I. Topological Spaces

A. Topologies

The topological structure of a space expresses the limit or convergence phenomena which occur within the space. There are many ways to specify this structure. For instance, one could specify which generalized sequences (sequences of possibly uncountable length) converge. Or one could specify a "closure operator" which assigns to each subset of the space its "closure". Or in special cases one could specify a metric (distance function) on the space. We follow the course that has through time become the standard approach. We specify the collection of "open sets" or the "topology" of the space.

Definition. A *topology* on a set X is a collection $\mathscr{T}$ of subsets of X satisfying the following three conditions.

a) $\emptyset, X \in \mathscr{T}$,.

b) if $\mathscr{U} \subset \mathscr{T}$, then $\bigcup \mathscr{U} \in \mathscr{T}$, and

c) if $U, V \in \mathscr{T}$, then $U \cap V \in \mathscr{T}$.

For definitions and illustrations of elementary usage of set theoretic notation like the union symbol "$\bigcup$" and the intersection symbol "$\cap$", we refer the student to Section A of Chapter 0.

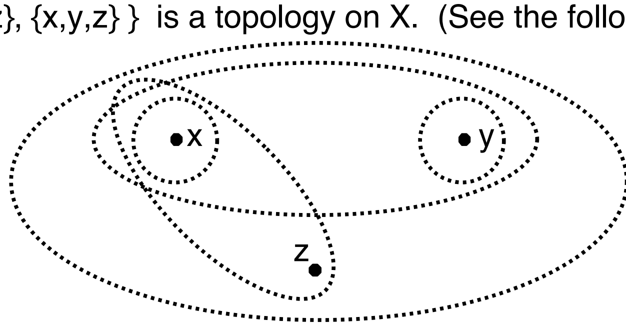
Observe that property c) together with an induction argument imply that if $U_1, U_2, \ldots, U_n \in \mathscr{T}$, then $U_1 \cap U_2 \cap \ldots \cap U_n \in \mathscr{T}$.

Definition. Let X be a set, and let $\mathscr{T}$ be a topology on X. Then the pair $(X, \mathscr{T})$ is called a *topological space.* When we don't wish to name the topology, we simply say "X is a topological space". The elements of $\mathscr{T}$ are called *open subsets* of X.

Example I.1. $\{ \emptyset, X \}$ is the called the *indiscrete topology* on the set X.

Example I.2. $\mathscr{P}(X) = \{ A : A \subset X \}$ is called the *discrete topology* on the set X.

Example I.3. Let $X = \{ x, y, z \}$ be a three-point set. Then $\{ \emptyset, \{x\}, \{y\}, \{x,y\}, \{x,z\}, \{x,y,z\} \}$ is a topology on X. (See the following figure.)



Exercise. Verify that the collections defined in examples I.1 through I.3 are topologies.

Example I.4. $\{ \varnothing \} \cup \{ U \subset X : X - U \text{ is finite} \}$ is called the *finite complement topology* on the set X.

Proof that the finite complement topology is a topology. Set $\mathscr{T} = \{ \varnothing \} \cup \{ U \subset X : X - U \text{ is finite} \}$.

a) Clearly $\varnothing \in \mathscr{T}$. $X \in \mathscr{T}$ because $X - X = \varnothing$ is a finite set.

b) Let $\mathscr{U} \subset \mathscr{T}$. If $\mathscr{U} = \varnothing$ or $\{ \varnothing \}$, then $\bigcup \mathscr{U} = \varnothing \in \mathscr{T}$. Now assume $\mathscr{U} \neq \varnothing$ or $\{ \varnothing \}$. Let $U_0 \in \mathscr{U} - \{ \varnothing \}$. By De Morgan's Laws, $X - (\bigcup \mathscr{U}) = \bigcap_{U \in \mathscr{U}} (X - U) \subset X - U_0$. Since $X - U_0$ is finite, then $X - (\bigcup \mathscr{U})$ is finite. So $\bigcup \mathscr{U} \in \mathscr{T}$.

c) Let $U, V \in \mathscr{T}$. If U or $V = \varnothing$, then $U \cap V = \varnothing \in \mathscr{T}$. Now assume $U \neq \varnothing \neq V$. By De Morgan's Laws, $X - (U \cap V) = (X - U) \cup (X - V)$. Since $X - U$ and $X - V$ are finite, then $(X - U) \cup (X - V)$ is finite. Hence, $X - (U \cap V)$ is finite. So $U \cap V \in \mathscr{T}$. ◻

Definition. Let $\mathscr{T}$ and $\mathscr{U}$ be topologies on a set X. If $\mathscr{T} \subset \mathscr{U}$, we say $\mathscr{T}$ is *smaller* or *coarser than* $\mathscr{U}$, and $\mathscr{U}$ is *larger* or *finer than* $\mathscr{T}$.

Observe that if $\mathscr{T}$ is any topology on a set X, then

$$\text{the indiscrete topology on } X \subset \mathscr{T} \subset \text{the discrete topology on } X.$$

B. Bases

To specify a topology on a set, one need not list the entire topology. It suffices to specify a generating subcollection or "basis" for the topology. In practice, a topology is often determined by specifying its basis, because it is easier or clearer to describe the basis than the entire topology.

Definition. Let $(X, \mathscr{T})$ be a topological space. A collection $\mathscr{B}$ of subsets of X is a *basis* for $\mathscr{T}$ if

a) $\mathscr{B} \subset \mathscr{T}$, and

b) for every $x \in U \in \mathscr{T}$, there is a $B \in \mathscr{B}$ such that $x \in B \subset U$.

Theorem I.1. Let (X, $\mathscr{T}$) be a topological space, and let $\mathscr{B}$ be a collection of subsets of X. Then $\mathscr{B}$ is a basis for $\mathscr{T}$ if and only if $\mathscr{T} = \{ \bigcup\mathscr{C} : \mathscr{C} \subset \mathscr{B} \}$.

Observe that this equation specifies the exact relationship between a basis and the topology it "generates".

Proof. First assume $\mathscr{B}$ is a basis for $\mathscr{T}$. Then by definition, $\mathscr{B} \subset \mathscr{T}$. So $\mathscr{C} \subset \mathscr{B} \Rightarrow \mathscr{C} \subset \mathscr{T} \Rightarrow \bigcup\mathscr{C} \in \mathscr{T}$. Hence, $\mathscr{T} \supset \{ \bigcup\mathscr{C} : \mathscr{C} \subset \mathscr{B} \}$.

Let $U \in \mathscr{T}$. Then for each $x \in U$, there is a $B_x \in \mathscr{B}$ such that $x \in B_x \subset U$. Set $\mathscr{D} = \{ B_x : x \in U \}$. Then $\mathscr{D} \subset \mathscr{B}$ and $\bigcup\mathscr{D} = \bigcup_{x \in U} B_x = U$. So U equals the union of the elements of the subset $\mathscr{D}$ of $\mathscr{B}$. In other words $U \in \{ \bigcup\mathscr{C} : \mathscr{C} \subset \mathscr{B} \}$. This proves $\mathscr{T} \subset \{ \bigcup\mathscr{C} : \mathscr{C} \subset \mathscr{B} \}$.

We conclude that $\mathscr{T} = \{ \bigcup\mathscr{C} : \mathscr{C} \subset \mathscr{B} \}$.

Second assume $\mathscr{T} = \{ \bigcup\mathscr{C} : \mathscr{C} \subset \mathscr{B} \}$.

To prove $\mathscr{B} \subset \mathscr{T}$, let $B \in \mathscr{B}$. Then $\{ B \} \subset \mathscr{B}$ and $\bigcup\{ B \} = B$. Hence, $B \in \{ \bigcup\mathscr{C} : \mathscr{C} \subset \mathscr{B} \}$. So $B \in \mathscr{T}$. This proves $\mathscr{B} \subset \mathscr{T}$.

Let $x \in U \in \mathscr{T}$. We must prove there is a $B \in \mathscr{B}$ such that $x \in B \subset U$. Since $U \in \mathscr{T} = \{ \bigcup\mathscr{C} : \mathscr{C} \subset \mathscr{B} \}$, then there is a $\mathscr{C} \subset \mathscr{B}$ such that $U = \bigcup\mathscr{C}$. So $x \in \bigcup\mathscr{C}$. Hence, there is a $B \in \mathscr{C}$ such that $x \in B$. Therefore, $B \in \mathscr{B}$ and $x \in B \subset \bigcup\mathscr{C} = U$.

We conclude that $\mathscr{B}$ is a basis for $\mathscr{T}$. □

Corollary. If $\mathscr{B}$ is a basis for a topology $\mathscr{T}$ on a set X, then $\mathscr{T}$ is uniquely determined by the formula $\mathscr{T} = \{ \bigcup\mathscr{C} : \mathscr{C} \subset \mathscr{B} \}$. □

According to this corollary, each basis determines a unique topology. However, simple examples show that a single topology may have many different bases.

Exercise. Prove that if the set X has more than one point, then the discrete topology on X has more than one basis.

We now present a criterion for a collection of subsets of a set to be a basis for *some* topology on the set.

Definition. If $\mathscr{C}$ is a collection of subsets of a set X such that $\bigcup\mathscr{C} = X$, then we say that $\mathscr{C}$ *covers* X and we call $\mathscr{C}$ a *cover* of X. Thus, $\mathscr{C}$ covers X $\Leftrightarrow$ for every $x \in X$, there is a $C \in \mathscr{C}$ such that $x \in C$.

