E. Closure and Convergence Properties

Topology provides a means of expressing concepts of limit, convergence and closure. Indeed, the basic definition of topology can easily be recast in terms of closed sets rather than open sets. The connection between topology and limit concepts accounts for topology's fundamental role in analysis.

Definition. A subset C of a topological space X is *closed* if X – C is open.

Theorem I.14. Let X be a topological space.

a) $\varnothing$ and X are closed subsets of X.

b) The intersection of any collection of closed subsets of X is a closed subset of X.

c) The union of any *finite* collection of closed subsets of X is a closed subset of X.

Proof. a) Since $X - \varnothing = X$ and $X - X = \varnothing$ are open subsets of X, then $\varnothing$ and X are closed subsets of X.

b) Suppose $\mathscr{C}$ is a collection of closed subsets of X. Then $\{ X - C : C \in \mathscr{C} \}$ is a collection of open subsets of X. Therefore $\bigcup \{ X - C : C \in \mathscr{C} \}$ is an open subset of X. By De Morgan's Laws $\bigcup \{ X - C : C \in \mathscr{C} \} = X - (\bigcap \mathscr{C})$. Thus, $X - (\bigcap \mathscr{C})$ is an open subset of X. Hence, $\bigcap \mathscr{C}$ is a closed subset of X.

c) Suppose $\mathscr{F}$ is a finite collection of closed subsets of X. Then $\{ X - F : F \in \mathscr{F} \}$ is a finite collection of open subsets of X. Therefore $\bigcap \{ X - F : F \in \mathscr{F} \}$ is an open subset of X. By De Morgan's Laws $\bigcap \{ X - F : F \in \mathscr{F} \} = X - (\bigcup \mathscr{F})$. Thus, $X - (\bigcup \mathscr{F})$ is an open subset of X. Hence, $\bigcup \mathscr{F}$ is a closed subset of X. □

Definition. Let X be a topological space. If $x \in X$, then any open subset of X which contains x is called a *neighborhood* of x in X. If $A \subset X$, then any open subset of X which contains A is called a *neighborhood* of A in X.

Theorem I.15. Let X be a topological space. Then:

a) A subset U of X is open if and only if each point of U has a neighborhood which is contained in U.

b) A subset C of X is closed if and only if each point of X – C has a neighborhood which is disjoint from C.

Problem I.9. Prove Theorem I.15. a)

Proof of Theorem I.15. b) Let $C \subset X$. Then: C is a closed subset of X
$\Leftrightarrow$ $X - C$ is an open subset of X
$\Leftrightarrow$ (by part a) each point of $X - C$ has a neighborhood which is contained in $X - C$
$\Leftrightarrow$ each point of $X - C$ has a neighborhood which is disjoint from C. ☐

Definition. Let X be a topological space, and let $A \subset X$.

The *interior* of A, denoted int(A) or $\mathring{A}$, is the set $\bigcup\{ U : U \text{ is an open set and } U \subset A \}$.

The *closure* of A, denoted cl(A) or $\overline{A}$, is the set $\bigcap\{ C : C \text{ is a closed set and } A \subset C \}$.

The *frontier* or *boundary* of A, denoted fr(A) or bdy(A), is the set $\mathrm{cl}(A) - \mathrm{int}(A) = \overline{A} - \mathring{A}$.

Theorem I.16. Let X be a topological space, and let $A \subset X$ and $x \in X$. Then:

a) $x \in \mathrm{int}(A)$ if and only if some neighborhood of x is contained in A.

b) $x \in \mathrm{cl}(A)$ if and only if every neighborhood of x intersects A.

c) $x \in \mathrm{fr}(A)$ if and only if every neighborhood of x intersects both A and $X - A$.

Proof of a). Observe that the following sequence of equivalences is valid:

$x \in \mathrm{int}(A)$ $\Leftrightarrow$ there is an open set U such that $U \subset A$ and $x \in U$ $\Leftrightarrow$

some neighborhood of x is contained in A.

This proves assertion a).

Proof of b). Observe that the following sequence of implications is valid:

There is a neighborhood U of x such that $U \cap A = \varnothing$ $\Rightarrow$

$X - U$ is a closed set and $A \subset X - U$ $\Rightarrow$

$\mathrm{cl}(A) \subset X - U$ $\Rightarrow$ $\mathrm{cl}(A) \cap U = \varnothing$ $\Rightarrow$ $x \notin \mathrm{cl}(A)$.

Hence, if $x \in \mathrm{cl}(A)$, then every neighborhood of x intersects A.

Next observe that the following sequence of implications is valid:

$x \notin \mathrm{cl}(A)$ $\Rightarrow$ there is a closed set C such that $A \subset C$ and $x \notin C$ $\Rightarrow$

$X - C$ is a neighborhood of x in X and $(X - C) \cap A = \varnothing$.

Hence, if every neighborhood of x intersects A, then $x \in \mathrm{cl}(A)$.

This completes the proof of assertion b).

Proof of c). By part b) of this theorem:

$x \in cl(A) \Leftrightarrow$ every neighborhood of x intersects A.

By part a) of this theorem:

$x \notin int(A) \Leftrightarrow$ no neighborhood of x is contained in A

$\Leftrightarrow$ every neighborhood of x intersects X – A.

Hence:

$x \in fr(A) \Leftrightarrow x \in cl(A) - int(A) \Leftrightarrow x \in cl(A)$ and $x \notin int(A) \Leftrightarrow$

every neighborhood of x intersects A and every neighborhood of x intersects X – A

$\Leftrightarrow$ every neighborhood of x intersects both A and X–A.

This completes the proof of assertion c). ◻

Corollary I.17. A subset D of a topological space X is dense if and only if cl(D) = X.

Exercise. Prove Corollary I.17.

Theorem I.18. Let X be a topological space, and let $A, B \subset X$.

a) int(A) is an open set, and cl(A) and fr(A) are closed sets.

b) $int(A) \subset A \subset cl(A)$, $int(A) \cap fr(A) = \emptyset$, and $int(A) \cup fr(A) = cl(A)$.

c) A is a closed set if and only if cl(A) = A if and only if $fr(A) \subset A$; and
A is an open set if and only if int(A) = A if and only if $fr(A) \cap A = \emptyset$.

d) $int(A) = X - cl(X - A)$, $cl(A) = X - int(X - A)$, and
$fr(A) = cl(A) \cap cl(X - A) = X - (int(A) \cup int(X - A))$.

e) If $A \subset B$, then $int(A) \subset int(B)$ and $cl(A) \subset cl(B)$.

f) $int(A \cap B) = int(A) \cap int(B)$, $int(A \cup B) \supset int(A) \cup int(B)$,
$cl(A \cup B) = cl(A) \cup cl(B)$, and $cl(A \cap B) \subset cl(A) \cap cl(B)$.

Proof of Theorem I.18. a), b), c), e) and f).

a) Since int(A) is, by definition, a union of open sets, it is an open set.

Since cl(A) is, by definition, an intersection of closed sets, then it is a closed set by Theorem I.14. b).

Since $fr(A) = cl(A) \cap (X - int(A))$, then fr(A) is an intersection of closed sets. So fr(A) is a closed set.

This completes the proof of a).

b) Since int(A) is, by definition, a union of subsets of A, then int(A) $\subset$ A.

Since cl(A) is, by definition, an intersection of sets which contain A, then A $\subset$ cl(A).

Since fr(A) = cl(A) – int(A), then fr(A) $\cap$ int(A) = $\emptyset$.

Since fr(A) = cl(A) – int(A), then cl(A) $\subset$ int(A) $\cup$ fr(A). Since int(A) $\subset$ A $\subset$ cl(A) and fr(A) = cl(A) – int(A) $\subset$ cl(A), then int(A) $\cup$ fr(A) $\subset$ cl(A). Thus, cl(A) = int(A) $\cup$ fr(A).

This completes the proof of b).

c) First we prove: A is a closed set $\Leftrightarrow$ cl(A) = A.

Assume A is a closed set. cl(A) is, by definition, the intersection of all the closed sets that contain A and A is such a set. Thus, cl(A) $\subset$ A. Since it is also true that A $\subset$ cl(A) by part b), then we conclude that cl(A) = A.

Conversely, if cl(A) = A, then A is a closed set by part a).

Next we prove: cl(A) = A $\Leftrightarrow$ fr(A) $\subset$ A.

Since fr(A) = cl(A) – int(A) $\subset$ cl(A), then cl(A) = A implies fr(A) $\subset$ A.

Assume fr(A) $\subset$ A. int(A) $\subset$ A by part b). Since cl(A) = int(A) $\cup$ fr(A), then it follows that cl(A) $\subset$ A. Since it is also true that A $\subset$ cl(A) by part b), then we conclude that cl(A) = A.

Third, we prove: A is an open set $\Leftrightarrow$ int(A) = A.

Assume A is an open set. int(A) is, by definition, the union of all the open sets that are contained in A, and A is such a set. Hence, A $\subset$ int(A). Since it is also true that int(A) $\subset$ A by part b), then we conclude that int(A) = A.

Conversely, if int(A) = A, then A is an open set by part a).

Finally, we prove: int(A) = A $\Leftrightarrow$ fr(A) $\cap$ A = $\emptyset$.

Since, by part b), int(A) $\cap$ fr(A) = $\emptyset$, then int(A) = A implies A $\cap$ fr(A) = $\emptyset$.

Assume $fr(A) \cap A = \emptyset$. Since, by part b), $A \subset cl(A) = int(A) \cup fr(A)$, then it follows that $A \subset int(A)$. Since it is also true that $int(A) \subset A$ by part b), then we conclude that $int(A) = A$.

This completes the proof of c).

e) Assume $A \subset B$.

Since $int(A) \subset A$, then $int(A) \subset B$. Also int(A) is an open set. Since int(B) is the union of all the open sets that are contained in B, it follows that $int(A) \subset int(B)$.

Since $B \subset cl(B)$, then $A \subset cl(B)$. Also cl(B) is a closed set. Since cl(A) is the intersection of all the closed sets that contain A, it follows that $cl(A) \subset cl(B)$.

This completes the proof of e).

f) Since $A \cap B \subset A$ and $A \cap B \subset B$, then part e) implies $int(A \cap B) \subset int(A)$ and $int(A \cap B) \subset int(B)$. Hence, $int(A \cap B) \subset int(A) \cap int(B)$. On the other hand, since by parts a) and b), int(A) is an open subset of A and int(B) is an open subset of B, then $int(A) \cap int(B)$ is an open subset of $A \cap B$. Since $int(A \cap B)$ is the union of all open subsets of $A \cap B$, then it follows that $int(A) \cap int(B) \subset int(A \cap B)$. We conclude that $int(A \cap B) = int(A) \cap int(B)$.

Since $A \subset A \cup B$ and $B \subset A \cup B$, then part e) implies $int(A) \subset int(A \cup B)$ and $int(B) \subset int(A \cup B)$. Hence, $int(A) \cup int(B) \subset int(A \cup B)$.

Since $A \subset A \cup B$ and $B \subset A \cup B$, then part e) implies $cl(A) \subset cl(A \cup B)$ and $cl(B) \subset cl(A \cup B)$. Hence, $cl(A) \cup cl(B) \subset cl(A \cup B)$. On the other hand, since by parts a) and b), cl(A) is a closed set containing A and cl(B) is a closed set containing B, then $cl(A) \cup cl(B)$ is a closed set containing $A \cup B$. Since $cl(A \cup B)$ is the intersection of all the closed sets that contain $A \cup B$, then it follows that $cl(A) \cup cl(B) \supset cl(A \cup B)$. We conclude that $cl(A) \cup cl(B) = cl(A \cup B)$.

Finally, since $A \cap B \subset A$ and $A \cap B \subset B$, then part e) implies $cl(A \cap B) \subset cl(A)$ and $cl(A \cap B) \subset cl(B)$. Hence, $cl(A \cap B) \subset cl(A) \cap cl(B)$.

This completes the proof of f). ◻

Problem I.10. Prove Theorem I.18 d). Also give examples which show that the inclusions $int(A \cup B) \supset int(A) \cup int(B)$ and $cl(A \cap B) \subset cl(A) \cap cl(B)$ in I.18 f) can't be replaced by equalities. In other words, give examples of subsets A and B of a topological space X such that $int(A \cup B) \neq int(A) \cup int(B)$ and $cl(A \cap B) \neq cl(A) \cap cl(B)$.

Suggestion: Look for ways to shorten and simplify your proof of I.18 d) by exploiting the parts of Theorem I.18 that are already proved.

Definition. Let X be a topological space, and let $A \subset X$ and $x \in X$. x is a *limit point* or *accumulation point* of A if for every neighborhood U of x in X, $U \cap (A - \{x\}) \neq \varnothing$. Thus, x is a limit point of A $\Leftrightarrow$ $x \in \text{cl}(A - \{x\})$. The set of all limit points of A is called the *derived set* of A and is denoted A´.

Theorem I.19. If A is a subset of a topological space X, then
a) $\text{cl}(A) = A \cup A'$, and **b)** A is a closed set if and only if $A' \subset A$.

Proof. a) $A \subset \text{cl}(A)$, by Theorem I.18 b). Also,

$x \in A'$ $\Rightarrow$ every neighborhood of x intersects $A - \{x\}$ $\Rightarrow$

every neighborhood of x intersects A $\Rightarrow$ $x \in \text{cl}(A)$ by Theorem I.16 b).

Hence, $A' \subset \text{cl}(A)$. It follows that $A \cup A' \subset \text{cl}(A)$.

