \times \mathscr{F}(\mathscr{F}(X))$.

i) Prove $X \approx Y$.

For every $A \subset X$, define $A^* \subset Y$ by $A^* = \{ (F, \Phi) \in Y : F \cap A \in \Phi \}$.

ii) Prove that the function $A \mapsto A^* : \mathscr{P}(X) \to \mathscr{P}(Y)$ is one-to-one.

iii) Prove that $\{ A^* : A \in \mathscr{P}(X) \}$ is an independent collection of subsets of Y.

iv) Prove that there exists an independent collection σ of subsets of X such that $\sigma \approx \mathscr{P}(X)$.

e) Prove that $\mathscr{U} \approx \mathscr{P}(\mathscr{P}(X))$.

f) Prove that $\mathscr{U}$ is an independent collection of subsets of $\mathscr{P}(X)$.

Set $\mathscr{B}$ =

$$\Big\{ \Big(\bigcap_{i=1}^{m} \alpha_i \Big) \cap \Big(\bigcap_{j=1}^{n} \big(\mathscr{P}(X) - \beta_j \big) \Big) : m, n \in \mathbb{N} \text{ and } \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \in \mathscr{U} \Big\},$$

and set $\mathscr{B}_p$ =

$$\Big\{ \Big(\bigcap_{i=1}^{m} \alpha_i \Big) \cap \Big(\bigcap_{j=1}^{n} \big(\mathscr{P}(X) - \beta_j \big) \Big) : m, n \in \mathbb{N} \text{ and } \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \in \mathscr{U}_p \Big\}.$$

g) Prove that $\mathscr{B}$ is a basis for a topology $\mathscr{T}$ on $\mathscr{P}(X)$, and $\mathscr{B}_p$ is a basis for a topology $\mathscr{T}_p$ on $\mathscr{P}(X)$ such that $\mathscr{T}_p \subset \mathscr{T}$.

h) Prove that for every $A \in \mathscr{P}(X)$, if $\mathscr{B}(A)$ is any basis for $\mathscr{T}$ at A, then $\mathscr{B}(A) \approx \mathscr{P}(\mathscr{P}(X))$.

i) Prove that $(\mathscr{P}(X), \mathscr{T}_p)$ and $(\mathscr{P}(X), \mathscr{T})$ are Hausdorff spaces.

(Parenthetical observation: Identify $\mathscr{P}(X)$ with $\{ 0, 1 \}^X$ by identifying each subset A of X with its characteristic function $\chi_A : X \to \{0,1\}$. ($\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \in X - A$.) Observe that this identification carries the topology $\mathscr{T}_p$ onto the topology defined on $\{ 0, 1 \}^X$ in Additional Problem I.18+.**)**

Now set $X = \mathbb{R}$.

j) Prove that $(\mathscr{P}(\mathbb{R}), \mathscr{T}_p)$ is separable.

Let Δ be a countable dense subset of $(\mathscr{P}(\mathbb{R}), \mathscr{T}_p)$.
Set $\mathscr{T}^\# = \{ (\alpha - \Delta) \cap (\beta \cap \Delta) : \alpha \in \mathscr{T}, \beta \in \mathscr{T}_p, \text{ and } \alpha \subset \beta \}$.

k) Prove that $\mathscr{T}^\#$ is a topology on $\mathscr{P}(\mathbb{R})$, and $\mathscr{T}_p \subset \mathscr{T}^\#$.

l) Prove that $(\mathscr{P}(\mathbb{R}), \mathscr{T}^\#)$ is a separable Hausdorff space.

m) Prove that for every $A \in \mathscr{P}(\mathbb{R}) - \Delta$, if $\mathscr{B}^{\#}(A)$ is any basis for $\mathscr{T}^{\#}$ at A, then $\mathscr{B}^{\#}(A) \approx \mathscr{P}(\mathscr{P}(\mathbb{R}))$. Conclude that if $\mathscr{B}^{\#}$ is any basis for $\mathscr{T}^{\#}$, then $\mathscr{B}^{\#} \approx \mathscr{P}(\mathscr{P}(\mathbb{R}))$. ◻

Problem I.20+. A space X is *completely normal* if it has the following property: if A and B are subsets of X satisfying $\mathrm{cl}(A) \cap B = \varnothing = A \cap \mathrm{cl}(B)$, then A and B have disjoint neighborhoods in X. Prove that every metric space is completely normal.