Theorem I.2. Let $\mathscr{B}$ be a collection of subsets of a set X. $\mathscr{B}$ is a basis for *some* topology on X if and only if
a) $\mathscr{B}$ covers X, and
b) for all $B_1, B_2 \in \mathscr{B}$, for every $x \in B_1 \cap B_2$, there is a $B_3 \in \mathscr{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Problem I.1. Prove Theorem I.2 $\Rightarrow$.

Proof of Theorem I.2 $\Leftarrow$. Assume $\mathscr{B}$ satisfies conditions a) and b). Set $\mathscr{T} = \{ \bigcup\mathscr{C} : \mathscr{C} \subset \mathscr{B} \}$. We will prove that $\mathscr{T}$ is a topology on X and that $\mathscr{B}$ is a basis for $\mathscr{T}$.

$\varnothing \in \mathscr{T}$ because $\varnothing \subset \mathscr{B}$ and $\bigcup\varnothing = \varnothing$. $X \in \mathscr{T}$ because $\mathscr{B} \subset \mathscr{B}$ and $\bigcup\mathscr{B} = X$.

Let $\mathscr{V} \subset \mathscr{T}$. Then for every $V \in \mathscr{V}$, there is a $\mathscr{C}_V \subset \mathscr{B}$ such that $V = \bigcup\mathscr{C}_V$. Set $\mathscr{C} = \bigcup_{V \in \mathscr{V}}\mathscr{C}_V$; i.e., $B \in \mathscr{C} \Leftrightarrow B \in \mathscr{C}_V$ for some $V \in \mathscr{V}$. Then $\mathscr{C} \subset \mathscr{B}$ and $\bigcup\mathscr{C} = \bigcup_{V \in \mathscr{V}}(\bigcup\mathscr{C}_V) = \bigcup\mathscr{V}$. Hence, $\bigcup\mathscr{V} \in \mathscr{T}$.

Let $U, V \in \mathscr{T}$. Then there are $\mathscr{C}, \mathscr{D} \subset \mathscr{B}$ such that $U = \bigcup\mathscr{C}$ and $V = \bigcup\mathscr{D}$. Let $x \in U \cap V$. Then $x \in (\bigcup\mathscr{C}) \cap (\bigcup\mathscr{D})$. So there is a $C_x \in \mathscr{C}$ and a $D_x \in \mathscr{D}$ such that $x \in C_x$ and $x \in D_x$. Furthermore, $C_x \subset \bigcup\mathscr{C} = U$ and $D_x \subset \bigcup\mathscr{D} = V$. Thus, $C_x \cap D_x \subset U \cap V$. Since $C_x, D_x \in \mathscr{B}$, then condition b) provides an $E_x \in \mathscr{B}$ such that $x \in E_x \subset C_x \cap D_x$. Hence, $x \in E_x \subset U \cap V$. Therefore, $\{ E_x : x \in U \cap V \} \subset \mathscr{B}$ and $U \cap V = \bigcup \{ E_x : x \in U \cap V \}$. Hence, $U \cap V \in \mathscr{T}$.

This proves $\mathscr{T}$ is a topology on X. Since $\mathscr{T} = \{ \bigcup\mathscr{C} : \mathscr{C} \subset \mathscr{B} \}$, then Theorem I.1 implies $\mathscr{B}$ is a basis for $\mathscr{T}$. ◻

Corollary. If $\mathscr{B}$ is a collection of subsets of a set X satisfying
a) $\mathscr{B}$ covers X, and
b) for all $B, C \in \mathscr{B}$, either $B \cap C = \varnothing$ or $B \cap C \in \mathscr{B}$,
then $\mathscr{B}$ is a basis for *some* topology on X. ◻

This corollary provides a sufficient (but not necessary) criterion for a collection of sets to be a basis for a topology, which is simpler than the one stated in Theorem I.2. Moreover, this simple criterion often applies. In particular, it applies in the following three examples. Thus, we define the topologies in these examples by specifying their bases and invoking the corollary.

The first two of these examples are the most important spaces in topology.

Example I.5. Let $\mathbb{R}$ denote the set of real numbers. For x, y $\in \mathbb{R}$, let

$$(x,y) = \{ z \in \mathbb{R} : x < z < y \},$$

and call this set an *open interval* in $\mathbb{R}$. Observe that the intersection of two open intervals is either the empty set or an open interval. Hence, the set of all open intervals in $\mathbb{R}$ is a basis for a topology on $\mathbb{R}$ called the *standard topology* on $\mathbb{R}$.

Example I.6. For $n \geq 1$, let

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R} \text{ (the Cartesian product of n copies of } \mathbb{R})$$

$$= \{ (x_1, x_2, \dots , x_n) : x_i \in \mathbb{R} \text{ for } 1 \leq i \leq n \}.$$

An *open box* in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ of the form $J_1 \times J_2 \times \dots \times J_n$ where J_i is an open interval in $\mathbb{R}$ for $1 \leq i \leq n$. The collection of open boxes covers $\mathbb{R}^n$. (Why?) Observe that if $J_i, K_i \subset \mathbb{R}$ for $1 \leq i \leq n$, then

$$(J_1 \times J_2 \times \dots \times J_n) \cap (K_1 \times K_2 \times \dots \times K_n) = (J_1 \cap K_1) \times (J_2 \cap K_2) \times \dots \times (J_n \cap K_n).$$

Since the intersection of two open intervals is either the empty set or an open interval, it follows that the intersection of two open boxes in $\mathbb{R}^n$ is either the empty set or an open box in $\mathbb{R}^n$. Hence, the set of all open boxes in $\mathbb{R}^n$ is a basis for a topology on $\mathbb{R}^n$ called the *standard topology* on $\mathbb{R}^n$.

Example I.7. For x, y $\in \mathbb{R}$, let

$$[x,y) = \{ z \in \mathbb{R} : x \leq z < y \},$$

and call this set a *closed–open interval* in $\mathbb{R}$. Observe that the intersection of two closed–open intervals is either the empty set or a closed–open intervals. Hence, the set of all closed-open intervals in $\mathbb{R}$ is a basis for a topology on $\mathbb{R}$ called the *closed–open interval topology* on $\mathbb{R}$. $\mathbb{R}$ with the closed-open interval topology is called $\mathbb{R}_{bad}$.

Two different bases on a set may or may not generate the same topology. The following theorem and its corollary provide a necessary and sufficient criterion for when two bases determine the same topology.

Theorem I.3. For i = 1, 2, suppose $\mathscr{B}_i$ is a basis for a topology $\mathscr{T}_i$ on a set X. Then $\mathscr{T}_1 \subset \mathscr{T}_2$ if and only if for every $x \in B_1 \in \mathscr{B}_1$, there is a $B_2 \in \mathscr{B}_2$ such that $x \in B_2 \subset B_1$.

Proof. First assume $\mathscr{T}_1 \subset \mathscr{T}_2$. Suppose $x \in B_1 \in \mathscr{B}_1$. Since $\mathscr{B}_1 \subset \mathscr{T}_1 \subset \mathscr{T}_2$, then $B_1 \in \mathscr{T}_2$. Since $\mathscr{B}_2$ is a basis for $\mathscr{T}_2$, then there is a $B_2 \in \mathscr{B}_2$ such that $x \in B_2 \subset B_1$.

Second assume that for every $x \in B_1 \in \mathscr{B}_1$, there is a $B_2 \in \mathscr{B}_2$ such that $x \in B_2 \subset B_1$. To prove $\mathscr{T}_1 \subset \mathscr{T}_2$, let $U \in \mathscr{T}_1$. Since $\mathscr{B}_1$ is a basis for $\mathscr{T}_1$, then for each $x \in U$, there is a $B_{1,x} \in \mathscr{B}_1$ such that $x \in B_{1,x} \subset U$. Then, by hypothesis, for each $x \in U$, there is a $B_{2,x} \in \mathscr{B}_2$ such that $x \in B_{2,x} \subset B_{1,x}$. Thus, for each $x \in U$, there is a $B_{2,x} \in \mathscr{B}_2$ such that $x \in B_{2,x} \subset U$. Then clearly, $U = \bigcup_{x \in U} B_{2,x}$. Since each $B_{2,x} \in \mathscr{B}_2 \subset \mathscr{T}_2$, then $U \in \mathscr{T}_2$. This proves $\mathscr{T}_1 \subset \mathscr{T}_2$. ◻

Corollary. For i = 1, 2, suppose $\mathscr{B}_i$ is a basis for a topology $\mathscr{T}_i$ on a set X. Then $\mathscr{T}_1 = \mathscr{T}_2$ if and only for every $x \in B_1 \in \mathscr{B}_1$, there is a $B_2 \in \mathscr{B}_2$ such that $x \in B_2 \subset B_1$, and for every $x \in B_2 \in \mathscr{B}_2$, there is a $B_1 \in \mathscr{B}_1$ such that $x \in B_1 \subset B_2$. ◻

Observe that Theorem I.3 implies that the standard topology on $\mathbb{R}$ is strictly smaller than the closed-open interval topology on $\mathbb{R}$. For if $x \in (y, z)$, then $x \in [x, z) \subset (y, z)$; but there is *no* open interval (u, v) in $\mathbb{R}$ such that $0 \in (u, v) \subset [0, 1)$. A fundamental question that can be asked about a topological space concerns the cardinality of its basis. How small a basis does a topology have? The following definitions provide terminology to discuss this issue.