Note that:

$x \in \text{cl}(A) - A$ $\Rightarrow$ $x \in \text{cl}(A)$ and $x \notin A$ $\Rightarrow$

every neighborhood of x intersects A (by Theorem I.15 b)) and $A = A - \{x\}$

$\Rightarrow$ every neighborhood of x intersects $A - \{x\}$ $\Rightarrow$ $x \in A'$.

Hence, $\text{cl}(A) - A \subset A'$. It follows that $\text{cl}(A) \subset A \cup A'$.

We conclude that $\text{cl}(A) = A \cup A'$. This completes the proof of a).

b) Observe that:

$A' \subset A$ $\Leftrightarrow$ $A \cup A' = A$ $\Leftrightarrow$ $\text{cl}(A) = A$ (by part a) of this theorem)

$\Leftrightarrow$ A is a closed set (by Theorem I.18 c)).

This completes the proof of b). ◻

Theorem I.20. In a second countable space, every uncountable subset contains a limit point of itself.

Problem I.11. Prove Theorem I.20.

Remark. This theorem can be strengthened. There is an Additional Problem that asks for a proof of a theorem with the same hypothesis as Theorem I.20 but with a significantly stronger conclusion.

Definition. Let X be a set. A *sequence* in X is a function from $\mathbb{N} = \{ 1, 2, 3, \cdots \}$ to X. If $x : \mathbb{N} \to X$ is a sequence in X, then we also denote x by $\{ x_n \}$ where $x_n = x(n)$ for $n \in \mathbb{N}$.

Definition. Let X be a topological space, let $\{ x_n \}$ be a sequence in X, and let $y \in X$. $\{ x_n \}$ *converges to* y if for every neighborhood U of y in X, there is an $n \in \mathbb{N}$ such that $i \geq n \Rightarrow x_i \in U$. If $\{ x_n \}$ converges to y, we also say y is a *limit* of $\{ x_n \}$, and we write $y = \lim x_n$ and $\{ x_n \} \to y$.

Theorem I.21. Let X be a topological space, and let $A \subset X$ and $x \in X$.

a) If x is a limit of a sequence in A, then $x \in \text{cl}(A)$.

b) Assume X is a first countable space. Then

$$x \in \text{cl}(A) \text{ if and only if } x \text{ is a limit of a sequence in A.}$$

Proof. a) If x is a limit of a sequence in A, then, clearly, every neighborhood of x intersects A. So $x \in \text{cl}(A)$ by Theorem I.16 b).

b) Assume X is a first countable space. Then there is a countable basis $\{ U_n : n \in \mathbb{N} \}$ for X at x. We can assume $U_1 \supset U_2 \supset U_3 \supset \cdots$ by replacing U_n by $U_1 \cap U_2 \cap \ldots \cap U_n$ for each $n \in \mathbb{N}$. Suppose $x \in \text{cl}(A)$. Then Theorem I.16 b) implies that each U_n intersects A. So for each $n \in \mathbb{N}$, we can choose a point $a_n \in U_n \cap A$. Then $\{ a_n \}$ is a sequence in A. To show $\{ a_n \} \to x$, let V be a neighborhood of x in X. Then there is an $n \in \mathbb{N}$ such that $U_n \subset V$. Now $i \geq n \Rightarrow x_i \in U_i \subset U_n \subset V$. This proves $\{ a_n \} \to x$. So x is a limit of a sequence in A. The converse direction of b) follows from a). □

Corollary I.22. Let X be a topological space, and let $A \subset X$.

a) If A is a closed set, then every point of X which is a limit of a sequence in A belongs to A.

b) Assume X is a first countable space. Then A is a closed set if and only if every point of X which is a limit of a sequence in A belongs to A. □

Problem I.12. This problem illustrates that the first countability hypothesis can't be omitted in either Theorem I.21 b) or its corollary. Recall the space $\Omega^+ = \Omega \cup \{ \omega^+ \}$ defined in Example I.11.

a) Prove that ω^+ is not the limit of any sequence in Ω.

b) Prove that $\omega^+ \in \text{cl}(\Omega)$.

Hence, every point of Ω^+ which is a limit of a sequence in Ω belongs to Ω, but Ω is not a closed subset of Ω^+.

F. Separation Properties

The separation properties – T_1, Hausdorff, regular and normal – are fundamental topological properties that can be used to distinguish among spaces. Linearly ordered spaces and metric spaces possess all four of these properties. Examples I.16 and I.17, presented below, describe spaces which satisfy one separation property but not another. These spaces are two of the more interesting examples in these notes.

Definition. The Separation Properties. Let X be a topological space.

a) X is a *T_1 space* if for all x, y ∈ X such that x ≠ y, there is a neighborhood U of x such that y ∉ U.

b) X is a *Hausdorff* or *T_2 space* if for all x, y ∈ X such that x ≠ y, there are neighborhoods U of x and V of y such that U ∩ V = ∅.

c) X is a *regular* or *T_3 space* if for every x ∈ X and for every closed subset C of X such that x ∉ C, there are neighborhoods U of x and V of C such that U ∩ V = ∅.

d) X is a *normal* or *T_4 space* if for all closed subsets C and D of X such that C ∩ D = ∅, there are neighborhoods U of C and V of D such that U ∩ V = ∅.

Theorem I.23. Let X be a topological space.

a) X is a T_1 space if and only if for every x ∈ X, { x } is a closed set.

b) X is a regular space if and only if for every x ∈ X and for every neighborhood U of x, there is a neighborhood V of x such that cl(V) ⊂ U.

c) X is a normal space if and only if for every closed subset C of X and for every neighborhood U of C, there is a neighborhood V of C such that cl(V) ⊂ U.

Proof. a) Assume X is a T_1 space. Let x ∈ X. Then for each y ∈ X – { x } , there is a neighborhood U_y of y such that x ∉ U_y. It follows that $\bigcup \{ U_y : y \in X - \{ x \} \}$ = X – { x }. Thus, X – { x } is a union of open sets. So X – { x } is an open set. We conclude that { x } is a closed set.

For the converse, assume that for each x ∈ X, { x } is a closed set. To prove that X is a T_1 space, let x and y be distinct points of X. Since { y } is a closed set, then X – { y } is a neighborhood of x that doesn't contain y. This proves X is a T_1 space.

b) Assume X is a regular space. Let x ∈ X and let U be a neighborhood of x. Set C = X – U. Then C is a closed set not containing the point x. Since X is a regular space, it follows that there are neighborhoods V of x and W of C such that V ∩ W = ∅. Since V ∩ W = ∅, then V ⊂ X – W. Since X – W is a closed set, it follows that cl(V) ⊂ X – W. Hence, cl(V) ∩ W = ∅. Since C ⊂ W, then cl(V) ∩ C = ∅. Thus, cl(V) ⊂ X – C =

U. We conclude that V is a neighborhood of x such that cl(V) $\subset$ U.

For the converse, assume that for every x $\in$ X and for every neighborhood U of x, there is a neighborhood V of x such that cl(V) $\subset$ U. To prove X is a regular space, let x $\in$ X and let C be a closed set not containing x. Then X – C is a neighborhood of x. Hence, there is a neighborhood V of x such that cl(V) $\subset$ X – C. Thus, C $\subset$ X – cl(V). Since V $\subset$ cl(V) and cl(V) is a closed set, then V and X – cl(V) are disjoint open sets. Furthermore, x $\in$ V and C $\subset$ X – cl(V). This proves X is regular.

c) Assume X is normal space. Let C be a closed subset of X and let U be a neighborhood of C. Set D = X – U. Then C and D are disjoint closed subsets of X. Since X is a normal space, it follows that there are neighborhoods V of C and W of D such that V $\cap$ W = $\varnothing$. Since V $\cap$ W = $\varnothing$, then V $\subset$ X – W. Since X – W is a closed set, it follows that cl(V) $\subset$ X – W. Hence, cl(V) $\cap$ W = $\varnothing$. Since D $\subset$ W, then cl(V) $\cap$ D = $\varnothing$. Thus, cl(V) $\subset$ X – D = U. We conclude that V is a neighborhood of C such that cl(V) $\subset$ U.

The following problem finishes this proof. ☐

Problem I.13. Prove Theorem I.23 c) $\Leftarrow$.

Theorem I.24. a) Every Hausdorff space is T_1.

b) Every regular T_1 space is Hausdorff.

c) Every normal T_1 space is regular.

Proof. a) Assume X is a Hausdorff space. If x and y are distinct points of X, then there are disjoint neighborhoods U of x and V of y. Hence, y $\notin$ U. This proves X is T_1.

b) Assume X is a regular T_1 space. To prove that X is Hausdorff, let x and y be distinct points of X. Then { y } is a closed subset of X by Theorem I.23 a). Since X is regular, it follows that x and { y } have disjoint neighborhoods. This proves X is Hausdorff.

c) Assume X is a normal T_1 space. To prove that X is regular, let x $\in$ X and let C be a closed set not containing x. Then { x } is a closed subset of X by Theorem I.23 a). Since X is normal, it follows that { x } and C have disjoint neighborhoods. This proves X is regular. ☐

Corollary. Every normal T_1 space is Hausdorff and regular.

Theorem I.25. Every metric space is T_1, Hausdorff, regular, and normal.

Proof. Since, according to Theorem I.24, a Hausdorff space is T_1 and a T_1 normal space is regular, then it suffices to prove that every metric space is Hausdorff and normal.

Let (X, ρ) be a metric space. To prove that X is Hausdorff, let x and y be distinct points of X. Set $\varepsilon = (^1/_2)\rho(x,y)$. $\varepsilon > 0$ because $x \neq y$. Set $U = N(x,\varepsilon)$ and $V = N(y,\varepsilon)$. Then U and V are neighborhoods of x and y, respectively. It remains to show that $U \cap V = \emptyset$. To this end, assume $U \cap V \neq \emptyset$. Then there is a point $z \in U \cap V$. Hence, $z \in N(x,\varepsilon)$ and $z \in N(y,\varepsilon)$. Therefore,

$$2\varepsilon \ = \ \rho(x,y) \ \leq \ \rho(x,z) + \rho(z,y) \ < \ \varepsilon + \varepsilon \ = \ 2\varepsilon.$$

Thus, $2\varepsilon < 2\varepsilon$. We have reached a contradiction. We conclude that $U \cap V = \emptyset$. This proves X is Hausdorff.

The following problem finishes this proof. □

Problem I.14. Prove that every metric space is normal.

Remark. Metric spaces satisfying a strong form of normality called complete normality. A space X is *completely normal* if for any two subsets A and B of X satisfying $cl(A) \cap B = \emptyset = A \cap cl(B)$, there are neighborhoods U of A and V of B such that $U \cap V = \emptyset$. The proof that every metric space is completely normal is assigned as an Additional Problem.

Exercise. Show that there are simple topological spaces having none of the separation properties by finding a topology on a three point set that is not T_1, not Hausdorff, not regular and not normal.

Next we explore the question of which of the separation properties are enjoyed by the spaces described in Examples I.1 through I.15.

We begin by observing that if A and B are disjoint subsets of a space X and if $B = \emptyset$, then X and $\emptyset$ are disjoint neighborhoods of A and B, respectively. Thus, when deciding whether a space is regular or normal, one need only consider *non-empty* closed sets. Armed with this observation, we note that, by default, a one-point space $X = \{ x \}$ has all the separation properties: T_1, Hausdorff, regular and normal, because X doesn't contain distinct points, X doesn't contain a point that is disjoint from a *non-empty* closed set, and X doesn't contain two disjoint *non-empty* closed sets.

If a space X has more than one point and is endowed with the indiscrete topology (Example I.1), then X is clearly not T_1 and not Hausdorff. However, X is regular and normal by default, because it doesn't contain a point that is disjoint from a non-empty closed set, and it doesn't contain two disjoint non-empty closed sets.

A space X with the discrete topology (Example I.2) enjoys all the separation properties because any two disjoint subsets of X, being open, are disjoint neighborhoods of themselves.

The space X = { x, y, z } described in Example I.3 is neither T_1 nor Hausdorff, because every neighborhood of the point z also contains the point x. This space is not regular because the point x and the closed set { y, z } don't have disjoint neighborhoods. However, X = { x, y, z } is normal. Indeed, the only pairs of non-empty disjoint closed sets in this space are { y }, { z } and { y }, { x, z }; and the pair { y }, { x, z } of disjoint open sets contains the sets of each of these pairs.

Problem I.15(4). Decide whether or under what conditions the set X endowed with the finite completement topology (described in Example I.4) is **a)** T_1, **b)** Hausdorff, **c)** regular, **d)** normal.

Since $\mathbb{R}$ and $\mathbb{R}^n$ (Examples I.5 and I.6) are metrizable, then Theorem I.25 implies that they enjoy all the separation properties.

Problem I.15(7). Decide whether the space $\mathbb{R}_{bad}$ (described in Example I.7) is **a)** T_1, **b)** Hausdorff, **c)** regular, **d)** normal.

The space X = ($\mathbb{N} \times \mathbb{N}$) ∪ { ∞ } described in Example I.8 is Hausdorff and normal (and, hence, T_1 and regular by Theorem I.24.) Recall that every subset of $\mathbb{N} \times \mathbb{N}$ is open in X. To verify the assertion that X is Hausdorff, first observe that if p = (m,n) ∈ $\mathbb{N} \times \mathbb{N}$ and we define the function f : $\mathbb{N} \to \mathbb{N}$ to be the constant function f(x) = n + 1, then X – { p } = (($\mathbb{N} \times \mathbb{N}$) – { p }) ∪ N(f). Since ($\mathbb{N} \times \mathbb{N}$) – { p } and N(f) are both open subsets of X, then we conclude that X – { p } is an open subset of X. Now suppose p and q are distinct points of X. We can assume p ≠ ∞. Then the preceding observation implies that { p } and X – { p } are disjoint neighborhoods of p and q, respectively. This proves X is a Hausdorff space. It is easier to verify the assertion that X is a normal space. Suppose A and B are disjoint closed subsets of X. We can assume ∞ ∉ A. Then A and X – A are disjoint neighborhoods of A and B, respectively, leading us to conclude that X is a normal space.