Problem I.21+. a) Let X be an infinite set with the finite complement topology, and let $\{ x_n \}$ be a sequence of distinct points of X. Observe that X is T_1 but not Hausdorff. Prove that $\{ x_n \}$ converges to every point of X.

b) Prove that if X is a Hausdorff space, then every converging sequence in X has a unique limit.

c) Assume X is a first countable space. Prove: X is Hausdorff $\Leftrightarrow$ every converging sequence in X has a unique limit.

d) Find a space which is not first countable and not Hausdorff in which every converging sequence has a unique limit.

Remark. The following two problems are related to Example I.17.

Problem I.22+. Suppose that the topological space X has a subset Y with the following two properties.

i) For each $y \in Y$, $\{ W_n(y) : n \in \mathbb{N} \}$ is a basis for X at y such that $W_{n+1}(y) \subset W_n(y)$ for each $n \in \mathbb{N}$.

ii) There is a bijection $t \mapsto y_t : \mathbb{R} \to Y$ satisfying the condition: for each $s \in \mathbb{R}$ and each $n \in \mathbb{N}$, there is a $\delta > 0$ such that if $t \in \mathbb{R}$ and $| s - t | < \delta$, then $W_n(y_s) \cap W_n(y_t) \neq \varnothing$.

a) Prove that the sets $\{ y_t : t \in \mathbb{Q} \}$ and $\{ y_t : t \in \mathbb{R} - \mathbb{Q} \}$ don't have disjoint neighborhoods in X. (Here $\mathbb{Q}$ denotes the set of rational numbers.)

Assume that the set Y has no limit points in X.

b) Prove that X is not normal.

Problem I.23+. Prove that if a separable space X has a subset Y which has no limit points in X such that $Y \approx \mathbb{R}$, then X is not normal.

G. Subspaces and Finite Product Spaces

Problem I.24+. Let X and Y be topological spaces and let $A \subset X$ and $B \subset Y$. Prove that $\text{fr}(A \times B) = (\text{fr}(A) \times B) \cup (A \times \text{fr}(B))$.

Problem I.25+. Recall the definition of *completely normal* from Additional Problem I.20+. A space X is *hereditarily normal* if each subspace of X is normal. Prove that for a space X, the following three assertions are equivalent.

i) X is completely normal.

ii) If Y is a subspace of X, then any two disjoint relatively closed subsets of Y have disjoint neighborhoods in X.

iii) X is hereditarily normal.

The following problem describes a space with properties similar to the space described in Example I.18. This problem is an alternative version of Problem I.21.

Problem I.26+. Let $\Omega^+ = \Omega \cup \{ \omega^+ \}$ be the space described in Example I.11. Set $Y = \{0\} \cup \{ {}^1/_n : n \in \mathbb{N} \}$, and regard Y as a subspace of $\mathbb{R}$. Let $\Omega^+ \times Y$ have the product topology.

a) Prove that $\Omega^+ \times Y$ is a normal Hausdorff space.

b) Prove that the subspace $(\Omega^+ \times Y) - \{ (\omega^+, 0) \}$ of $\Omega^+ \times Y$ is not normal.

c) Is the space $\Omega^+ \times Y$ completely normal?

Problem I.27+. Is every linearly ordered space completely normal? (Equivalently: is every linearly ordered space hereditarily normal?)

Problem I.28+. A space is *hereditarily separable* if each of its subspaces is separable. (Problem I.16 asks for a separable space which is not hereditarily separable.) Is every separable linearly ordered space hereditarily separable?