Definition. A topological space is *second countable* (or *satisfies the second axiom of countability*) if its topology has a countable basis.

Definition. Let (X, $\mathscr{T}$) be a topological space. Let $x \in X$. A collection $\mathscr{B}_x$ of subsets of X is a *basis for $\mathscr{T}$ at x* if

a) $\mathscr{B}_x \subset \mathscr{T}$,

b) $x \in B$ for every $B \in \mathscr{B}_x$, and

c) for every $U \in \mathscr{T}$ such that $x \in U$, there is a $B \in \mathscr{B}_x$ such that $B \subset U$.

Observe that if for every $x \in X$, $\mathscr{B}_x$ is a basis for $\mathscr{T}$ at x, then $\bigcup_{x \in X} \mathscr{B}_x$ is a basis for $\mathscr{T}$.

Definition. A topological space (X, $\mathscr{T}$) is *first countable* (or *satisfies the first axiom of countability*) if for every $x \in X$, there is a countable basis for $\mathscr{T}$ at x.

Definition. Let X be a topological space. A subset D of X is *dense* in X if D intersects every non-empty open subset of X. X is *separable* if it has a countable dense subset.

Exercise. Let X be a topological space. Formulate conjectures about the possible logical relationships between the statements: "X is second countable", "X is first countable", and "X is separable". Keep these conjectures in mind when working Problems I.3 and I.4(n) below.

Theorem I.4. Every second countable topological space is first countable.

Proof. Assume (X, $\mathscr{T}$) is a second countable space. Then X has a countable basis $\mathscr{B}$. Let $x \in X$. Define

$$\mathscr{B}_x = \{ B \in \mathscr{B} : x \in B \}.$$

We will prove that $\mathscr{B}_x$ is a countable basis for $\mathscr{T}$ at x. Since $\mathscr{B}_x \subset \mathscr{B}$ and $\mathscr{B}$ is countable, then $\mathscr{B}_x$ is countable. Also since $\mathscr{B} \subset \mathscr{T}$, then $\mathscr{B}_x \subset \mathscr{T}$. By definition, every element of $\mathscr{B}_x$ contains the point x. If $x \in U \in \mathscr{T}$, then $x \in B \subset U$ for some $B \in \mathscr{B}$, because $\mathscr{B}$ is a basis for $\mathscr{T}$. But then $B \in \mathscr{B}_x$ and $B \subset U$. This completes the proof that $\mathscr{B}_x$ is a countable basis for $\mathscr{T}$ at x. It follows that X is first countable. □

Theorem I.5. Every second countable topological space is separable.

Problem I.2. Prove Theorem I.5.

Exercise. Prove that a topological space which contains uncountably many pairwise disjoint non-empty open subsets is not separable.

We now consider whether the topological spaces described in Examples I.1 through I.7 are second countable, first countable or separable. We will settle this issue for some of these spaces and leave the others as problems for the student.

Examples I.1 and I.3 are second countable, first countable and separable. This is because in both these examples, the topology $\mathscr{T}$ on the set X is a finite set. Hence, $\mathscr{T}$ itself is a countable (in fact, finite) basis for the topology $\mathscr{T}$. So X is second countable. Hence, Theorems I.4 and I.5 imply that X is first countable and separable.

Problem I.3(2). Decide whether or under what conditions the space described in Example I.2 (the set X with the discrete topology) is
a) second countable,
b) first countable,
c) separable.

Problem I.3(4). Decide whether or under what conditions the space described in Example I.4 (the set X with the finite complement topology) is
a) second countable,
b) first countable,
c) separable.

Problems I.3(2) and I.3(4) are the first two in a sequence of problems. In general, Problem I.3(n) asks whether the space described in Example I.n is second countable, first countable or separable. We will formulate this problem for many of the examples subsequently introduced in this chapter.

Example I.5 – $\mathbb{R}$ with the standard topology – is second countable, first countable and separable. Our candidate for a countable basis for $\mathbb{R}$ is the collection

$$\mathscr{B} = \{ (u, v) : (u, v) \in \mathbb{Q} \times \mathbb{Q} \text{ and } u < v \}.$$

Here $\mathbb{Q} = \{ {}^m/_n : m, n \in \mathbb{Z} \text{ and } n \neq 0 \}$ denotes the set of all rational numbers. Unfortunately, in this formula for $\mathscr{B}$, the notation (u, v) is used ambiguously: the first instance of (u, v) represents an *open interval* and the second instance represents an *ordered pair*. Since $\mathbb{Q}$ is a countable set (Theorem 0.18), then $\mathbb{Q} \times \mathbb{Q}$ is countable (Theorem 0.17). Therefore, the subset $\{ (u, v) \in \mathbb{Q} \times \mathbb{Q} : u < v \}$ of $\mathbb{Q} \times \mathbb{Q}$ is countable. Since the elements of the latter set index the elements of $\mathscr{B}$, then it follows that $\mathscr{B}$ is a countable set. We now argue that $\mathscr{B}$ is a basis for the standard topology on $\mathbb{R}$. First notice that since each element of $\mathscr{B}$ is an open interval, then $\mathscr{B}$ is a subset of the standard topology on $\mathbb{R}$. Now suppose $x \in U$ where U is an element of the standard topology on $\mathbb{R}$. Then, by definition, there is an open interval (y, z) such that $x \in (y, z) \subset U$. Hence, $y < x < z$. Since, the set $\mathbb{Q}$ of rational numbers is dense in the set $\mathbb{R}$ of real numbers, then there exist $u, v \in \mathbb{Q}$ such that $u \in (y, x)$ and $v \in (x, z)$. Thus, $(u, v) \in \mathbb{Q} \times \mathbb{Q}$ and $u < v$. Therefore, $(u, v) \in \mathscr{B}$ and $x \in (u, v) \subset (y, z) \subset U$. This completes the proof that $\mathscr{B}$ is a basis for the standard topology on $\mathbb{R}$. Since $\mathscr{B}$ is a countable set, then it follows that $\mathbb{R}$ with the standard topology is second countable. It follows from Theorems I.4 and I.5 imply that $\mathbb{R}$ with the standard topology is first countable and separable.

Example I.6 – $\mathbb{R}^n$ with the standard topology – is second countable, first countable and separable. Let $\mathscr{B}$ denote the countable basis for the standard topology on $\mathbb{R}$ described in the preceding paragraph. Our candidate for a countable basis for $\mathbb{R}^n$ is the collection

$$\mathscr{B}^* = \{ J_1 \times J_2 \times \ldots \times J_n : (J_1, J_2, \ldots , J_n) \in \mathscr{B} \times \mathscr{B} \times \ldots \times \mathscr{B} \}.$$

Since $\mathscr{B}$ is a countable set, then $\mathscr{B} \times \mathscr{B} \times \ldots \times \mathscr{B}$ (n copies) is also countable. (This follows by induction from Theorem 0.17.) Since the elements of $\mathscr{B} \times \mathscr{B} \times \ldots \times \mathscr{B}$ index the elements of $\mathscr{B}^*$, then $\mathscr{B}^*$ must be countable. We now argue that $\mathscr{B}^*$ is a basis for the standard topology on $\mathbb{R}^n$. First notice that if $J_1 \times J_2 \times \ldots \times J_n \in \mathscr{B}^*$, then each J_i is an open interval in $\mathbb{R}$ with rational endponts. Thus, each $J_1 \times J_2 \times \ldots \times J_n \in \mathscr{B}^*$ is an open box in $\mathbb{R}^n$. Therefore, $\mathscr{B}^*$ is a subset of the standard topology on $\mathbb{R}^n$. Now suppose $(x_1, x_2, \ldots , x_n) \in U$ where U is an element of the standard topology on $\mathbb{R}^n$. Then, by definition, there is an open box in $\mathbb{R}^n$ of the form $J_1 \times J_2 \times \ldots \times J_n$ such that $(x_1, x_2, \ldots , x_n) \in J_1 \times J_2 \times \ldots \times J_n \subset U$. Hence, for $1 \leq i \leq n$, $x_i \in J_i$ where J_i, being an open interval in $\mathbb{R}$, is an element of the standard topology on $\mathbb{R}$. Since $\mathscr{B}$ is a basis for the standard topology on $\mathbb{R}$, then there is a $K_i \in \mathscr{B}$ such that $x_i \in K_i \subset J_i$ for $1 \leq i \leq n$. Hence, $K_1 \times K_2 \times \ldots \times K_n \in \mathscr{B}^*$ and $(x_1, x_2, \ldots , x_n) \in K_1 \times K_2 \times \ldots \times K_n \subset J_1 \times J_2 \times \ldots \times J_n \subset U$. This completes the proof that $\mathscr{B}^*$ is a basis for the standard topology on $\mathbb{R}^n$. Since $\mathscr{B}^*$ is a countable set, then it follows that $\mathbb{R}^n$ with the standard topology is second countable. It follows from Theorems I.4 and I.5 imply that $\mathbb{R}^n$ with the standard topology is first countable and separable.

Problem I.3(7). Decide whether the space $\mathbb{R}_{bad}$ described in Example I.7 is
a) second countable,
b) first countable,
c) separable.