Next we consider three linearly ordered spaces: $[0, 1]^2$ with the lexicographic order described in Example I.9, and the well ordered spaces Ω and Ω^+ described in Examples I.10 and I.11 respectively. The status of these three spaces with respect to the separation properties is settled by the following result: all linearly ordered spaces are all Hausdorff and normal (and, hence, T_1 and regular by Theorem I.24). Here is a proof that every linearly ordered space is Hausdorff. Assume that (X, <) is a linearly ordered space and that x and y are distinct points of X. We can assume that x < y. We must consider two cases: either the open interval (x, y) is empty or non-empty. If

(x, y) ≠ ∅, choose $z \in$ (x, y). In this case (–∞, z) and (z, ∞) are disjoint neighborhoods of x and y, respectively. On the other hand, if (x, y) = ∅, then (–∞, y) and (x, ∞) are disjoint neighborhoods of x and y, respectively. Since x and y have disjoint neighborhoods in either case, we conclude that X is a Hausdorff space. The proof that linearly ordered spaces are normal is long and logically complicated, requiring the consideration of a complex tree of alternative cases. (We just saw that the proof that linearly ordered spaces are Hausdorff requires a bifurcation into two cases. The proof of normality is much more involved.) The proof of normality of linearly ordered spaces is assigned as an Additional Problem. We can give ad hoc arguments to settle the question of normality for $[0, 1]^2$, Ω and Ω^+ without appealing to the general result that all linearly ordered spaces are normal. For instance, information acquired in a later chapter will allow us to give straightforward proofs of the normality of $[0, 1]^2$ and Ω^+, because both these spaces are compact and Hausdorff. Also the solution to Problem I.15(7) which settles the normality of $\mathbb{R}_{bad}$ can be adapted to show that the well ordered spaces Ω and Ω^+ are normal. This is because well ordered spaces and $\mathbb{R}_{bad}$ have a common characteristic that is instrumental in settling the issue of their normality.

Exercise. Modify the solution to Problem I.15(7) settling the normality of $\mathbb{R}_{bad}$ into a proof that every well ordered linearly ordered space is normal.

The spaces described in Examples I.12 through I.15 – a set with the discrete metric, $\mathbb{R}$ with the standard metric, $\mathbb{R}^n$ with either the taxicab, Euclidean or supremum metric, and B(X) with the supremum metric – are all metrizable. Hence, they are T_1, Hausdorff, regular and normal by Theorem I.25.

Theorem I.24 establishes various logical connections between the separation properties. We could conjecture other relationships between the separation properties such as: every Hausdorff space is regular, and every regular space is normal. The following two examples illustrate the limitations on such conjectures.

Example I.16. Let $\mathscr{T}$ denote the standard topology on $\mathbb{R}$, and let $\mathbb{Q}$ denote the set of rational numbers. Set $\mathscr{B}_{\mathbb{Q}} = \{ \{x\} \cup (U \cap \mathbb{Q}) : x \in U \in \mathscr{T} \}$. Then $\mathscr{B}_{\mathbb{Q}}$ is a basis for a non-standard topology on $\mathbb{R}$ called the *rational topology* on $\mathbb{R}$.

Exercise. Verify that $\mathscr{B}_{\mathbb{Q}}$ is a basis for a topology on $\mathbb{R}$.

Observe that $\mathbb{R}$ with the rational topology is separable. Indeed, since every element of the basis $\mathscr{B}_{\mathbb{Q}}$ has non-empty intersection with $\mathbb{Q}$, then $\mathbb{Q}$ is a countable dense subset of this space. Also observe that for each $x \in \mathbb{R}$, the countable collection

$$\{ \{x\} \cup ((x - {}^1/_n, x + {}^1/_n) \cap \mathbb{Q}) : n \in \mathbb{N} \}$$

is a basis for the rational topology at x. Hence, $\mathbb{R}$ with the rational topology is first countable. We assert that $\mathbb{R}$ with the rational topology is not second countable. To prove this, suppose $\mathscr{B}$ is any basis for $\mathbb{R}$ with the rational topology. We will prove that $\mathscr{B}$ is uncountable. Since $\mathscr{B}$ is a basis for $\mathbb{R}$ with the rational topology, then for each irrational number $x \in \mathbb{R} - \mathbb{Q}$, there is a $B_x \in \mathscr{B}$ such that

$$x \in B_x \subset \{x\} \cup ((x - 1, x + 1) \cap \mathbb{Q}).$$

Observe that for each $x \in \mathbb{R} - \mathbb{Q}$, x is the only irrational number which is an element of the set B_x. Thus, if x and y are distinct elements of $\mathbb{R} - \mathbb{Q}$, then $B_x \neq B_y$. Therefore, the function $x \mapsto B_x : (\mathbb{R} - \mathbb{Q}) \to \mathscr{B}$ is injective. Since $\mathbb{R} - \mathbb{Q}$ is uncountable, the $\mathscr{B}$ must also be uncountable. This proves that $\mathbb{R}$ with the rational topology does not have a countable basis. We conclude that $\mathbb{R}$ with the rational topology is not second countable. Finally, since $\mathbb{R}$ with the rational topology is separable but not second countable, then $\mathbb{R}$ with the rational topology is not metrizable by Theorem I.13.

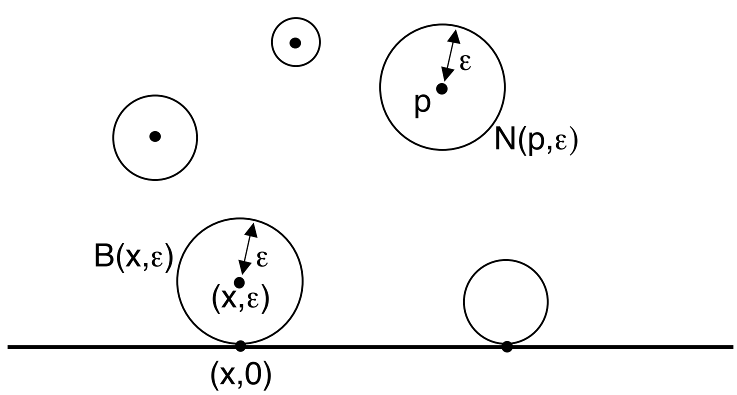
Problem I.15(16). Decide whether $\mathbb{R}$ with the rational topology (described in Example I.16) is **a)** T_1, **b)** Hausdorff, **c)** regular, **d)** normal.

Example I.17. Let $\mathbb{R}^2_+ = \mathbb{R} \times [0,\infty) = \{ (x,y) \in \mathbb{R}^2 : y \geq 0 \}$. Let ρ_2 denote the Euclidean metric on $\mathbb{R}^2$; and for $p \in \mathbb{R}^2$ and $\varepsilon > 0$, let $N(p,\varepsilon) = \{ q \in \mathbb{R}^2 : \rho_2(p,q) < \varepsilon \}$. Observe that if $(x,y) \in \mathbb{R}^2$ and $0 < \varepsilon \leq y$, then $N((x,y),\varepsilon) \subset \mathbb{R}^2_+$. For $x \in \mathbb{R}$ and $\varepsilon > 0$, set $B(x,\varepsilon) = \{ (x,0) \} \cup N((x,\varepsilon),\varepsilon)$. Then the collection

$$\{ N((x,y),\varepsilon) : (x,y) \in \mathbb{R}^2_+ \text{ and } 0 < \varepsilon \leq y \} \cup \{ B(x,\varepsilon) : x \in \mathbb{R} \text{ and } \varepsilon > 0 \}$$

is a basis for a (non-standard) topology on $\mathbb{R}^2_+$ called the *bubble topology* on $\mathbb{R}^2_+$.

Exercise. Verify that this collection is actually a basis for some topology on $\mathbb{R}^2_+$.



We observe the countable set D = { (x,y) ∈ $\mathbb{Q} \times \mathbb{Q}$: y > 0 } is a dense subset of $\mathbb{R}^2_+$ with the bubble topology. This is because D has non-empty intersection with each neighborhood of the form N((x,y),ε) where (x,y) ∈ $\mathbb{R}^2_+$ and 0 < ε ≤ y. [If we choose rational numbers a ∈ (x – ε/$\sqrt{2}$, x + ε/$\sqrt{2}$) and b ∈ (y – ε/$\sqrt{2}$, y + ε/$\sqrt{2}$), then (a,b) ∈ D ∩ N((x,y),ε).] Hence, D has non-empty intersection with every element of the given basis for $\mathbb{R}^2_+$ with the bubble topology. This makes D a countable dense subset of $\mathbb{R}^2_+$ with the bubble topology. It follows that $\mathbb{R}^2_+$ with the bubble topology is separable. For (x,y) ∈ $\mathbb{R}^2_+$ where y > 0, it is clear that the countable collection

$$\{ N((x,y),{}^1/_n) : n \in \mathbb{N} \text{ and } {}^1/_n \le y \}$$

is a basis for $\mathbb{R}^2_+$ with the bubble topology at (x,y). Also for x ∈ $\mathbb{R}$, it is clear that the countable collection { B(x,$^1/_n$) : n ∈ $\mathbb{N}$ } is a basis for $\mathbb{R}^2_+$ with the bubble topology at (x,0). Hence, $\mathbb{R}^2_+$ with the bubble topology is first countable. We assert that $\mathbb{R}^2_+$ with the bubble topology is not second countable. To prove this, suppose $\mathscr{B}$ is any basis for $\mathbb{R}^2_+$ with the bubble topology. We will prove that $\mathscr{B}$ is uncountable. Since $\mathscr{B}$ is a basis for $\mathbb{R}^2_+$ with the bubble topology, then for each x ∈ $\mathbb{R}$, there is a $B_x \in \mathscr{B}$ such that (x,0) ∈ B_x ⊂ B(x,1). Observe that for each x ∈ $\mathbb{R}$, (x,0) is the only point on the x-axis $\mathbb{R} \times \{ 0 \}$ which is an element of the set B_x. Thus, if x and y are distinct elements of $\mathbb{R}$, then $B_x \neq B_y$. Therefore, the function $x \mapsto B_x : \mathbb{R} \to \mathscr{B}$ is injective. Since $\mathbb{R}$ is uncountable, the $\mathscr{B}$ must also be uncountable. This proves that $\mathbb{R}^2_+$ with the bubble topology does not have a countable basis. We conclude that $\mathbb{R}^2_+$ with the bubble topology is not second countable. Finally, since $\mathbb{R}^2_+$ with the bubble topology is separable but not second countable, then $\mathbb{R}^2_+$ with the bubble topology is not metrizable by Theorem I.13.

Problem I.15(17). Decide whether $\mathbb{R}^2_+$ with the bubble topology (described in Example I.17) is **a)** T_1, **b)** Hausdorff, **c)** regular, **d)** normal.

The Additional Problems include two questions which might give the student further insight into $\mathbb{R}^2_+$ with the bubble topology.

G. Subspaces and Finite Product Spaces

There are a number of natural ways to generate new topological spaces from given ones. Two of the most fundamental of these are the formation of subspaces and finite products of spaces.

Definition. Let (X, $\mathscr{T}$) be a topological space, and let Y ⊂ X. Let

$$\mathscr{T}|Y = \{ U \cap Y : U \in \mathscr{T}\}.$$

Then $\mathscr{T}|Y$ is a topology on Y called the *subspace topology* or *relative topology* on Y. The topological space (Y, $\mathscr{T}|Y$) is called a *subspace* of (X, $\mathscr{T}$). The elements of $\mathscr{T}|Y$ are called (*relatively*) *open subsets of Y* and are said to be *open in Y.* If C ⊂ Y and Y – C ∈ $\mathscr{T}|Y$, then C is called a (*relatively*) *closed subset of Y* and is said to be *closed in Y.* **By convention,** a subset of a topological space is automatically assigned the subspace topology, unless otherwise specified.

Theorem I.26. Let (X, $\mathscr{T}$) be a topological space, and let Y ⊂ X.

a) $\mathscr{T}|Y = \{ U \cap Y : U \in \mathscr{T}\}$ is a topology on Y.

b) If $\mathscr{B}$ is a basis for $\mathscr{T}$, then $\mathscr{B}|Y = \{ B \cap Y : B \in \mathscr{B} \}$ is a basis for $\mathscr{T}|Y$.

c) If y ∈ Y and $\mathscr{B}_y$ is a basis for $\mathscr{T}$ at y, then $\mathscr{B}_y|Y = \{ B \cap Y : B \in \mathscr{B}_y \}$ is a basis for $\mathscr{T}|Y$ at y.

d) Let C ⊂ Y. C is a closed subset of Y if and only if there is a closed subset D of X such that C = D ∩ Y.

e) If Z ⊂ Y, then $\mathscr{T}|Z = (\mathscr{T}|Y)|Z$.

f) Suppose Y is an open subset of X. Let U ⊂ Y. Then U is an open subset of Y if and only if U is an open subset of X.

g) Suppose Y is a closed subset of X. Let C ⊂ Y. Then C is a closed subset of Y if and only if C is a closed subset of X.

h) If ρ is a metric on X which induces the topology $\mathscr{T}$, then $\rho|Y \times Y$ is a metric on Y which induces the topology $\mathscr{T}|Y$.

Proof. a) i) Since ∅ and X ∈ $\mathscr{T}$, then ∅ = ∅ ∩ Y and Y = X ∩ Y ∈ $\mathscr{T}|Y$.

ii) Assume $\mathscr{U} \subset \mathscr{T}|Y$. Then for each U ∈ $\mathscr{U}$, there is a $U^* \in \mathscr{T}$ such that $U = U^* \cap Y$. Hence, $\cup\mathscr{U} = \cup\{ U^* \cap Y : U \in \mathscr{U}\} = (\cup\{ U^* : U \in \mathscr{U}\}) \cap Y$, and $\cup\{ U^* : U \in \mathscr{U}\} \in \mathscr{T}$. Therefore, $\cup\mathscr{U} \in \mathscr{T}|Y$.

iii) Assume U and V $\in \mathscr{T}|Y$. Then $U = U^* \cap Y$ and $V = V^* \cap Y$ where U^* and $V^* \in \mathscr{T}$. Hence, $U \cap V = (U^* \cap Y) \cap (V^* \cap Y) = (U^* \cap V^*) \cap Y$, and $U^* \cap V^* \in \mathscr{T}$. Therefore, $U \cap V \in \mathscr{T}|Y$.

It follows that $\mathscr{T}|Y$ is a topology on Y.

b) Assume that $\mathscr{B}$ is a basis for $\mathscr{T}$.

i) Let $B \in \mathscr{B}|Y$. Then $B = B^* \cap Y$ where $B^* \in \mathscr{B}$. Since $\mathscr{B} \subset \mathscr{T}$, then $B^* \in \mathscr{T}$. Hence, $B = B^* \cap Y \in \mathscr{T}|Y$. This proves $\mathscr{B}|Y \subset \mathscr{T}|Y$.

ii) Let $y \in U \in \mathscr{T}|Y$. Then $U = U^* \cap Y$ where $U \in \mathscr{T}$. Hence, $y \in U^* \in \mathscr{T}$. Since $\mathscr{B}$ is a basis for $\mathscr{T}$, then there is a $B^* \in \mathscr{B}$ such that $y \in B^* \subset U^*$. Set $B = B^* \cap Y$. Then $y \in B \in \mathscr{B}|Y$ and $B = B^* \cap Y \subset U^* \cap Y = U$.

It follows that $\mathscr{B}|Y$ is a basis for $\mathscr{T}|Y$.

c) Assume $y \in Y$ and $\mathscr{B}_y$ is a basis for $\mathscr{T}$ at y.

i) Let $B \in \mathscr{B}_y|Y$. Then $B = B^* \cap Y$ where $B^* \in \mathscr{B}_y$. Since $\mathscr{B}_y \subset \mathscr{T}$, then $B^* \in \mathscr{T}$. Hence, $B = B^* \cap Y \in \mathscr{T}|Y$. This proves $\mathscr{B}_y|Y \subset \mathscr{T}|Y$.

ii) Let $B \in \mathscr{B}_y|Y$. Then $B = B^* \cap Y$ where $B^* \in \mathscr{B}_y$. Hence, $y \in B^*$ and $y \in Y$. Therefore, $y \in B^* \cap Y = B$.

iii) Let $y \in U \in \mathscr{T}|Y$. Then $U = U^* \cap Y$ where $U \in \mathscr{T}$. Hence, $y \in U^* \in \mathscr{T}$. Since $\mathscr{B}_y$ is a basis for $\mathscr{T}$ at y, then there is a $B^* \in \mathscr{B}_y$ such that $B^* \subset U^*$. Set $B = B^* \cap Y$. Then $B \in \mathscr{B}_y|Y$ and $B = B^* \cap Y \subset U^* \cap Y = U$.

It follows that $\mathscr{B}_y|Y$ is a basis for $\mathscr{T}|Y$ at y.

d) Let $C \subset Y$.

i) Assume C is a closed subset of Y. Then $Y - C \in \mathscr{T}|Y$. So $Y - C = U \cap Y$ where $U \in \mathscr{T}$. Set $D = X - U$. Then D is a closed subset of X, and

$$C = Y - (Y - C) = Y - (U \cap Y) = Y - U = (X - U) \cap Y = D \cap Y.$$

ii) Conversely, assume there is a closed subset D of X such that $C = D \cap Y$. Set $U = X - D$. Then $U \in \mathscr{T}$. So $U \cap Y \in \mathscr{T}|Y$. Also

$$C = D \cap Y = (X - U) \cap Y = Y - U = Y - (U \cap Y).$$

Hence, C is a closed subset of Y.

e) Let $Z \subset Y$.

i) Assume $U \in \mathscr{T}|Z$. Then $U = U^* \cap Z$ where $U^* \in \mathscr{T}$. Hence, $U^* \cap Y \in \mathscr{T}|Y$. Since $U = U^* \cap Z = U^* \cap (Y \cap Z) = (U^* \cap Y) \cap Z$, then it follows that $U \in (\mathscr{T}|Y)|Z$. This proves $\mathscr{T}|Z \subset (\mathscr{T}|Y)|Z$.

ii) Assume $U \in (\mathscr{T}|Y)|Z$. Then $U = U^* \cap Z$ where $U^* \in \mathscr{T}|Y$. Hence $U^* = U^{**} \cap Y$ where $U^{**} \in \mathscr{T}$. Since $U = U^* \cap Z = (U^{**} \cap Y) \cap Z = U^{**} \cap (Y \cap Z) = U^{**} \cap Z$, then $U \in \mathscr{T}|Z$. This proves $(\mathscr{T}|Y)|Z \subset \mathscr{T}|Z$.

We have shown that $\mathscr{T}|Z = (\mathscr{T}|Y)|Z$.

f) Assume Y is an open subset of X and $U \subset Y$.

i) Assume U is an open subset of Y. Then $U = U^* \cap Y$ where $U^* \in \mathscr{T}$. Since both U^* and Y are open subsets of X, then so is U.

ii) Conversely, assume U is an open subset of X. Since $U = U \cap Y$ and $U \in \mathscr{T}$, then U is an open subset of Y.

g) Assume Y is a closed subset of X and $C \subset Y$.

i) Assume C is a closed subset of Y. Then by part d) of this theorem, $C = D \cap Y$ where D is a closed subset of X. Since both D and Y are closed subsets of X, then so is C.

ii) Conversely, assume C is a closed subset of X. Since $C = C \cap Y$ and C is a closed subset of X, then part d) of this theorem implies that C is a closed subset of Y.

h) Assume ρ is a metric on X which induces the topology $\mathscr{T}$. Set $\sigma = \rho|Y \times Y$. Then for y, z and $x \in Y$:

i) $\sigma(y,z) = 0 \Leftrightarrow \rho(y,z) = 0 \Leftrightarrow y = z$,

ii) $\sigma(y,z) = \rho(y,z) = \rho(z,y) = \sigma(z,y)$, and

iii) $\sigma(y,z) = \rho(y,z) \le \rho(y,x) + \rho(x,z) = \sigma(y,x) + \sigma(x,z)$.

Hence, σ is a metric on Y.

For $y \in Y$ and $\varepsilon > 0$, set $N_\sigma(y,\varepsilon) = \{ z \in Y : \sigma(y,z) < \varepsilon \}$; and for $x \in X$ and $\varepsilon > 0$, set $N_\rho(x,\varepsilon) = \{ y \in X : \rho(x,y) < \varepsilon \}$. Set $\mathscr{B}_\sigma = \{ N_\sigma(y,\varepsilon) : y \in Y \text{ and } \varepsilon > 0 \}$. Then $\mathscr{B}_\sigma$ is a basis for the topology on Y induced by the metric σ.

We will now prove that $\mathscr{B}_\sigma$ is also a basis for $\mathscr{T}|Y$. To begin, observe that for each $y \in Y$ and $\varepsilon > 0$,

$$N_\sigma(y,\varepsilon) = \{ z \in Y : \sigma(y,z) < \varepsilon \} = \{ z \in Y : \rho(y,z) < \varepsilon \}$$

$$= \{ z \in X : \rho(y,z) < \varepsilon \} \cap Y = N_\rho(y,\varepsilon) \cap Y.$$

i) For each $y \in Y$ and $\varepsilon > 0$, since $N_\rho(x,\varepsilon) \in \mathscr{T}$ and $N_\sigma(y,\varepsilon) = N_\rho(y,\varepsilon) \cap Y$, then $N_\sigma(y,\varepsilon) \in \mathscr{T}|Y$. Hence, $\mathscr{B}_\sigma \subset \mathscr{T}|Y$.

ii) Let $y \in U \in \mathscr{T}|Y$. Then $U = U^* \cap Y$ where $U^* \in \mathscr{T}$. Since $y \in U^* \in \mathscr{T}$, then by Theorem I.10, $N_\rho(y,\varepsilon) \subset U^*$ for some $\varepsilon > 0$. Therefore,

$$y \in N_\sigma(y,\varepsilon) = N_\rho(y,\varepsilon) \cap Y \subset U^* \cap Y = U.$$

This completes the proof that $\mathscr{B}_\sigma$ is a basis for $\mathscr{T}|Y$.

Since $\mathscr{B}_\sigma$ is a basis for both the topology on Y induced by the metric σ and for $\mathscr{T}|Y$, then the Corollary to Theorem I.1 implies that these two topologies are equal; i.e., the topology on Y induced by the metric σ equals $\mathscr{T}|Y$. Therefore, $\sigma = \rho|Y \times Y$ induces the topology $\mathscr{T}|Y$ on Y. □

Next we consider the extent to which the topological properties we have been studying – separability, first and second countability, the separation properties and metrizability – are inherited by subspaces.

Theorem I.27. Let X be a topological space, and let Y be a subset of X with the subspace topology.

a) If X is second countable, then so is Y.

b) If X is first countable, then so is Y.

c) If X is T_1, then so is Y.

d) If X is Hausdorff, then so is Y.

e) If X is regular, then so is Y.

f) If X is metrizable, then so is Y.

Proof. a) Assume X is second countable. Then the topology on X has a countable basis $\mathscr{B}$. Theorem I.26 b) implies that $\mathscr{B}|Y = \{ B \cap Y : B \in \mathscr{B} \}$ is a basis for the subspace topology on Y. Since $\mathscr{B}$ is countable, then $\mathscr{B}|Y$ is also clearly countable. Hence, Y is second countable.

b) Assume X is first countable. To prove that Y is first countable, let $y \in Y$. Then the topology on X has a countable basis $\mathscr{B}_y$ at y. Theorem I.26 c) implies that $\mathscr{B}_y|Y = \{ B \cap Y : B \in \mathscr{B}_y \}$ is a basis for the subspace topology on Y at y. Since $\mathscr{B}_y$ is countable, then $\mathscr{B}_y|Y$ is also clearly countable. This proves Y is first countable.

c) Assume X is T_1. To prove Y is T_1, let $y, z \in Y$ such that $y \neq z$. Since X is T_1, there is an open subset U of X such that $y \in U$ and $z \notin U$. Therefore, $U \cap Y$ is an open subset of Y such that $y \in U \cap Y$ and $z \notin U \cap Y$. This proves Y is T_1.

d) Assume X is Hausdorff. To prove Y is Hausdorff, let $y, z \in Y$ such that $y \neq z$. Since X is Hausdorff, there are disjoint open subsets U and V of X such that $y \in U$ and $z \in V$. Therefore, $U \cap Y$ and $V \cap Y$ are disjoint open subsets of Y such that $y \in U \cap Y$ and $z \in V \cap Y$. This proves Y is Hausdorff.

e) Assume X is regular. To prove Y is regular, let $y \in Y$ and let C be a closed subset of Y such that $y \notin C$. Then by Theorem I.26.d), $C = D \cap Y$ where D is a closed subset of X. Since $y \in Y$ and $y \notin C$, then $y \notin D$. Since X is regular, it follows that there are disjoint open subsets U and V of X such that $y \in U$ and $D \subset V$. Then $U \cap Y$ and $V \cap Y$ are disjoint open subsets of Y such that $y \in U \cap Y$ and $C = D \cap Y \subset V \cap Y$. This proves Y is regular.

f) Assume X is metrizable. Then there is a metric ρ on X that induces the topology on X. Therefore, Theorem I.26 h) implies that $\rho|Y \times Y$ is a metric on Y that induces the subspace topology on Y. Hence, Y is metrizable. ◻

Observe that Theorem I.27 does not assert that the properties of separability and normality are inherited by subspaces. Indeed, these properties are not inherited by subspaces, as the following problem and remark reveal.

Problem I.16. Among the spaces described in Examples I.1 through I.17, find a separable space which contains a non-separable subspace.

Remark. Problem I.21 below provides an example of a normal space which contains a non-normal subspace.

Corollary I.28. Every subspace of a separable metrizable space is separable.

Problem I.17. Prove Corollary I.28.

Exercise. Let (X, <) be a linearly ordered set, let $Y \subset X$, and let $<_Y$ denote the restriction of < to Y (In other words, if we regard < as a subset of $X \times X$, then $<_Y = < \cap (Y \times Y)$.) In general, the order topology on Y determined by $<_Y$ does not coincide with the subspace topology on Y. Find a subset of $\mathbb{R}$ which illustrates this phenomenon.