Example I.8. Let $\mathbb{N} = \{ 1, 2, 3, \ldots \}$, let ∞ denote a point which is not an element of $\mathbb{N} \times \mathbb{N}$, and let X denote the countable set $(\mathbb{N} \times \mathbb{N}) \cup \{ \infty \}$. We now describe a basis $\mathscr{B}$ for a topology on X. First, let $\mathbb{N}^{\mathbb{N}}$ denote the set of all functions from $\mathbb{N}$ to itself; and for each $f \in \mathbb{N}^{\mathbb{N}}$, define the set N(f) containing ∞ by the formula

$$N(f) = \{ \infty \} \cup \{ (x, y) \in \mathbb{N} \times \mathbb{N} : f(x) \leq y \}.$$

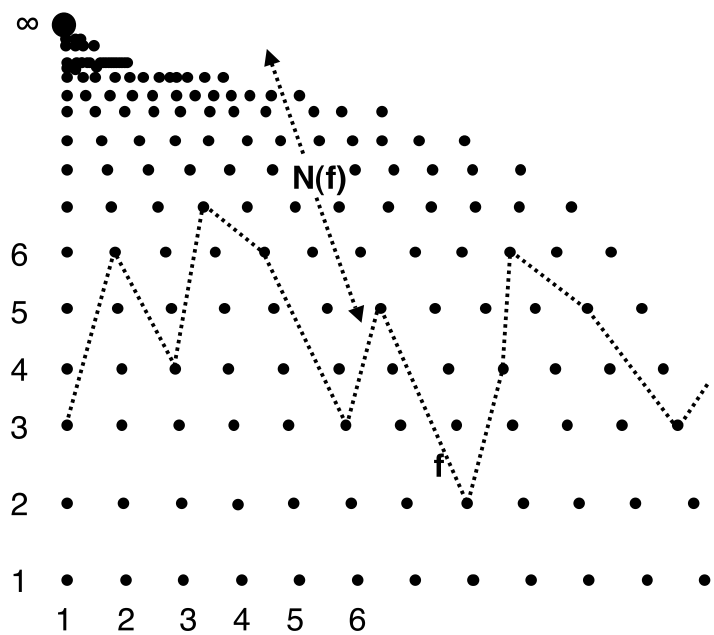
Let

$$\mathscr{B} = \{ \{ p \} : p \in \mathbb{N} \times \mathbb{N} \} \cup \{ N(f) : f \in \mathbb{N}^{\mathbb{N}} \}.$$

To prove that $\mathscr{B}$ is a basis for a topology on X, we will prove that the intersection of any two elements of $\mathscr{B}$ is either the empty set or an element of $\mathscr{B}$. Indeed, observe that if $p, q \in \mathbb{N} \times \mathbb{N}$ and $f, g \in \mathbb{N}^{\mathbb{N}}$, then:

- $\{p\} \cap \{q\} = \emptyset$ or $\{p\}$,
- $\{p\} \cap N(f) = \emptyset$ or $\{p\}$, and
- $N(f) \cap N(g) = N(h)$ where h is the element of $\mathbb{N}^{\mathbb{N}}$ defined by the formula $h(x) = \max \{ f(x), g(x) \}$ for $x \in \mathbb{N}$.

Since the intersection of any two elements of $\mathscr{B}$ is either an element of $\mathscr{B}$ or empty, it follows from Corollary following Theorem I.2 that $\mathscr{B}$ is a basis for a topology on X = $(\mathbb{N} \times \mathbb{N}) \cup \{ \infty \}$. We endow X with this topology.



Exercise. Formulate a conjecture involving first countability to which Example I.8 is relevant.

Problem I.3(8). Decide whether the space describe in Example I.8 is
a) second countable,
b) first countable,
c) separable.

C. Linearly Ordered Spaces

The linearly ordered spaces form a class of topological spaces which are easily visualized because of their linear character. This simplicity is deceptive. This class contains some very interesting spaces.

Definition. Let X be a set. Any subset of X × X is called a *relation* on X. If R is a relation on X and (x, y) ∈ R, we write xRy.

Definition. A relation $<$ on a set X is a *linear order* on X if it satisfies the following two conditions.

a) *Transitivity:* For all $x, y, z \in X$, $x < y$ and $y < z \Rightarrow x < z$.

b) *Trichotomy:* For all $x, y \in X$, exactly one of the following holds: $x < y$, $x = y$, $y < x$.

Definition. Let $<$ be a linear order on a set X. Then the pair $(X,<)$ is called a *linearly ordered set.* In this situation, the relation $\leq$ on X is defined by:
$x \leq y \Leftrightarrow x < y$ or $x = y$.

Definition. Let $(X, <)$ be a linearly ordered set. Let $x, y \in X$. Let

$$(x, y) = \{ z \in X : x < z < y \}, \quad (x, \infty) = \{ z \in X : x < z \} \text{ and } (-\infty, y) = \{ z \in X : z < y \};$$

and call these sets *open intervals* in X. Let

$$[x, y] = \{ z \in X : x \leq z \leq y \}, \quad [x, \infty) = \{ z \in X : x \leq z \} \text{ and } (-\infty, y] = \{ z \in X : z \leq y \};$$

and call these sets *closed intervals* in X. Let

$$[x, y) = \{ z \in X : x \leq z < y \} \text{ and } (x, y] = \{ z \in X : x < z \leq y \};$$

and call these sets *half-open intervals* in X.

Definition. Let $(X, <)$ be a linearly ordered set. Observe that the intersection of two open intervals in X is either the empty set or an open interval in X. Hence, the set

$$\{ (x, y) : x, y \in X \} \cup \{ (x, \infty) : x \in X \} \cup \{ (-\infty, y) : y \in X \}$$

of all open intervals in X is a basis for a topology on X called the *order topology* on X. A linearly ordered set with the order topology is called a *linearly ordered space.*

Observe that the order topology on $\mathbb{R}$ is the standard topology.

Example I.9. Let $[0, 1] = \{ x \in \mathbb{R} : 0 \leq x \leq 1 \}$ and $[0, 1]^2 = [0, 1] \times [0, 1]$. A linear order on $[0, 1]^2$, called the *lexicographic order,* is defined as follows. For (x, y) and $(x', y') \in [0, 1]^2$,

$$(x, y) < (x', y') \quad \text{if either} \quad x < x' \text{ or } (x = x' \text{ and } y < y').$$

The order topology on $[0, 1]^2$ associated to the lexicographic order on $[0, 1]^2$ is called the *lexicographic order topology* on $[0, 1]^2$.

Next we observe that the space described in Example I.9 – $[0, 1]^2$ with the lexicographic order topology – is neither second countable nor separable. First note that for all distinct $x, y \in [0, 1]$, the open intervals $((x, 0), (x, 1)) = \{ x \} \times (0, 1)$ and $((y, 0), (y, 1)) = \{ y \} \times (0, 1)$ are disjoint open subsets of $[0, 1]^2$. Hence,

$\{ ((x, 0), (x, 1)) : x \in [0, 1] \}$ is an uncountable pairwise disjoint collection of open subsets of $[0, 1]^2$. Every dense subset of $[0, 1]^2$ must intersect each element of this collection. Consequently no countable subset of $[0, 1]^2$ can be dense. Thus, $[0, 1]^2$ is not separable. It follows by Theorem I.5 that $[0, 1]^2$ is not second countable.

On the other hand, $[0, 1]^2$ with the lexicographic order topology is first countable. To prove this, we will describe a countable basis at each point of $[0, 1]^2$. There are five different types of points in $[0, 1]^2$, and the description of the countable basis at a point depends on the type of the point. First consider a point of the type $(x, y) \in [0, 1]^2$ where $x \in [0, 1]$ and $0 < y < 1$. At such a point, the countable collection of open intervals

$$\{ ((x, u), (x, v)) : \ 0 < u < y < v < 1 \text{ and } u, v \in \mathbb{Q} \}$$

forms a basis. We leave it to the student to verify that any open subset of $[0, 1]^2$ that contains (x, y) also contains an interval from this collection. Second we consider a point of the type $(x, 0) \in [0, 1]^2$ where $x \in (0, 1]$. At such a point, the countable collection of open intervals

$$\{ ((u, 1), (x, v)) : \ 0 < u < x, 0 < v < 1 \text{ and } u, v \in \mathbb{Q} \}$$

forms a basis. Again, we leave it to the student to verify that any open subset of $[0, 1]^2$ that contains $(x, 0)$ also contains an interval from this collection. Similarly, at a point of the type $(x, 1) \in [0, 1]^2$ where $x \in [0, 1)$, the countable collection of open intervals

$$\{ ((x, u), (v, 0)) : \ 0 < u < 1, x < v < 1 \text{ and } u, v \in \mathbb{Q} \}$$

forms a basis. (Again the student should verify this assertion.) At the point $(0, 0) \in [0, 1]^2$, the countable collection of open intervals

$$\{ (-\infty, (0, v)) : 0 < v < 1 \text{ and } v \in \mathbb{Q} \}$$

forms a basis; and at the point $(1, 1) \in [0, 1]^2$, the countable collection of open intervals

$$\{ ((1, v), \infty) : 0 < v < 1 \text{ and } v \in \mathbb{Q} \}$$

forms a basis. (Again verify these assertions.) We conclude that there is a countable basis at every point of $[0, 1]^2$. Hence, $[0, 1]^2$ with the lexicographic order topology is first countable.