Definition. Let $X_1, X_2, \ldots, X_n$ be topological spaces. The *Cartesian product* of $X_1, X_2, \ldots, X_n$ is the set of all *n-tuples* $(x_1, x_2, \ldots, x_n)$ such that $x_i \in X_i$ for $1 \le i \le n$. It is denoted $X_1 \times X_2 \times \ldots \times X_n$. Thus,

$$X_1 \times X_2 \times \ldots \times X_n = \{ (x_1, x_2, \ldots, x_n) : x_i \in X_i \text{ for } 1 \le i \le n \}.$$

An *open box* in $X_1 \times X_2 \times \ldots \times X_n$ is an subset of $X_1 \times X_2 \times \ldots \times X_n$ of the form $U_1 \times U_2 \times \ldots \times U_n$ where U_i is an open subset of X_i for $1 \le i \le n$. Observe that if $U_i, V_i \subset X_i$ for $1 \le i \le n$, then

$$(U_1 \times U_2 \times \ldots \times U_n) \cap (V_1 \times V_2 \times \ldots \times V_n) = (U_1 \cap V_1) \times (U_2 \cap V_2) \times \ldots \times (U_n \cap V_n).$$

Since the intersection of two open subsets of X_i is an open subset of X_i for $1 \le i \le n$, then it follows that the intersection of two open boxes in $X_1 \times X_2 \times \ldots \times X_n$ is an open box in $X_1 \times X_2 \times \ldots \times X_n$. Hence, the Corollary to Theorem I.2 implies that the set of all open boxes in $X_1 \times X_2 \times \ldots \times X_n$ is a basis for a topology on $X_1 \times X_2 \times \ldots \times X_n$. This topology is called the *product topology* on $X_1 \times X_2 \times \ldots \times X_n$.

Theorem I.29. Let $(X_1, \mathscr{T}_1), (X_2, \mathscr{T}_2), \ldots, (X_n, \mathscr{T}_n)$ be topological spaces, and let $\mathscr{T}$ denote the product topology on $X_1 \times X_2 \times \ldots \times X_n$.

a) If $\mathscr{B}_i$ is a basis for $\mathscr{T}_i$ for $1 \le i \le n$, then $\{ B_1 \times B_2 \times \ldots \times B_n : B_i \in \mathscr{B}_i \text{ for } 1 \le i \le n \}$ is a basis for $\mathscr{T}$.

b) If $x_i \in X_i$ and $\mathscr{B}_i$ is a basis for $\mathscr{T}_i$ at x_i for $1 \le i \le n$, then $\{ B_1 \times B_2 \times \ldots \times B_n : B_i \in \mathscr{B}_i \text{ for } 1 \le i \le n \}$ is a basis for $\mathscr{T}$ at the point $(x_1, x_2, \ldots, x_n)$.

c) If C_i is a closed subset of X_i for $1 \le i \le n$, then $C_1 \times C_2 \times \ldots \times C_n$ is a closed subset of $X_1 \times X_2 \times \ldots \times X_n$.

Proof. a) Assume $\mathscr{B}_i$ is a basis for $\mathscr{T}_i$ for $1 \le i \le n$, and set

$$\mathscr{B} = \{ B_1 \times B_2 \times \ldots \times B_n : B_i \in \mathscr{B}_i \text{ for } 1 \le i \le n \}.$$

We must prove $\mathscr{B}$ is a basis for the product topology $\mathscr{T}$.

Since each element of $\mathscr{B}$ is an open box and each open box is an element of the product topology $\mathscr{T}$, then $\mathscr{B} \subset \mathscr{T}$.

Suppose $x = (x_1, x_2, \ldots, x_n) \in U \in \mathscr{T}$. Since the collection of all open boxes is a basis for $\mathscr{T}$, then there is an open box $V_1 \times V_2 \times \ldots \times V_n$ such that $(x_1, x_2, \ldots, x_n) \in V_1 \times V_2 \times \ldots \times V_n \subset U$. Therefore, $x_i \in V_i \in \mathscr{T}_i$ for $1 \le i \le n$. For $1 \le i \le n$, since $\mathscr{B}_i$ is a basis for $\mathscr{T}_i$, then there is a $B_i \in \mathscr{B}_i$ such that $x_i \in B_i \subset V_i$. It follows that $B_1 \times B_2 \times \ldots \times B_n \in \mathscr{B}$ and $x = (x_1, x_2, \ldots , x_n) \in B_1 \times B_2 \times \ldots \times B_n \subset V_1 \times V_2 \times \ldots \times V_n \subset U$.

This completes the proof that $\mathscr{B}$ is a basis for $\mathscr{T}$.

b) Assume $x_i \in X_i$ and $\mathscr{B}_i$ is a basis for $\mathscr{T}_i$ at x_i for $1 \leq i \leq n$, and set

$$\mathscr{B} = \{ B_1 \times B_2 \times \ldots \times B_n : B_i \in \mathscr{B}_i \text{ for } 1 \leq i \leq n \}.$$

We must prove that $\mathscr{B}$ is a basis for the product topology $\mathscr{T}$ at the point $x = (x_1, x_2, \ldots , x_n)$.

Since each element of $\mathscr{B}$ is an open box and each open box is an element of the product topology $\mathscr{T}$, then $\mathscr{B} \subset \mathscr{T}$.

Let $B_1 \times B_2 \times \ldots \times B_n \in \mathscr{B}$. For $1 \leq i \leq n$, since $B_i \in \mathscr{B}_i$ and $\mathscr{B}_i$ is a basis for $\mathscr{T}_i$ at x_i, then $x_i \in B_i$. Hence, $x = (x_1, x_2, \ldots , x_n) \in B_1 \times B_2 \times \ldots \times B_n$.

Suppose $x = (x_1, x_2, \ldots , x_n) \in U \in \mathscr{T}$. Since the collection of all open boxes is a basis for $\mathscr{T}$, then there is an open box $V_1 \times V_2 \times \ldots \times V_n$ such that $(x_1, x_2, \ldots , x_n) \in V_1 \times V_2 \times \ldots \times V_n \subset U$. Therefore, $x_i \in V_i \in \mathscr{T}_i$ for $1 \leq i \leq n$. For $1 \leq i \leq n$, since $\mathscr{B}_i$ is a basis for $\mathscr{T}_i$ at x_i, then there is a $B_i \in \mathscr{B}_i$ such that $x_i \in B_i \subset V_i$. It follows that $B_1 \times B_2 \times \ldots \times B_n \in \mathscr{B}$ and $x = (x_1, x_2, \ldots , x_n) \in B_1 \times B_2 \times \ldots \times B_n \subset V_1 \times V_2 \times \ldots \times V_n \subset U$.

This completes the proof that $\mathscr{B}$ is a basis for the product topology $\mathscr{T}$ at the point $x = (x_1, x_2, \ldots , x_n)$.

c) Assume C_i is a closed subset of X_i for $1 \leq i \leq n$. For $1 \leq i \leq n$, set

$$U_i = X_1 \times \cdots \times X_{i-1} \times (X_i - C_i) \times X_{i+1} \times \cdots \times X_n.$$

Then each U_i is an open box and, hence, an open subset of $X_1 \times X_2 \times \ldots \times X_n$. Furthermore, for $1 \leq i \leq n$, $x = (x_1, x_2, \ldots , x_n) \in U_i$ if and only if $x_i \notin C_i$. Hence, a point $x = (x_1, x_2, \ldots , x_n)$ lies in the complement of $C_1 \times C_2 \times \ldots \times C_n$ if and only if $x_i \notin C_i$ for some i between 1 and n if and only if $x \in U_i$ for some i between 1 and n. Therefore,

$$(X_1 \times X_2 \times \ldots \times X_n) - (C_1 \times C_2 \times \ldots \times C_n) = U_1 \cup U_2 \cup \ldots \cup U_n.$$

Thus, the complement of $C_1 \times C_2 \times \ldots \times C_n$ is an open subset of $X_1 \times X_2 \times \ldots \times X_n$. Hence, $C_1 \times C_2 \times \ldots \times C_n$ is a closed subset of $X_1 \times X_2 \times \ldots \times X_n$. □

Exercise. Let $\mathbb{R}$ have the standard topology. Using Theorem I.29. a), observe that for $n \geq 1$, the product topology on $\mathbb{R}^n$ is the standard topology.

Now we consider the issue of whether the topological properties we have been studying – separability, first and second countability, the separation properties and metrizability – can pass from a finite collection of spaces to their Cartesian product. We also consider whether these properties pass in the reverse direction – from the Cartesian product of finitely many spaces to the individual factor spaces. To explore these issues, it is convenient to introduce the following terminology and prove a lemma.

Definition. Let $X_1, X_2, \dots, X_n$ be topological spaces. For $1 \le i \le n$, define the *i^{th} projection function*

$$\pi_i : X_1 \times X_2 \times \dots \times X_n \to X_i$$

by the equation $\pi_i(x_1, x_2, \dots, x_n) = x_i$ for each $(x_1, x_2, \dots, x_n) \in X_1 \times X_2 \times \dots \times X_n$. For each $\mathbf{a} = (a_1, a_2, \dots, a_n) \in X_1 \times X_2 \times \dots \times X_n$ and for $1 \le i \le n$, define the *i^{th} injection function*

$$e_{\mathbf{a},i} : X_i \to X_1 \times X_2 \times \dots \times X_n$$

by the equation $e_{\mathbf{a},i}(x) = (a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)$ for each $x \in X_i$.

Lemma I.30. Let $X_1, X_2, \dots, X_n$ be topological spaces, let $X_1 \times X_2 \times \dots \times X_n$ have the product topology, and let $\mathbf{a} = (a_1, a_2, \dots, a_n) \in X_1 \times X_2 \times \dots \times X_n$.

a) $\pi_i \circ e_{\mathbf{a},i} = id_{X_i}$ for $1 \le i \le n$.

b) For $1 \le i \le n$, if V is an open subset of X_i, then $\pi_i^{-1}(V)$ is an open subset of $X_1 \times X_2 \times \dots \times X_n$; and if D is a closed subset of X_i, then $\pi_i^{-1}(D)$ is a closed subset of $X_1 \times X_2 \times \dots \times X_n$.

c) For $1 \le i \le n$, if U is an open subset of $X_1 \times X_2 \times \dots \times X_n$, then $e_{\mathbf{a},i}^{-1}(U)$ is an open subset of X_i; and if C is a closed subset of $X_1 \times X_2 \times \dots \times X_n$, then $e_{\mathbf{a},i}^{-1}(C)$ is a closed subset of X_i.

Remark. When continuity is defined in Chapter II, we will see that assertions b) and c) are equivalent to the statements that $\pi_i : X_1 \times X_2 \times \dots \times X_n \to X_i$ and $e_{\mathbf{a},i} : X_i \to X_1 \times X_2 \times \dots \times X_n$ are continuous functions.

Proof. a) Let $1 \le i \le n$. For $x \in X_i$, $\pi_i \circ e_{\mathbf{a},i}(x) = \pi_i(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n) = x$. Thus, $\pi_i \circ e_{\mathbf{a},i} = id_{X_i}$.

b) Let $1 \le i \le n$.

Suppose V is an open subset of X_i. Then

$$\pi_i^{-1}(V) = X_1 \times \dots \times X_{i-1} \times V \times X_{i+1} \times \dots \times X_n.$$

Hence, $\pi_i^{-1}(V)$ is an open box. Therefore, $\pi_i^{-1}(V)$ is an open subset of $X_1 \times X_2 \times \dots \times X_n$.

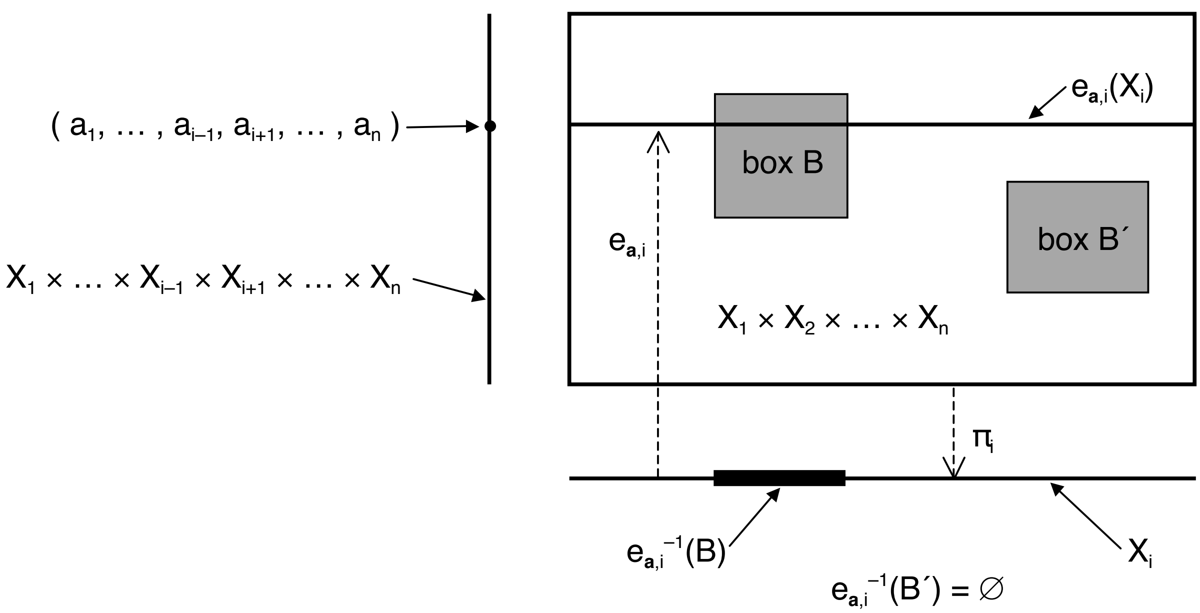
Next suppose D is a closed subset of X_i. Let $V = X_i - D$. Then V is an open subset of X_i, and $D = X_i - V$. Hence, $\pi_i^{-1}(V)$ is an open subset of $X_1 \times X_2 \times \ldots \times X_n$. Furthermore, (by Theorem 0.10.d)

$$\pi_i^{-1}(D) = \pi_i^{-1}(X_i) - \pi_i^{-1}(V) = (X_1 \times X_2 \times \ldots \times X_n) - \pi_i^{-1}(V).$$

Thus, $\pi_i^{-1}(D)$ is a closed subset of $X_1 \times X_2 \times \ldots \times X_n$.

c) Let $1 \le i \le n$.

First we show that if B is an open box in $X_1 \times X_2 \times \ldots \times X_n$, then $e_{\mathbf{a},i}^{-1}(B)$ is an open subset of X_i. Suppose $B = V_1 \times V_2 \times \ldots \times V_n$ is an open box in $X_1 \times X_2 \times \ldots \times X_n$. For $x \in X_i$, since $e_{\mathbf{a},i}(x) = (a_1, \ldots , a_{i-1}, x, a_{i+1}, \ldots , a_n)$, then $e_{\mathbf{a},i}(x) \in V_1 \times V_2 \times \ldots \times V_n = B$ if and only if $a_j \in V_j$ for all $j \neq i$ and $x \in V_i$. Therefore, $x \in e_{\mathbf{a},i}^{-1}(B)$ if and only if $a_j \in V_j$ for all $j \neq i$ and $x \in V_i$. Hence, $e_{\mathbf{a},i}^{-1}(B) = V_i$ if $a_j \in V_j$ for all $j \neq i$, and $e_{\mathbf{a},i}^{-1}(B) = \varnothing$ if $a_j \notin V_j$ for some $j \neq i$. Consequently, either $e_{\mathbf{a},i}^{-1}(B) = V_i$ or $e_{\mathbf{a},i}^{-1}(B) = \varnothing$. This proves $e_{\mathbf{a},i}^{-1}(B)$ is an open subset of X_i.



Now suppose U is an arbitrary open subset of $X_1 \times X_2 \times \ldots \times X_n$. Since the open boxes in $X_1 \times X_2 \times \ldots \times X_n$ form a basis for the product topology, then Theorem I.1 implies that U can be expressed as a union of open boxes. Thus, there is a collection $\mathscr{C}$ of open boxes in $X_1 \times X_2 \times \ldots \times X_n$ such that $U = \bigcup\mathscr{C}$. Then (by Theorem 0.10.b)

$$e_{\mathbf{a},i}^{-1}(U) = \bigcup \{ e_{\mathbf{a},i}^{-1}(B) : B \in \mathscr{C} \}.$$

Since each $B \in \mathscr{C}$ is an open box, then $e_{\mathbf{a},i}^{-1}(B)$ is an open subset of X_i for each $B \in \mathscr{C}$. Hence, $e_{\mathbf{a},i}^{-1}(U)$ is an open subset of X_i.

Finally, suppose C is a closed subset of $X_1 \times X_2 \times \ldots \times X_n$. Let U = $(X_1 \times X_2 \times \ldots \times X_n) - C$. Then U is an open subset of $X_1 \times X_2 \times \ldots \times X_n$, and C = $(X_1 \times X_2 \times \ldots \times X_n) - U$. Hence, $e_{\mathbf{a},i}^{-1}(U)$ is an open subset of X_i. Furthermore, (by Theorem 0.10.d) $e_{\mathbf{a},i}^{-1}(C) = e_{\mathbf{a},i}^{-1}(X_1 \times X_2 \times \ldots \times X_n) - e_{\mathbf{a},i}^{-1}(U) = X_i - e_{\mathbf{a},i}^{-1}(U)$. Thus, $e_{\mathbf{a},i}^{-1}(C)$ is a closed subset of X_i. ☐

Next we state and prove a theorem which tells us the extent to which the topological properties we have been studying pass between a finite collection of spaces and their Cartesian product.

Theorem I.31. Let $X_1, X_2, \ldots , X_n$ be topological spaces, and let $X_1 \times X_2 \times \ldots \times X_n$ have the product topology. Then:

a) $X_1 \times X_2 \times \ldots \times X_n$ is second countable if and only if each of $X_1, X_2, \ldots , X_n$ is second countable.

b) $X_1 \times X_2 \times \ldots \times X_n$ is first countable if and only if each of $X_1, X_2, \ldots , X_n$ is first countable.

c) $X_1 \times X_2 \times \ldots \times X_n$ is separable if and only if each of $X_1, X_2, \ldots , X_n$ is separable.

d) $X_1 \times X_2 \times \ldots \times X_n$ is T_1 if and only if each of $X_1, X_2, \ldots , X_n$ is T_1.

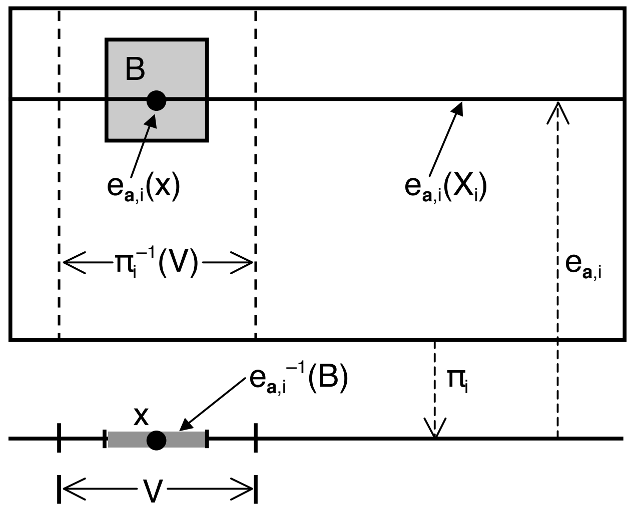
e) $X_1 \times X_2 \times \ldots \times X_n$ is Hausdorff if and only if each of $X_1, X_2, \ldots , X_n$ is Hausdorff.

f) $X_1 \times X_2 \times \ldots \times X_n$ is regular if and only if each of $X_1, X_2, \ldots , X_n$ is regular.

g) If $X_1 \times X_2 \times \ldots \times X_n$ is normal, then each of $X_1, X_2, \ldots , X_n$ is normal.

Proof. For $1 \le i \le n$, let $\mathscr{T}_i$ denote the topology on X_i and let $\mathscr{T}$ denote the product topology on $X_1 \times X_2 \times \ldots \times X_n$. Also choose $\mathbf{a} = (a_1, a_2, \ldots , a_n) \in X_1 \times X_2 \times \ldots \times X_n$. We now prove the various parts of Theorem I.31.

Proof of a) ⇒. Assume that $X_1 \times X_2 \times \ldots \times X_n$ is second countable. Then the product topology $\mathscr{T}$ has a countable basis $\mathscr{B}$. Let $1 \le i \le n$. We will prove that X_i is second countable by showing that the countable set $\{ e_{\mathbf{a},i}^{-1}(B) : B \in \mathscr{B} \}$ is a basis for the topology $\mathscr{T}_i$ on X_i. Since $\mathscr{B} \subset \mathscr{T}$, then Lemma I.30.c implies that $e_{\mathbf{a},i}^{-1}(B) \in \mathscr{T}_i$ for each $B \in \mathscr{B}$. Hence, $\{ e_{\mathbf{a},i}^{-1}(B) : B \in \mathscr{B} \} \subset \mathscr{T}_i$. Now let $x \in V \in \mathscr{T}_i$. Then $\pi_i^{-1}(V) \in \mathscr{T}$ by Lemma 1.30.c. Also, $e_{\mathbf{a},i}(x) \in \pi_i^{-1}(V)$. (**Proof:** $\pi_i \circ e_{\mathbf{a},i} = id_{X_i}$ (by Lemma I.30.a) $\Rightarrow$ $\pi_i(e_{\mathbf{a},i}(x)) = x \in V$ $\Rightarrow$ $e_{\mathbf{a},i}(x) \in \pi_i^{-1}(V)$.) Since $\mathscr{B}$ is a basis for $\mathscr{T}$, it follows that there is a $B \in \mathscr{B}$ such that $e_{\mathbf{a},i}(x) \in B \subset \pi_i^{-1}(V)$. Hence, $x \in e_{\mathbf{a},i}^{-1}(B)$. Furthermore, $e_{\mathbf{a},i}^{-1}(B) \subset V$. (**Proof:** $B \subset \pi_i^{-1}(V)$ $\Rightarrow$ $e_{\mathbf{a},i}^{-1}(B) \subset e_{\mathbf{a},i}^{-1}(\pi_i^{-1}(V)) = (\pi_i \circ e_{\mathbf{a},i})^{-1}(V) = (id_{X_i})^{-1}(V) = V$.) Thus, $x \in e_{\mathbf{a},i}^{-1}(B) \subset V$. This completes the proof that $\{ e_{\mathbf{a},i}^{-1}(B) : B \in \mathscr{B} \}$ is a basis for $\mathscr{T}_i$. We conclude that X_i is second countable. (See the figure below.)



Proof of a) ⇐. Assume that X_i is second countable for each i between 1 and n. Then for $1 \le i \le n$, the topology $\mathscr{T}_i$ on X_i has a countable basis $\mathscr{B}_i$. Since $\mathscr{B}_1, \mathscr{B}_2, \cdots, \mathscr{B}_n$ are all countable sets, then their Cartesian product $\mathscr{B}_1 \times \mathscr{B}_2 \times \ldots \times \mathscr{B}_n$ is also a countable set (according to Theorem 0.17). Hence, the collection

$$\mathscr{B} = \{ B_1 \times B_2 \times \cdots \times B_n : (B_1, B_2, \cdots, B_n) \in \mathscr{B}_1 \times \mathscr{B}_2 \times \ldots \times \mathscr{B}_n \}$$

is countable as well. According to Theorem I.27.a, the collection $\mathscr{B}$ is a basis for the product topology $\mathscr{T}$ on $X_1 \times X_2 \times \ldots \times X_n$. Thus, $\mathscr{T}$ has a countable basis. We conclude that $X_1 \times X_2 \times \ldots \times X_n$ is second countable.

Proof of b) ⇒. Assume that $X_1 \times X_2 \times \ldots \times X_n$ is first countable. Let $1 \le i \le n$ and let $x \in X_i$. We will show that $\mathscr{T}_i$ has a countable basis at x. Let $y = e_{\mathbf{a},i}(x)$. Since $y \in X_1 \times X_2 \times \ldots \times X_n$ and $X_1 \times X_2 \times \ldots \times X_n$ is first countable, then $\mathscr{T}$ has a countable basis $\mathscr{B}_y$ at y. We will prove that the countable set $\{ e_{\mathbf{a},i}^{-1}(B) : B \in \mathscr{B}_y \}$ is a basis for $\mathscr{T}_i$ at x. The proof is essentially the same as the proof of a) ⇒. First since $\mathscr{B}_y \subset \mathscr{T}$, then Lemma I.30.c implies that $e_{\mathbf{a},i}^{-1}(B) \in \mathscr{T}_i$ for each $B \in \mathscr{B}_y$. Hence, $\{ e_{\mathbf{a},i}^{-1}(B) : B \in \mathscr{B}_y \} \subset \mathscr{T}_i$. Second, for each $B \in \mathscr{B}_y$, since $e_{\mathbf{a},i}(x) = y \in B$, then $x \in e_{\mathbf{a},i}^{-1}(B)$. Third and last, let $x \in V \in \mathscr{T}_i$. Then, as in the proof of a) ⇒, $\pi_i^{-1}(V) \in \mathscr{T}$ and $y = e_{\mathbf{a},i}(x) \in \pi_i^{-1}(V)$. Since $\mathscr{B}_y$ is a basis for $\mathscr{T}$ at y, it follows that there is a $B \in \mathscr{B}$ such that $e_{\mathbf{a},i}(x) = y \in B \subset \pi_i^{-1}(V)$. Therefore, as in the proof of a) ⇒, $x \in e_{\mathbf{a},i}^{-1}(B)$ and $e_{\mathbf{a},i}^{-1}(B) \subset V$. This completes the proof that $\{ e_i^{-1}(B) : B \in \mathscr{B}_y \}$ is a countable basis for $\mathscr{T}_i$ at x. We have shown that $\mathscr{T}_i$ has a countable basis at every point of X_i. We conclude that X_i is first countable.

Proof of b) ⇐. Assume that X_i is first countable for each i between 1 and n. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \ldots \times X_n$. We will show that $\mathscr{T}$ has a countable basis at $\mathbf{x}$.

For $1 \le i \le n$, since X_i is first countable, then $\mathscr{T}_i$ has a countable basis $\mathscr{B}_i$ at x_i. Since the sets $\mathscr{B}_1, \mathscr{B}_2, \dots, \mathscr{B}_n$ are all countable, then their Cartesian product $\mathscr{B}_1 \times \mathscr{B}_2 \times \dots \times \mathscr{B}_n$ is also countable (according to Theorem 0.17). Hence, the collection

$$\mathscr{B} = \{ B_1 \times B_2 \times \dots \times B_n : (B_1, B_2, \dots , B_n) \in \mathscr{B}_1 \times \mathscr{B}_2 \times \dots \times \mathscr{B}_n \}$$

is countable as well. According to Theorem I.29.b, the collection $\mathscr{B}$ is a basis for $\mathscr{T}$ at $\mathbf{x}$. Thus, $\mathscr{T}$ has a countable basis at $\mathbf{x}$. We have shown that $\mathscr{T}$ has a countable basis at every point of $X_1 \times X_2 \times \dots \times X_n$. We conclude that $X_1 \times X_2 \times \dots \times X_n$ is first countable.