Definition. Let $(X, <)$ be a linearly ordered set. Let $Y \subset X$ and let $x \in X$. x is the *minimum* or *least element* of Y (abbreviated $x = \min(Y)$) if $x \in Y$ and $x \leq y$ for every $y \in Y$. x is the *maximum* or *greatest element* of Y (abbreviated $x = \max(Y)$) if $x \in Y$ and $y \leq x$ for every $y \in Y$. If every non-empty subset of X has a least element, then we say that $<$ *well orders* X, we call $<$ a *well ordering* of X, and we call $(X, <)$ a *well ordered set.*

Observe that $\mathbb{N} = \{ 1, 2, 3, \cdots \}$ is well ordered by its natural linear order. However, $\mathbb{Z} = \{ \cdots , -2, -1, 0, 1, 2, \cdots \}$, $\mathbb{Q} = \{ {}^m/_n : m, n \in \mathbb{Z} \text{ and } n \neq 0 \}$, and $\mathbb{R}$ are *not* well ordered by their natural linear orders.

The question of whether the set of real numbers $\mathbb{R}$ admits a (non-natural) well-ordering, or more generally, whether every non-empty set can be well-ordered is the subject of the following set-theoretic principle.

Zermelo's Well Ordering Principle. Every non-empty set can be well ordered.

The Well Ordering Principle is one of the powerful set theoretic principles mentioned in Chapter 0, Section B that is logically equivalent to the Axiom of Choice. In other words, the Well Ordering Principle can be proved from the Axiom of Choice and vice versa. At first glance, this equivalence may seem remarkable because the two propositions seem unrelated. As was noted in Chapter 0, these principles are not obviously self evident. In fact, neither these principles nor their negations can be proved from more self evident set theoretic propositions. None the less, they are considered to be true statements which can be used in a proof when no other approach to the proof can be found. In such a case, however, the dependence of the proof on the Axiom of Choice or the Well Ordering Principle is usually remarked upon.

Example I.10. There is a well ordered set $(\Omega, <)$ such that Ω is uncountable, but $(-\infty, x)$ is countable for every $x \in \Omega$. We give Ω the order topology.

The construction of Ω. The construction begins with an uncountable well ordered set $(X, <)$. To obtain X quickly, we simply use Zermelo's Well Ordering Principle to well order $\mathbb{R}$. (It is, in fact, possible to prove the existence of an uncountable well ordered set without invoking Zermelo's Principle or any other equivalent of the Axiom of Choice. However, for the sake of brevity, we avoid this approach.) Next let $Y = \{ x \in X : (-\infty, x) \text{ is uncountable} \}$. If $Y = \varnothing$, set $\Omega = X$. On the other hand, if $Y \neq \varnothing$, then Y has a least element y_0, because X is well ordered. In this case, set $\Omega = (-\infty, y_0)$. In either case, it is easily verified that Ω is an uncountable set such that $(-\infty, x)$ is countable for every $x \in \Omega$. To obtain a well ordering of Ω, restrict the well ordering $<$ on X to Ω; i.e., replace $<$ by $< \cap (\Omega \times \Omega)$. Then, clearly, $(\Omega, <)$ is a well-ordered set. ◻

We establish two useful properties of Ω for future reference.

Lemma I.6. a) For each $x \in \Omega$, there is an $x^+ \in \Omega$ such that $x < x^+$ and there is no $y \in \Omega$ such that $x < y < x^+$. x^+ is called an *immediate successor* of x.

b) If A is a countable subset of Ω, then there is an $x \in \Omega$ such that $A \subset (-\infty, x)$.

Proof of a). Let $x \in \Omega$. Since $(-\infty, x] = (-\infty, x) \cup \{ x \}$ is a countable set and Ω is an uncountable set, then $(x, \infty) = \Omega - (-\infty, x]$ is non-empty. Since Ω is well ordered, then (x, ∞) has a least element which we denote x^+. Then clearly $x < x^+$ and there is no $y \in \Omega$ such that $x < y < x^+$.

Proof of b). Let A be a countable subset of Ω. Then $\bigcup_{a \in A} (-\infty, a]$ is the union of a countable collection of countable sets. Hence, $\bigcup_{a \in A} (-\infty, a]$ is itself countable by Theorem 0.19. Since Ω is uncountable , then $\Omega - (\bigcup_{a \in A} (-\infty, a])$ is non-empty. Choose $x \in \Omega - (\bigcup_{a \in A} (-\infty, a])$. Then $a \in A \Rightarrow x \notin (-\infty, a] \Rightarrow a \in (-\infty, x)$. This proves $A \subset (-\infty, x)$. ☐

Lemma I.6 helps us establish that Ω is first countable but neither second countable nor separable. To see that Ω isn't separable, suppose that D is a countable subset of Ω. Then Lemma I.6 b) provides an $x \in \Omega$ such that $D \subset (-\infty, x)$. Since $x^+ \in (x, \infty)$ by Lemma I.6 a), then (x,∞) is a non-empty open subset of Ω that is disjoint from D. Hence, D isn't a dense subset of Ω. Thus, no countable subset of Ω is dense. Consequently, Ω is not separable. It then follows by Theorem I.5 that Ω is not second countable. Let $x \in \Omega$. To prove Ω is first countable at x, we must consider two cases. If $x = \min(\Omega)$, then $\{ (-\infty, x^+) \}$ is a one–element basis at x. On the other hand, if $x > \min(\Omega)$, then $\{ (y, x^+) : y \in (-\infty, x) \}$ is a countable basis at x. (The student should verify these assertions.) Hence, Ω is first countable.

Example I.11. Let ω^+ be a point not in Ω. Set $\Omega^+ = \Omega \cup \{ \omega^+ \}$. Extend $<$ to a well ordering of Ω^+ by declaring that $x < \omega^+$ for every $x \in \Omega$. We give Ω^+ the order topology.

Problem I.3(11). Decide whether the space Ω^+ describe in Example I.11 is
a) second countable,
b) first countable,
c) separable.

Definition. Let (X, <) be a linearly ordered set. Let $Y \subset X$ and let $x \in X$.

x is a *lower bound* of Y if $x \leq y$ for every $y \in Y$.

x is an *upper bound* of Y if $y \leq x$ for every $y \in Y$.

Y is *bounded below* if it has a lower bound.

Y is *bounded above* if it has an upper bound.

Y is *bounded* if it has both a lower bound and an upper bound.

x is the *greatest lower bound* or *infimum* of Y (abbreviated x = inf(Y)) if x is a lower bound of Y and if $x' \leq x$ for every other lower bound x' of Y.

x is a *least upper bound* or *supremum* of Y (abbreviated x = sup(Y)) if x is an upper bound of Y and if $x \leq x'$ for every other upper bound x' of Y.

We call < a *complete* linear order on X, and we call (X, <) a *complete* linearly ordered set if every non-empty subset of X which is bounded below has a greatest lower bound.

Observe that the natural linear orders on $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ are complete; but the natural linear order on $\mathbb{Q}$ is not complete.

We chose to define completeness in terms of lower bounds rather than upper bounds. The following lemma shows that it doesn't matter which choice we make.

Lemma I.7. A linearly ordered set (X, <) is complete if and only if every non-empty subset of X which is bounded above has a least upper bound.

Proof. Let us define a linearly ordered set (X, <) to be *L-complete* if every non-empty subset of X which is bounded below has a greatest lower bound, and define (X, <) to be *U-complete* if every non-empty subset of X which is bounded above has a least upper bound. Thus "L-complete" is the same as "complete", and we must prove that (X, <) is L-complete if and only if it is U-complete.

Aside. In the special case that X is $\mathbb{R}$ with the natural linear ordering, the proof is simpler than in the general case. The reason is that $\mathbb{R}$ has an order reversing bijection: multiplication by –1. (In general, linearly ordered sets don't have order reversing bijections; for example, $\mathbb{N} = \{ 1, 2, 3, \cdots \}$ has none.) In this special case, the proof goes as follows. Assume $\mathbb{R}$ is L-complete. To prove it is U-complete, let A be a non-empty subset of $\mathbb{R}$ that is bounded above. Then $-A = \{ -x : x \in A \}$ is a non-empty subset of $\mathbb{R}$ that is bounded below. (If u is an upper bound of A, then –u is a lower bound of –A.) Since $\mathbb{R}$ is L-complete, then –A has a greatest lower bound b. It then follows that –b is a least upper bound of A. (Verify this.) This proves $\mathbb{R}$ is U-complete. The proof that U-completeness implies L-completeness in this special case is similar.

Back to the proof in the general case. Assume (X, <) is L-complete. To prove (X, <) is U-complete, let A be a non-empty subset of X that is bounded above. Let B be the (non-empty) set of all upper bounds of A. Then every element of A is a lower bound of B. (Verify this.) Hence, B is a non-empty subset of X that is bounded below. Since X is L-complete, then B has a greatest lower bound c. Since every element of A is a lower bound of B and c is the *greatest* lower bound of B, then c is an upper bound of A. Since c is a lower bound of B and B is the set of all upper bounds of A, then c is ≤ every upper bound of A. We conclude that c is a least upper bound of A. This proves (X, <) is U-complete.

Exercise. Finish this proof by showing that if a linearly ordered set is U-complete, then it is L-complete. □

Problem I.4. Assume that (X, <) is a complete linearly ordered set. Prove that if $\{ I_n : n \in \mathbb{N} \}$ is a collection of non-empty bounded closed intervals in X such that $I_1 \supset I_2 \supset I_3 \supset \ldots$, then $\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$.

Well ordered sets are complete. This is because in a well ordered set, every non-empty subset has a least element, and this least element is the greatest lower bound of the subset. Hence, the well ordered sets Ω and Ω^+ are complete.

Problem I.5. Is $[0, 1]^2$ with the lexicographic order topology (described in Example I.9) complete?

D. Metric Spaces

Metric spaces form perhaps the most useful and important class of topological spaces. If a space admits a *metric* or distance function, then geometric intuition can be applied to analyze the space. Sets of functions frequently have natural metrics that allowgeometric concepts to illuminate their structure. This observation plays a fundamental and crucial role in analysis.

Definition. A *metric* on a set X is a function $\rho : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$:

a) $\rho(x,y) = 0 \Leftrightarrow x = y$,

b) $\rho(x,y) = \rho(y,x)$, and

c) the triangle inequality: $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$.

Definition. If ρ is a metric on a set X, then the pair (X, ρ) is called a *metric space.*

Definition. Let (X, ρ) be a metric space. For $x \in X$ and $\varepsilon > 0$, the *ε–neighborhood of x in X* is the set $N(x,\varepsilon) = N_\rho(x,\varepsilon) = \{ y \in X : \rho(x,y) < \varepsilon \}$.

Lemma I.8. Let (X, ρ) be a metric space. If $x, y \in X$ such that $\rho(x,y) < \varepsilon$ and $0 < \delta \leq \varepsilon - \rho(x,y)$, then $N(y,\delta) \subset N(x,\varepsilon)$.

Proof. By hypothesis, $\rho(x,y) + \delta \leq \varepsilon$. Consequently,
$z \in N(y,\delta) \Rightarrow \rho(x,z) \leq \rho(x,y) + \rho(y,z) < \rho(x,y) + \delta \leq \varepsilon \Rightarrow z \in N(x,\varepsilon)$.
This proves $N(y,\delta) \subset N(x,\varepsilon)$. □

Theorem I.9. If (X, ρ) is a metric space, then $\{ N(x,\varepsilon) : x \in X \text{ and } \varepsilon > 0 \}$ is a basis for a topology on X.

Proof. We will verify that $\{ N(x,\varepsilon) : x \in X \text{ and } \varepsilon > 0 \}$ satisfies the criterion for a basis expressed in Theorem I.2.

First, $\{ N(x,\varepsilon) : x \in X \text{ and } \varepsilon > 0 \}$ covers X because $x \in N(x,1)$ for each $x \in X$.

Second, if $z \in N(x,\delta) \cap N(y,\varepsilon)$, set $\gamma = \min \{ \delta - \rho(x,z), \varepsilon - \rho(y,z) \}$. Then $\gamma > 0$ and Lemma I.8 implies $z \in N(z,\gamma) \subset N(x,\delta) \cap N(y,\varepsilon)$. □

Definition. Let (X, ρ) be a metric space. The topology on X with basis $\{ N(x,\varepsilon) : x \in X \text{ and } \varepsilon > 0 \}$ is called the *metric topology* on X or the *topology on* X *induced by* the metric ρ.

Theorem I.10. Let (X, ρ) be a metric space. Then for every $x \in X$, $\{ N(x,{}^1/_n) : n \in \mathbb{N} \}$ is a countable basis for the metric topology at x.

Proof. Let $x \in X$. Let U be an open subset of X such that $x \in U$. Then $x \in N(y,\varepsilon) \subset U$ for some $y \in X$ and $\varepsilon > 0$. There is an $n \in \mathbb{N}$ such that ${}^1/_n \leq \varepsilon - \rho(x,y)$. Then Lemma I.8 implies $N(x,{}^1/_n) \subset N(y,\varepsilon)$. So $x \in N(x,{}^1/_n) \subset U$. Consequently, $\{ N(x,{}^1/_n) : n \in \mathbb{N} \}$ is a basis for the metric topology at x. □

Corollary. Every metric space is first countable. □

Definition. A topological space X is *metrizable* if there is a metric on X which induces the given topology.

Example I.12. The *discrete metric* on a set X is defined by

$$\rho(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Exercise. Verify that ρ is a metric.

Observe that the discrete metric on a set X induces the discrete topology on X. Indeed, for each $x \in X$, $N(x,1) = \{ x \}$; so $\{ x \}$ is an open set. It follows that every subset of X is open with respect to the discrete metric. In other words, the discrete metric induces the discrete topology. Consequently every topological space with the discrete topology is metrizable.

An important class of metric spaces. Normed vector spaces (which are defined in Section 0.D) form an important class of metric spaces. If $(V, \| \ \|)$ is a normed vector space, then a metric ρ is defined on V by the formula

$$\rho(v,w) = \| v - w \|$$

for $v, w \in V$.

We verify that this formula defines a metric on V.

a) $\rho(v,w) = 0 \Leftrightarrow \| v - w \| = 0 \Leftrightarrow v - w = 0 \Leftrightarrow v = w$.

b) $\rho(v,w) = \| v - w \| = \| (-1) (w - v) \| = |-1| \| w - v \| = \| w - v \| = \rho(w,v)$.

c) $\rho(u,w) = \| u - w \| = \| (u - v) + (v - w) \| \leq \| u - v \| + \| v - w \| = \rho(u,v) + \rho(v,w)$. □

Definition. If $(V, \| \ \|)$ is a normed vector space, then the topology on V induced by the metric $\rho(v,w) = \| v - w \|$ is called the *norm topology* on V or the *topology* on V *induced by* the norm $\| \ \|$.

Example I.13. The *standard metric* on $\mathbb{R}$ is defined by

$$\rho(x,y) = | x - y | \quad \text{for } x, y \in \mathbb{R}.$$

Since $(\mathbb{R}, | \ |)$ is a normed vector space, it follows that ρ is a metric on $\mathbb{R}$. Moreover, ρ induces the standard topology on $\mathbb{R}$.

We verify that the standard metric ρ on $\mathbb{R}$ induces the standard topology on $\mathbb{R}$. First we note that every ε–neighborhood in $\mathbb{R}$ is a bounded open interval. Indeed, for $x \in \mathbb{R}$ and $\varepsilon > 0$, $N(x,\varepsilon) = (x - \varepsilon, x + \varepsilon)$. Second we note that every bounded open interval in $\mathbb{R}$ is an ε–neighborhood of some point. Indeed, for $a < b$ in $\mathbb{R}$, if we let $x = {}^{(a+b)}/_2$ and $\varepsilon = {}^{(b-a)}/_2$, then $(a, b) = N(x,\varepsilon)$. Thus, the basis for the metric topology on $\mathbb{R}$ is identical with the basis of the standard topology on $\mathbb{R}$. Consequently, the metric topology on $\mathbb{R}$ and the standard topology on $\mathbb{R}$ are equal. It follows that $\mathbb{R}$ (with the standard topology) is metrizable. □

Example I.14. For $n \geq 1$, we define three metrics on $\mathbb{R}^n$.

a) The *taxicab* or *1-metric:* $\rho_1(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} | x_i - y_i |$.

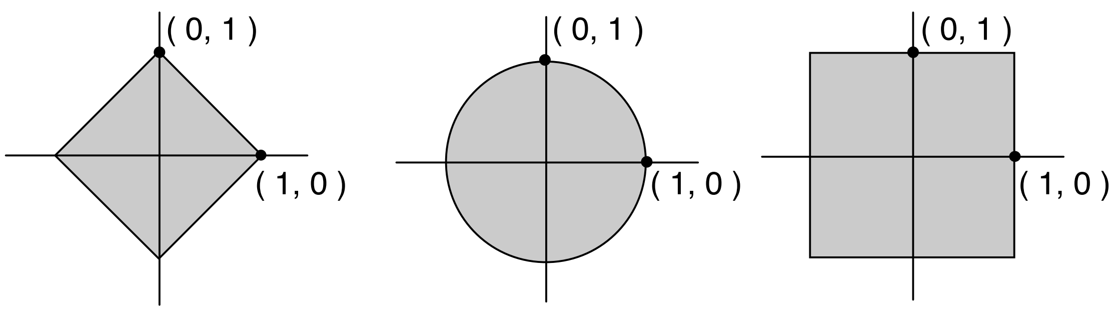
b) The *Euclidean* or *2-metric:* $\rho_2(\mathbf{x},\mathbf{y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$.

c) The *supremum* or *∞-metric:* $\rho_\infty(\mathbf{x},\mathbf{y}) = \max \{ | x_i - y_i | : 1 \leq i \leq n \}$.

Here, $\mathbf{x} = (x_1, x_2, \ldots , x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots , y_n) \in \mathbb{R}^n$.

Exercise. Verify that these three formulas actually define metrics on $\mathbb{R}^n$. (Observe that these metrics are directly related to the three norms on $\mathbb{R}^n$ defined in Section 0.D. Use the information in Section 0.D about norms and inner products on vector spaces to help with these verifications.)

Here are pictures of $N_{\rho_1}((0,0),1)$, $N_{\rho_2}((0,0),1)$ and $N_{\rho_\infty}((0,0),1)$ in $\mathbb{R}^2$.