Proof of d) ⇒. Assume $X_1 \times X_2 \times \dots \times X_n$ is T_1. Let $1 \le i \le n$ and let x and y be distinct points of X_i. Then

$$e_{\mathbf{a},i}(x) = (a_1, \cdots , a_{i-1}, x, a_{i+1}, \cdots , a_n) \quad \text{and} \quad e_{\mathbf{a},i}(y) = (a_1, \cdots , a_{i-1}, y, a_{i+1}, \cdots , a_n)$$

are distinct points of $X_1 \times X_2 \times \dots \times X_n$. Since $X_1 \times X_2 \times \dots \times X_n$ is T_1, there is an open subset U of $X_1 \times X_2 \times \dots \times X_n$ such that $e_{\mathbf{a},i}(x) \in U$ and $e_i(y) \notin U$. Then $x \in e_{\mathbf{a},i}{}^{-1}(U)$ and $y \notin e_{\mathbf{a},i}{}^{-1}(U)$. Furthermore, Lemma I.30.c implies that $e_{\mathbf{a},i}{}^{-1}(U)$ is an open subset of X_i. This proves X_i is T_1.

Proof of d) ⇐. Assume that X_i is T_1 for each i between 1 and n. Let $\mathbf{x} = (x_1, x_2, \dots , x_n) \in X_1 \times X_2 \times \dots \times X_n$. Then $\{ x_i \}$ is a closed subset of X_i for $1 \le i \le n$, by Theorem I.23.a. Observe that $\{ \mathbf{x} \} = \{ (x_1, x_2, \dots , x_n) \} = \{ x_1 \} \times \{ x_2 \} \times \dots \times \{ x_n \}$. Therefore, Theorem I.29.c implies that $\{ \mathbf{x} \}$ is a closed subset of $X_1 \times X_2 \times \dots \times X_n$. According to Theorem I.23.a, this proves $X_1 \times X_2 \times \dots \times X_n$ is T_1.

Proof of e) ⇒. Assume $X_1 \times X_2 \times \dots \times X_n$ is Hausdorff. Let $1 \le i \le n$ and let x and y be distinct points of X_i. Then

$$e_{\mathbf{a},i}(x) = (a_1, \cdots , a_{i-1}, x, a_{i+1}, \cdots , a_n) \quad \text{and} \quad e_{\mathbf{a},i}(y) = (a_1, \cdots , a_{i-1}, y, a_{i+1}, \cdots , a_n)$$

are distinct points of $X_1 \times X_2 \times \dots \times X_n$. Since $X_1 \times X_2 \times \dots \times X_n$ is Hausdorff, there are disjoint open subsets U and V of $X_1 \times X_2 \times \dots \times X_n$ such that $e_{\mathbf{a},i}(x) \in U$ and $e_{\mathbf{a},i}(y) \in V$. Then $x \in e_{\mathbf{a},i}{}^{-1}(U)$, $y \in e_{\mathbf{a},i}{}^{-1}(V)$, and (by Theorem 0.10.c)

$$e_{\mathbf{a},i}{}^{-1}(U) \cap e_{\mathbf{a},i}{}^{-1}(V) = e_{\mathbf{a},i}{}^{-1}(U \cap V) = e_{\mathbf{a},i}{}^{-1}(\varnothing) = \varnothing.$$

Furthermore, Lemma I.30.c implies that $e_{\mathbf{a},i}{}^{-1}(U)$ and $e_{\mathbf{a},i}{}^{-1}(V)$ are open subsets of X_i. This proves X_i is Hausdorff.

Proof of e) ⇐. Assume that X_i is Hausdorff for each i between 1 and n. Let $\mathbf{x} = (x_1, x_2, \dots , x_n)$ and $\mathbf{y} = (y_1, y_2, \dots , y_n)$ be distinct points of $X_1 \times X_2 \times \dots \times X_n$. Since $\mathbf{x} \ne \mathbf{y}$, then $x_i \ne y_i$ for some i between 1 and n. Since X_i is Hausdorff and x_i and y_i are distinct points of X_i, then there are disjoint open subsets U and V of X_i such that $x_i \in U$ and $y_i \in V$. Since $\pi_i(\mathbf{x}) = x_i$ and $\pi_i(\mathbf{y}) = y_i$, then $\pi_i(\mathbf{x}) \in U$ and $\pi_i(\mathbf{y}) \in V$. Therefore, $\mathbf{x} \in \pi_i^{-1}(U)$ and $\mathbf{y} \in \pi_i^{-1}(V)$. Also (by Theorem 0.10.c)

$$\pi_i^{-1}(U) \cap \pi_i^{-1}(V) = \pi_i^{-1}(U \cap V) = \pi_i^{-1}(\varnothing) = \varnothing.$$

Furthermore, Lemma I.30.b implies that $\pi_i^{-1}(U)$ and $\pi_i^{-1}(V)$ are open subsets of $X_1 \times X_2 \times \ldots \times X_n$. This proves $X_1 \times X_2 \times \ldots \times X_n$ is Hausdorff.

The following three problems complete this proof. □

Problem I.18. Prove part c) of Theorem I.31.

Problem I.19. Prove part f) of Theorem I.31.

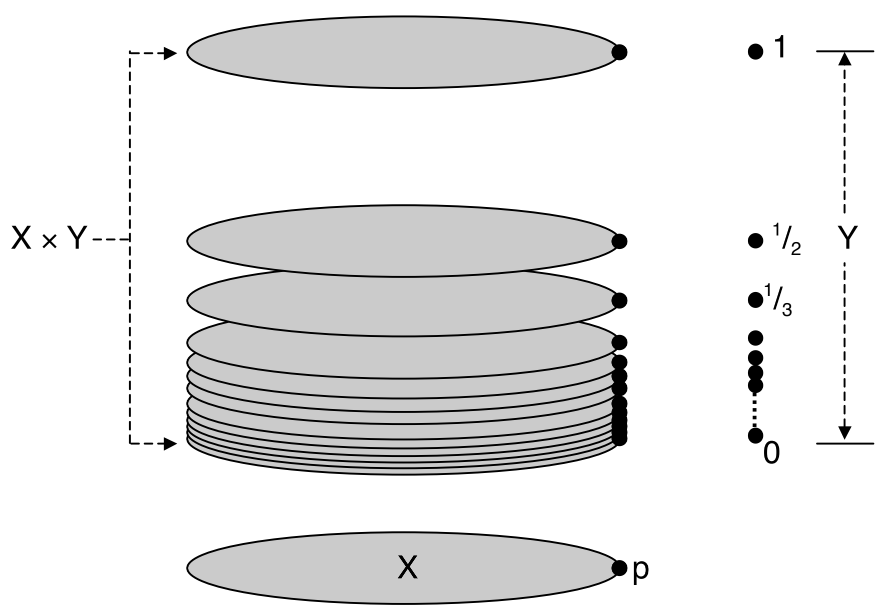
Problem I.20. Prove part g) of Theorem I.31.

Theorems I.27 and I.31, by failing to make obvious assertions about normal spaces, raise two natural questions. Are all subspaces of normal spaces normal? Are all Cartesian products of two normal spaces normal? The following two examples shed light on these questions.

Example I.18. Let X be an uncountable set, let $p \in X$, and set

$$\mathscr{T} = \mathscr{P}(X - \{p\}) \cup \{ U \subset X : X - U \text{ is finite} \}.$$

Then $\mathscr{T}$ is a topology on X. Endow X with the topology $\mathscr{T}$. Set $Y = \{0\} \cup \{ {}^1/_n : n \in \mathbb{N} \}$, and regard Y as a subspace of $\mathbb{R}$. Let $X \times Y$ have the product topology.



Problem I.21. Parts a, b and c of this problem refer to the space X × Y described in Example I.18.

a) Verify that $\mathscr{T}$ is a topology on X.

b) Prove that X × Y is a normal Hausdorff space.

(The fact that X × Y is normal will follow without difficulty from results in a later chapter. This is because X and Y are easily seen to be compact Hausdorff spaces. Then theorems in a subsequent chapter will imply that X × Y is a compact Hausdorff space and, hence, a normal space. However, for the present problem, you are asked to prove that X × Y is normal from the definition of "normal" without appealing to the notion of *compactness* which will be introduced later.)

c) Prove that the subspace (X × Y) – { (p,0) } of X × Y is not normal.

Hint: Consider the two disjoint subsets (X – { p }) × { 0 } and { p } × (Y – { 0 }).

d) Prove that X × Y is not completely normal. (The definition of "completely normal" follows Problem I.14.)

Problem I.21 reveals that a subspace of a normal space need not be normal.

Recall the space $\mathbb{R}_{bad}$ described in Example I.7. In Problem I.15(7) the question of whether $\mathbb{R}_{bad}$ is normal was decided.

Example I.19. Consider the Cartesian product $\mathbb{R}_{bad} \times \mathbb{R}_{bad}$ with the product topology.

Problem I.22. Is $\mathbb{R}_{bad} \times \mathbb{R}_{bad}$ normal?

Problem I.22 resolves the question of whether the Cartesian product of two normal spaces must be normal.

Theorem I.32. Let $(X_1, \rho_1), (X_2, \rho_2), \dots , (X_n, \rho_n)$ be metric spaces. Then three metrics on $X_1 \times X_2 \times \dots \times X_n$ are defined by the following formulas. For $\mathbf{x} = (x_1, x_2, \dots , x_n)$ and $\mathbf{y} = (y_1, y_2, \dots , y_n) \in X_1 \times X_2 \times \dots \times X_n$:

a) $\sigma_1(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} \rho_i(x_i,y_i)$,

b) $\sigma_2(\mathbf{x},\mathbf{y}) = \left(\sum_{i=1}^{n} (\rho_i(x_i,y_i))^2\right)^{1/2}$,

c) $\sigma_\infty(\mathbf{x},\mathbf{y}) = \max \{ \rho_i(x_i,y_i) : 1 \le i \le n \}$.

Moreover, σ_1, σ_2 and σ_∞ are equivalent metrics and induce the product topology on $X_1 \times X_2 \times \dots \times X_n$.

Proof. For $\lambda \in \{ 1, 2, \infty \}$, $\mathbf{x} \in X_1 \times X_2 \times \dots \times X_n$ and $\varepsilon > 0$, let

$$N_\lambda(\mathbf{x},\varepsilon) = \{ \mathbf{y} \in X_1 \times X_2 \times \dots \times X_n : \sigma_\lambda(\mathbf{x},\mathbf{y}) < \varepsilon \}.$$

Recall that the taxicab norm $\| \ \|_1$, the Euclidean norm $\| \ \|_2$ and the supremum norm $\| \ \|_\infty$ on $\mathbb{R}^n$ are defined and shown to be norms in 0.D. Observe that if $\mathbf{x} = (x_1, x_2, \dots , x_n)$ and $\mathbf{y} = (y_1, y_2, \dots , y_n) \in X_1 \times X_2 \times \dots \times X_n$, and if $\mathbf{r} = (\rho_1(x_1,y_1), \rho_2(x_2,y_2), \dots , \rho_n(x_n,y_n)) \in \mathbb{R}^n$, then for $\lambda \in \{ 1, 2, \infty \}$,

$$\sigma_\lambda(\mathbf{x},\mathbf{y}) = \| \mathbf{r} \|_\lambda.$$

We now verify that σ_1, σ_2 and σ_∞ are metrics on $X_1 \times X_2 \times \dots \times X_n$. Let $\lambda \in \{ 1, 2, \infty \}$. Let $\mathbf{x} = (x_1, x_2, \dots , x_n)$, $\mathbf{y} = (y_1, y_2, \dots , y_n)$ and $\mathbf{z} = (z_1, z_2, \dots , z_n) \in X_1 \times X_2 \times \dots \times X_n$. Then clearly

$$\mathbf{x} = \mathbf{y} \Leftrightarrow x_i = y_i \text{ for } 1 \le i \le n \Leftrightarrow \rho_i(x_i,y_i) = 0 \text{ for } 1 \le i \le n \Leftrightarrow \sigma_\lambda(\mathbf{x},\mathbf{y}) = 0.$$

Also since $\rho_i(x_i,y_i) = \rho_i(y_i,x_i)$ for $1 \le i \le n$, then $\sigma_\lambda(\mathbf{x},\mathbf{y}) = \sigma_\lambda(\mathbf{y},\mathbf{x})$. To prove the triangle inequality, define $\mathbf{r}$, $\mathbf{s}$ and $\mathbf{t} \in \mathbb{R}^n$ by

$$\mathbf{r} = (\rho_1(x_1,y_1), \rho_2(x_2,y_2), \dots , \rho_n(x_n,y_n)),$$

$$\mathbf{s} = (\rho_1(y_1,z_1), \rho_2(y_2,z_2), \dots , \rho_n(y_n,z_n)) \text{ and}$$

$$\mathbf{t} = (\rho_1(x_1,z_1), \rho_2(x_2,z_2), \dots , \rho_n(x_n,z_n)).$$

Then $\sigma_\lambda(\mathbf{x},\mathbf{y}) = \| \mathbf{r} \|_\lambda$, $\sigma_\lambda(\mathbf{y},\mathbf{z}) = \| \mathbf{s} \|_\lambda$ and $\sigma_\lambda(\mathbf{x},\mathbf{z}) = \| \mathbf{t} \|_\lambda$. For $1 \le i \le n$, since $\rho_i(x_i,z_i)$ is the i^{th} coordinate of $\mathbf{t}$, $\rho_i(x_i,y_i) + \rho_i(y_i,z_i)$ is the i^{th} coordinate of $\mathbf{r} + \mathbf{s}$, and $0 \le \rho_i(x_i,z_i) \le \rho_i(x_i,y_i) + \rho_i(y_i,z_i)$, then clearly $\| \mathbf{t} \|_\lambda \le \| \mathbf{r} + \mathbf{s} \|_\lambda$. Since the norm $\| \ \|_\lambda$ satisfies the triangle inequality, it follows that

$$\sigma_\lambda(\mathbf{x},\mathbf{z}) = \| \mathbf{t} \|_\lambda \le \| \mathbf{r} + \mathbf{s} \|_\lambda \le \| \mathbf{r} \|_\lambda + \| \mathbf{s} \|_\lambda = \sigma_\lambda(\mathbf{x},\mathbf{y}) + \sigma_\lambda(\mathbf{y},\mathbf{z}).$$

This completes the proof that σ_1, σ_2 and σ_∞ are metrics on $X_1 \times X_2 \times \dots \times X_n$.