$N_{\rho_1}((0,0),1)$ $N_{\rho_2}((0,0),1)$ $N_{\rho_\infty}((0,0),1)$

Definition. Two metrics on a set X are *equivalent* if they induce the same topology on X.

Theorem I.11. Two metrics ρ and σ on a set X are equivalent if and only if $\forall x \in X$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $N_\rho(x,\delta) \subset N_\sigma(x,\varepsilon)$ and $N_\sigma(x,\delta) \subset N_\rho(x,\varepsilon)$.

Proof. First assume that the metrics ρ and σ on the set X are equivalent. Let $x \in X$ and let $\varepsilon > 0$. Since ρ and σ are equivalent metrics, then $N_\sigma(x,\varepsilon)$ is an open subset of X with respect to the topology induced by ρ. Since $x \in N_\sigma(x,\varepsilon)$, then Theorem I.10 implies there is a $\delta' > 0$ such that $N_\rho(x,\delta') \subset N_\sigma(x,\varepsilon)$. Similarly, since ρ and σ are equivalent metrics, then $N_\rho(x,\varepsilon)$ is an open subset of X with respect to the topology induced by σ. Since $x \in N_\rho(x,\varepsilon)$, then Theorem I.10 implies there is a $\delta'' > 0$ such that $N_\sigma(x,\delta'') \subset N_\rho(x,\varepsilon)$. Now set $\delta = \min \{ \delta', \delta'' \}$. Then $N_\rho(x,\delta) \subset N_\rho(x,\delta') \subset N_\sigma(x,\varepsilon)$ and $N_\sigma(x,\delta) \subset N_\sigma(x,\delta'') \subset N_\rho(x,\varepsilon)$.

Second assume that $\forall$ x $\in$ X, $\forall$ $\varepsilon > 0$, $\exists$ $\delta > 0$ such that $N_\rho(x,\delta) \subset N_\sigma(x,\varepsilon)$ and $N_\sigma(x,\delta) \subset N_\rho(x,\varepsilon)$. Let $\mathscr{B}_\rho = \{ N_\rho(x,\varepsilon) : x \in X \text{ and } \varepsilon > 0 \}$ and let $\mathscr{B}_\sigma = \{ N_\sigma(x,\varepsilon) : x \in X \text{ and } \varepsilon > 0 \}$. Then $\mathscr{B}_\rho$ is a basis for the topology on X induced by ρ, and $\mathscr{B}_\sigma$ is a basis for the topology on X induced by σ. We will rely on the Corollary to Theorem I.3 to establish that $\mathscr{B}_\rho$ and $\mathscr{B}_\sigma$ generate the same topology on X. To this end, let $x \in B_1 \in \mathscr{B}_\rho$. Then $B_1 = N_\rho(y,\gamma)$ for some $y \in X$ and some $\gamma > 0$. Let $\varepsilon = \gamma - \rho(x,y)$. Then $\varepsilon > 0$ and $N_\rho(x,\varepsilon) \subset N_\rho(y,\gamma)$ by Lemma I.8. Now, by hypothesis, there is a , $\exists$ $\delta > 0$ such that $N_\sigma(x,\delta) \subset N_\rho(x,\varepsilon)$. Let $B_2 = N_\sigma(x,\delta)$. Then $B_2 \in \mathscr{B}_\sigma$ and $x \in B_2 = N_\sigma(x,\delta) \subset N_\rho(x,\varepsilon) \subset N_\rho(y,\gamma) = B_1$. This completes half of the proof. The other half is similar. Let $x \in B_2 \in \mathscr{B}_\sigma$. Then $B_2 = N_\sigma(y,\gamma)$ for some $y \in X$ and some $\gamma > 0$. Let $\varepsilon = \gamma - \sigma(x,y)$. Then $\varepsilon > 0$ and $N_\sigma(x,\varepsilon) \subset N_\sigma(y,\gamma)$ by Lemma I.8. Now, by hypothesis, there is a , $\exists$ $\delta > 0$ such that $N_\rho(x,\delta) \subset N_\sigma(x,\varepsilon)$. Let $B_1 = N_\rho(x,\delta)$. Then $B_1 \in \mathscr{B}_\rho$ and $x \in B_1 = N_\rho(x,\delta) \subset N_\sigma(x,\varepsilon) \subset N_\sigma(y,\gamma) = B_2$. This completes the second half of the proof. At this point, the Corollary to Theorem I.3 tells us that $\mathscr{B}_\rho$ and $\mathscr{B}_\sigma$ generate the same topology on X. In other words, the metrics ρ and σ induce the same topology on X. ◻

The taxicab, Euclidean and supremum metrics on $\mathbb{R}^n$ are equivalent metrics that induce the standard topology on $\mathbb{R}^n$. Thus, $\mathbb{R}^n$ with the standard topology is metrizable. We will verify part of this assertion and leave part of it as a problem for students.

We first show that the taxicab and supremum metrics on $\mathbb{R}^n$ are equivalent. To begin, we observe that for $\mathbf{x} = (x_1, x_2, \ldots , x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots , y_n) \in \mathbb{R}^n$,

$$\rho_\infty(\mathbf{x},\mathbf{y}) \leq \rho_1(\mathbf{x},\mathbf{y}) \leq n\rho_\infty(\mathbf{x},\mathbf{y}).$$

We claim these inequalities are obvious consequences of the definitions of ρ_1 and ρ_∞ and say no more about it. It follows that for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\varepsilon > 0$, $\rho_\infty(\mathbf{x},\mathbf{y}) < \varepsilon/n \Rightarrow \rho_1(\mathbf{x},\mathbf{y}) < \varepsilon$ and $\rho_1(\mathbf{x},\mathbf{y}) < \varepsilon \Rightarrow \rho_\infty(\mathbf{x},\mathbf{y}) < \varepsilon$. Hence, for each $\mathbf{x} \in \mathbb{R}^n$ and each $\varepsilon > 0$, $N_{\rho_\infty}(\mathbf{x},\varepsilon/n) \subset N_{\rho_1}(\mathbf{x},\varepsilon)$ and $N_{\rho_1}(\mathbf{x},\varepsilon/n) \subset N_{\rho_\infty}(\mathbf{x},\varepsilon)$. It now follows from Theorem I.11 that the taxicab metric ρ_1 and the supremum metric ρ_∞ are equivalent.

Problem I.6. Prove that the taxicab metric ρ_1 and the Euclidean metric ρ_2 are equivalent metrics on $\mathbb{R}^n$.

Now we show that the supremum metric induces the standard topology on $\mathbb{R}^n$. Let $\mathscr{B}_{\rho_\infty} = \{ N_{\rho_\infty}(\mathbf{x},\varepsilon) : \mathbf{x} \in \mathbb{R}^n \text{ and } \varepsilon > 0 \}$ and let $\mathscr{B}_{box}$ denote the collection of all open boxes in $\mathbb{R}^n$. Then $\mathscr{B}_{\rho_\infty}$ is a basis for the topology induced on $\mathbb{R}^n$ by the supremum metric ρ_∞, and $\mathscr{B}_{box}$ is a basis for the standard topology on $\mathbb{R}^n$. We will use the Corollary to Theorem I.3 to show that $\mathscr{B}_{\rho_\infty}$ and $\mathscr{B}_{box}$ generate the same topology on $\mathbb{R}^n$. First suppose $\mathbf{x} \in B_1 \in \mathscr{B}_{\rho_\infty}$. Then there is a $\mathbf{y} = (y_1, y_2, \ldots , y_n) \in \mathbb{R}^n$ and an $\varepsilon > 0$ such that $B_1 = N_{\rho_\infty}(\mathbf{y},\varepsilon)$. Observe that

$\mathbf{z} = (z_1, z_2, \ldots , z_n) \in B_1 = N_{\rho_\infty}(\mathbf{y},\varepsilon) \Leftrightarrow \rho_\infty(\mathbf{y},\mathbf{z}) < \varepsilon \Leftrightarrow$

$| y_i - z_i | < \varepsilon$ for $1 \le i \le n \Leftrightarrow z_i \in (y_i - \varepsilon, y_i + \varepsilon)$ for $1 \le i \le n \Leftrightarrow$

$\mathbf{z} \in (y_1 - \varepsilon, y_1 + \varepsilon) \times (y_2 - \varepsilon, y_2 + \varepsilon) \times \ldots \times (y_n - \varepsilon, y_n + \varepsilon)$.