Observe that the definitions of the taxicab, Euclidean and supremum norms imply that for $\mathbf{r} \in \mathbb{R}^n$,

$$\| \mathbf{r} \|_\infty \le \| \mathbf{r} \|_2 \le \| \mathbf{r} \|_1 \le n \| \mathbf{r} \|_\infty.$$

($\| \mathbf{r} \|_2 \le \| \mathbf{r} \|_1$ follows from the obvious fact that $(\| \mathbf{r} \|_2)^2 \le (\| \mathbf{r} \|_1)^2$.) Hence, for $\mathbf{x}$ and $\mathbf{y} \in X_1 \times X_2 \times \dots \times X_n$, we have

$$\sigma_\infty(\mathbf{x},\mathbf{y}) \le \sigma_2(\mathbf{x},\mathbf{y}) \le \sigma_1(\mathbf{x},\mathbf{y}) \le n\, \sigma_\infty(\mathbf{x},\mathbf{y}).$$

Thus, for $\mathbf{x}$ and $\mathbf{y} \in X_1 \times X_2 \times \dots \times X_n$ and $\varepsilon > 0$,

$$\sigma_2(\mathbf{x},\mathbf{y}) < \varepsilon \Rightarrow \sigma_\infty(\mathbf{x},\mathbf{y}) < \varepsilon,\ \sigma_1(\mathbf{x},\mathbf{y}) < \varepsilon \Rightarrow \sigma_2(\mathbf{x},\mathbf{y}) < \varepsilon \text{ and } \sigma_\infty(\mathbf{x},\mathbf{y}) < \varepsilon/n \Rightarrow \sigma_1(\mathbf{x},\mathbf{y}) < \varepsilon.$$

Therefore, for $\mathbf{x} \in X_1 \times X_2 \times \dots \times X_n$ and $\varepsilon > 0$,

$$N_\infty(\mathbf{x},\varepsilon/n) \subset N_1(\mathbf{x},\varepsilon) \subset N_2(\mathbf{x},\varepsilon) \subset N_\infty(\mathbf{x},\varepsilon).$$

Consequently, Theorem I.11 implies that σ_1, σ_2 and σ_∞ are equivalent metrics on $X_1 \times X_2 \times \ldots \times X_n$.

Lastly, we prove that σ_∞ induces the product topology on $X_1 \times X_2 \times \ldots \times X_n$. Let $\mathscr{B}_\infty = \{ N_\infty(\mathbf{x},\varepsilon) : \mathbf{x} \in X_1 \times X_2 \times \ldots \times X_n \text{ and } \varepsilon > 0 \}$ and let $\mathscr{B}_{box}$ denote the collection of all open boxes in $X_1 \times X_2 \times \ldots \times X_n$. Then $\mathscr{B}_\infty$ is a basis for the topology induced on $X_1 \times X_2 \times \ldots \times X_n$ by the metric σ_∞, and $\mathscr{B}_{box}$ is a basis for the product topology on $X_1 \times X_2 \times \ldots \times X_n$. We will use the Corollary to Theorem I.3 to show that $\mathscr{B}_\infty$ and $\mathscr{B}_{box}$ generate the same topology on $X_1 \times X_2 \times \ldots \times X_n$. To accomplish this, we must prove:

- if $\mathbf{x} \in B_1 \in \mathscr{B}_\infty$, then there is a $B_2 \in \mathscr{B}_{box}$ and $\mathbf{x} \in B_2 \subset B_1$, and
- if $\mathbf{x} \in B_2 \in \mathscr{B}_{box}$, then there is a $B_1 \in \mathscr{B}_\infty$ such that $\mathbf{x} \in B_1 \subset B_2$.

We begin by introducing more notation. For $1 \le i \le n$, $x \in X_i$ and $\varepsilon > 0$, let

$$M_i(x,\varepsilon) = \{ y \in X_i : \rho_i(x,y) < \varepsilon \}.$$

Next observe that for $\mathbf{x} = (x_1, x_2, \ldots , x_n) \in X_1 \times X_2 \times \ldots \times X_n$ and $\varepsilon > 0$:

$\mathbf{y} = (y_1, y_2, \ldots , y_n) \in N_\infty(\mathbf{x},\varepsilon) \;\Leftrightarrow\; \sigma_\infty(\mathbf{x},\mathbf{y}) < \varepsilon \;\Leftrightarrow$

$\rho_i(x_i,y_i) < \varepsilon$ for $1 \le i \le n \;\Leftrightarrow\; y_i \in M_i(x_i,\varepsilon)$ for $1 \le i \le n \;\Leftrightarrow$

$\mathbf{y} \in M_1(x_1,\varepsilon) \times M_2(x_2,\varepsilon) \times \ldots \times M_n(x_n,\varepsilon)$.

This proves that for $\mathbf{x} = (x_1, x_2, \ldots , x_n) \in X_1 \times X_2 \times \ldots \times X_n$ and $\varepsilon > 0$,

$$N_\infty(\mathbf{x},\varepsilon) = M_1(x_1,\varepsilon) \times M_2(x_2,\varepsilon) \times \ldots \times M_n(x_n,\varepsilon).$$

Hence, for each $\mathbf{x} \in X_1 \times X_2 \times \ldots \times X_n$ and $\varepsilon > 0$, $N_\infty(\mathbf{x},\varepsilon)$ is an open box. Consequently, $\mathscr{B}_\infty \subset \mathscr{B}_{box}$. Now suppose $\mathbf{x} \in B_1 \in \mathscr{B}_\infty$. Then $B_1 \in \mathscr{B}_{box}$. So if we set $B_2 = B_1$, then $B_2 \in \mathscr{B}_{box}$ and $\mathbf{x} \in B_2 \subset B_1$. This completes the first half of the proof that $\mathscr{B}_\infty$ and $\mathscr{B}_{box}$ generate the same topology on $X_1 \times X_2 \times \ldots \times X_n$. Second suppose $\mathbf{x} = (x_1, x_2, \ldots , x_n) \in B_2 \in \mathscr{B}_{box}$. Since B_2 is an open box, then $B_2 = U_1 \times U_2 \times \ldots \times U_n$ where U_i is an open subset X_i for $1 \le i \le n$. Thus, $(x_1, x_2, \ldots , x_n) \in U_1 \times U_2 \times \ldots \times U_n$. Therefore, $x_i \in U_i$ for $1 \le i \le n$. At this point, Theorem I.10 implies that for $1 \le i \le n$, there is an $\varepsilon_i > 0$ such that $M_i(x_1,\varepsilon_i) \subset U_i$. Let $\varepsilon = \min \{ \varepsilon_1, \varepsilon_2, \ldots , \varepsilon_n \}$. Then $\varepsilon > 0$ and $x_i \in M_i(x_1,\varepsilon) \subset M_i(x_1,\varepsilon_i) \subset U_i$ for $1 \le i \le n$. Hence,

$$\mathbf{x} = (x_1, x_2, \ldots , x_n) \in M_1(x_1,\varepsilon) \times M_2(x_2,\varepsilon) \times \ldots \times M_n(x_n,\varepsilon) \subset U_1 \times U_2 \times \ldots \times U_n = B_2.$$

We know that $M_1(x_1,\varepsilon) \times M_2(x_2,\varepsilon) \times \ldots \times M_n(x_n,\varepsilon) = N_\infty(\mathbf{x},\varepsilon)$ and $N_\infty(\mathbf{x},\varepsilon) \in \mathscr{B}_\infty$. So if we set $B_1 = N_\infty(\mathbf{x},\varepsilon)$, then $B_1 \in \mathscr{B}_\infty$ and $\mathbf{x} \in B_1 \subset B_2$. This completes the second half of the proof that $\mathscr{B}_\infty$ and $\mathscr{B}_{box}$ generate the same topology on $X_1 \times X_2 \times \ldots \times X_n$. We conclude that the metric σ_∞ induces the product topology on $X_1 \times X_2 \times \ldots \times X_n$. Since the metrics σ_1, σ_2 and σ_∞ are equivalent, then all three metrics induce the product topology on

$X_1 \times X_2 \times \ldots \times X_n$. □

Theorem I.33. If $X_1, X_2, \ldots, X_n$ are topological spaces and $X_1 \times X_2 \times \ldots \times X_n$ has the product topology, then $X_1 \times X_2 \times \ldots \times X_n$ is metrizable if and only if each of $X_1, X_2, \ldots, X_n$ is metrizable.

Proof. The $\Leftarrow$ direction of the proof follows from Theorem I.32. The following problem completes this proof. □

Problem I.23. Suppose if $X_1, X_2, \ldots, X_n$ are topological spaces and $X_1 \times X_2 \times \ldots \times X_n$ has the product topology. Prove that if $X_1 \times X_2 \times \ldots \times X_n$ is metrizable, then each of $X_1, X_2, \ldots, X_n$ is metrizable.

Theorem I.34. Let $X_1, X_2, \ldots, X_n$ be topological spaces, and let $X_1 \times X_2 \times \ldots \times X_n$ have the product topology. For $1 \leq i \leq n$, let $x_i : \mathbb{N} \to X_i$ be a sequence in X_i and let $y_i \in X_i$. Define the sequence $\mathbf{x} : \mathbb{N} \to X_1 \times X_2 \times \ldots \times X_n$ in $X_1 \times X_2 \times \ldots \times X_n$ by

$$\mathbf{x}(k) = (x_1(k), x_2(k), \ldots, x_n(k))$$

for $k \in \mathbb{N}$, and set $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in X_1 \times X_2 \times \ldots \times X_n$. Then $\mathbf{x}$ converges to $\mathbf{y}$ in $X_1 \times X_2 \times \ldots \times X_n$ if and only if x_i converges to y_i in X_i for $1 \leq i \leq n$.

Proof. First assume that $\mathbf{x}$ converges to $\mathbf{y}$ in $X_1 \times X_2 \times \ldots \times X_n$. We must prove that x_i converges to y_i in X_i for $1 \leq i \leq n$. To this end, let $1 \leq i \leq n$ and let V be a neighborhood of y_i in X_i. Then Lemma I.30.b implies that $\pi_i^{-1}(V)$ is an open subset of $X_1 \times X_2 \times \ldots \times X_n$. Furthermore, since $\pi_i(\mathbf{y}) = y_i$, then $\mathbf{y} \in \pi_i^{-1}(V)$. Since $\mathbf{x}$ converges to $\mathbf{y}$, then there is a $k \in \mathbb{N}$ such that $\mathbf{x}(j) \in \pi_i^{-1}(V)$ for all $j \in \mathbb{N}$ such that $j \geq k$. Therefore, $\pi_i(\mathbf{x}(j)) \in V$ for all $j \geq k$. Since $\pi_i(\mathbf{x}(j)) = x_i(j)$, then $x_i(j) \in V$ for all $j \geq k$. We conclude that x_i converges to y_i in X_i.

Second assume that x_i converges to y_i in X_i for $1 \leq i \leq n$. We must prove that $\mathbf{x}$ converges to $\mathbf{y}$ in $X_1 \times X_2 \times \ldots \times X_n$. To this end, let U be a neighborhood of $\mathbf{y}$ in $X_1 \times X_2 \times \ldots \times X_n$. Since $X_1 \times X_2 \times \ldots \times X_n$ has the product topology, then there is an open box $V_1 \times V_2 \times \ldots \times V_n$ in $X_1 \times X_2 \times \ldots \times X_n$ such that $\mathbf{y} \in V_1 \times V_2 \times \ldots \times V_n \subset U$. Since $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, then $(y_1, y_2, \ldots, y_n) \in V_1 \times V_2 \times \ldots \times V_n$. Hence, $y_i \in V_i$ for $1 \leq i \leq n$. For $1 \leq i \leq n$, since x_i converges to y_i in X_i, then there is a $k_i \in \mathbb{N}$ such that $x_i(j) \in V_i$ for all $j \in \mathbb{N}$ such that $j \geq k_i$. Let $k = \max\{ k_1, k_2, \ldots, k_n \}$. It follows that if $j \in \mathbb{N}$ and $j \geq k$, then $x_i(j) \in V_i$ for $1 \leq i \leq n$. Therefore, if $j \in \mathbb{N}$ and $j \geq k$, then $\mathbf{x}(j) = (x_1(j), x_2(j), \ldots, x_n(j)) \in V_1 \times V_2 \times \ldots \times V_n$. Thus, $\mathbf{x}(j) \in U$ for all $j \in \mathbb{N}$ such that $j \geq k$. We conclude that $\mathbf{x}$ converges to $\mathbf{y}$ in $X_1 \times X_2 \times \ldots \times X_n$. □