Thus, $B_1 = (y_1 - \varepsilon, y_1 + \varepsilon) \times (y_2 - \varepsilon, y_2 + \varepsilon) \times \ldots \times (y_n - \varepsilon, y_n + \varepsilon)$. In other words, B_1 is an open box. So if we set $B_2 = B_1$, then $B_2 \in \mathscr{B}_{box}$ and $\mathbf{x} \in B_2 \subset B_1$. This completes half of the proof. Second suppose $\mathbf{x} = (x_1, x_2, \ldots , x_n) \in B_2 \in \mathscr{B}_{box}$. Since B_2 is an open box, then B_2 is a Cartesian product of open intervals. Therefore, for $1 \le i \le n$, there are real numbers $a_i < b_i$ such that $B_2 = (a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_n, b_n)$. Hence, $x_i \in (a_i, b_i)$ for $1 \le i \le n$. Let $\varepsilon = \min \{ x_1 - a_1, b_1 - x_1, x_2 - a_2, b_2 - x_2, \ldots , x_n - a_n, b_n - x_n \}$. Then $(x_i - \varepsilon, x_i + \varepsilon) \subset (a_i, b_i)$ for $1 \le i \le n$. Hence,

$$N_{\rho_\infty}(\mathbf{x},\varepsilon) = (x_1 - \varepsilon, x_1 + \varepsilon) \times (x_2 - \varepsilon, x_2 + \varepsilon) \times \ldots \times (x_n - \varepsilon, x_n + \varepsilon) \subset$$

$$(a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_n, b_n) = B_2.$$

Let $B_1 = N_{\rho_\infty}(\mathbf{x},\varepsilon)$. Then $B_1 \in \mathscr{B}_{\rho_\infty}$ and $\mathbf{x} \in B_1 \subset B_2$. This completes the second half of the proof. It now follows from the Corollary to Theorem I.3 that $\mathscr{B}_{\rho_\infty}$ and $\mathscr{B}_{box}$ generate the same topology on $\mathbb{R}^n$. Hence, the supremum metric ρ_∞ induces the standard topology on $\mathbb{R}^n$. Consequently, the standard topology on $\mathbb{R}^n$ is metrizable.

Definition. Let (X, ρ) be a metric space. Let $A \subset X$. The *diameter* of A is $\sup \{ \rho(x,y) : x, y \in A \} \in [0,\infty]$, and is denoted *diam(A)* or *diam*$_\rho$*(A)*. A is *bounded* if $\text{diam}(A) < \infty$. If X is bounded, we say that the metric ρ is *bounded.*

Theorem I.12. Let (X, ρ) be a metric space. Define $\bar{\rho} : X \times X \to [0,1]$ by

$$\bar{\rho}(x,y) = \min \{ \rho(x,y), 1 \}.$$

Then $\bar{\rho}$ is a bounded metric on X which is equivalent to ρ.

Proof. First we verify that $\bar{\rho}$ is a metric on X. Let x, y and $z \in X$. Clearly $\bar{\rho}$

a) $\bar{\rho}(x,y) = 0 \Leftrightarrow \min \{ \rho(x,y), 1 \} = 0 \Leftrightarrow \rho(x,y) = 0 \Leftrightarrow x = y$, and

b) $\bar{\rho}(x,y) = \min \{ \rho(x,y), 1 \} = \min \{ \rho(y,x), 1 \} = \bar{\rho}(y,x)$.

Next we prove the triangle inequality. First assume $\rho(x,y) \le 1$ and $\rho(y,z) \le 1$. Then $\bar{\rho}(x,y) = \rho(x,y)$ and $\bar{\rho}(y,z) = \rho(y,z)$. Hence,

$$\bar{\rho}(x,z) \le \rho(x,z) \le \rho(x,y) + \rho(y,z) = \bar{\rho}(x,y) + \bar{\rho}(y,z).$$

Second assume that either $\rho(x,y) > 1$ or $\rho(y,z) > 1$. Then $\bar{\rho}(x,y) = 1$ or $\bar{\rho}(y,z) = 1$. Hence, $\bar{\rho}(x,z) \le 1 \le \bar{\rho}(x,y) + \bar{\rho}(y,z)$. It follows that $\bar{\rho}$ satisfies the triangle inequality and is, therefore, a metric on X.

Since $\bar{\rho}(x,y) \le 1$ for all x, y $\in$ X, then diam(X) $\le$ 1. Hence, $\bar{\rho}$ is a bounded metric on X.

Finally we rely on Theorem I.11 to show that $\bar{\rho}$ is equivalent to ρ. Let $\varepsilon > 0$. Set $\delta = \min\{\varepsilon, 1\}$. Let $x \in X$. Then:

$$y \in N_\rho(x,\delta) \quad \Rightarrow \quad \rho(x,y) < \delta \quad \Rightarrow \quad \rho(x,y) < \varepsilon$$

$$\Rightarrow \quad \bar{\rho}(x,y) < \varepsilon \text{ (because } \bar{\rho} \le \rho) \quad \Rightarrow \quad y \in N_{\bar{\rho}}(x,\varepsilon).$$

This proves $N_\rho(x,\delta) \subset N_{\bar{\rho}}(x,\varepsilon)$. Also:

$$y \in N_{\bar{\rho}}(x,\delta) \quad \Rightarrow \quad \bar{\rho}(x,y) < \delta \quad \Rightarrow \quad \bar{\rho}(x,y) < 1 \quad \Rightarrow \quad \bar{\rho}(x,y) = \rho(x,y)$$

$$\Rightarrow \quad \rho(x,y) < \delta \quad \Rightarrow \quad \rho(x,y) < \varepsilon \quad \Rightarrow \quad y \in N_\rho(x,\varepsilon).$$

This proves $N_{\bar{\rho}}(x,\delta) \subset N_\rho(x,\varepsilon)$. Now the equivalence of the metrics ρ and $\bar{\rho}$ follows by Theorem I.11. □

Definition. Let (Y, ρ) be a metric space. A function f : X → Y is *bounded* if f(X) is a bounded subset of Y (i.e., if diam(f(X)) < ∞).

Example I.15. Let X be a set, let $\mathbb{R}$ have the standard metric, and let B(X) denote the set of all bounded functions from X to $\mathbb{R}$. Define the *supremum metric* σ on B(X) by the formula

$$\sigma(f,g) = \sup\{\, | f(x) - g(x) | : x \in X \,\}$$

for f, g $\in$ B(X).

We verify that the supremum metric σ is indeed a metric on B(X).

First we show that $\sigma(f,g) < \infty$ for all f, g $\in$ B(X). Let f, g $\in$ B(X) and fix $x_0 \in X$. Then for all $x \in X$,

$$| f(x) - g(x) | \le | f(x) - f(x_0) | + | f(x_0) - g(x_0) | + | g(x_0) - g(x) | \le$$

$$\operatorname{diam}(f(X)) + | f(x_0) - g(x_0) | + \operatorname{diam}(g(X)).$$

Thus, $\sigma(f,g) \le \operatorname{diam}(f(X)) + | f(x_0) - g(x_0) | + \operatorname{diam}(g(X)) < \infty$.

Next we verify that σ satisfies the three defining conditions for a metric. Let f, g and h $\in$ B(X). Then:

- $\sigma(f,g) = 0 \iff | f(x) - g(x) | = 0$ for all $x \in X \iff f = g$.
- $\sigma(f,g) = \sup\{\, | f(x) - g(x) | : x \in X \,\} = \sup\{\, | g(x) - f(x) | : x \in X \,\} = \sigma(g,f)$.

- $\sigma(f,h) = \sup\{ | f(x) - h(x) | : x \in X \} \le \sup\{ | f(x) - g(x) | + | g(x) - h(x) | : x \in X \} \le$ $\sup\{ | f(x) - g(x) | : x \in X \} + \sup\{ | g(x) - h(x) | : x \in X \} = \sigma(f,g) + \sigma(g,h)$.

This finishes the proof that σ is a metric on B(X).

The Corollary to Theorem I.10 tells us that every metric space is first countable. In general, second countable topological spaces are separable but not vice versa. Metric spaces need not be separable or second countable. However, for metric spaces, separability and second countability are equivalent.

Theorem I.13. A metric space is second countable if and only if it is separable.

Remark. Theorem I.5 tells us that every second countable topological space is separable. So it remains to prove only that every separable metric space is second countable.

Problem I.7. Prove that every separable metric space is second countable.

Problem I.3(15). Decide whether or under what conditions the metric space B(X) described in Example I.15 is
a) second countable,
b) first countable,
c) separable.

We now discuss the question of which of the spaces described in Examples I.1 through I.15 are metrizable. Some of these spaces can be seen to be non-metrizable by using the following simple observation: if x and y are distinct points of a metric space (X, ρ), then there are disjoint open subsets U and V of X such that $x \in U$ and $y \in V$.

[**Proof.** Assume x and y are distinct points of a metric space (X, ρ). Let $\delta = (1/2)\rho(x,y)$, let $U = N(x,\delta)$ and $V = N(y,\delta)$. Then U and V are open subsets of X such that $x \in U$ and $y \in V$. We assert that U and V are disjoint. Suppose not. Then there is a point $z \in U \cap V$. Hence, $\rho(x,z) < \delta$ and $\rho(y,z) < \delta$. Therefore, $2\delta = \rho(x,y) \le \rho(x,z) + \rho(z,y) < 2\delta$. Since $2\delta < 2\delta$ is false, we have reached a contradiction. We must conclude that $U \cap V = \varnothing$. $\square$]

Next consider Example I.1: the set X with the indiscrete topology. If $X = \{ x \}$, a one-point space, then X is metrizable by the metric ρ which is defined by $\rho(x,x) = 0$. However, if X has more than one point and is endowed with the indiscrete topology, then the preceding observation implies that X is non-metrizable because distinct points in X do not lie in disjoint open sets.

A set X with the discrete topology (Example I.2) is always metrizable by the discrete metric. (See the remarks accompanying Example I.12.)

The three–point space described in Example I.3 is not metrizable because the distinct points x and z do not lie in disjoint open sets.

Problem I.8(4). Decide whether or under what conditions the space described in Example I.4 (the set X with the finite complement topology) is metrizable.

We saw that the space $\mathbb{R}$ of real numbers with the standard topology described in Example I.5 is metrizable using the standard metric (Example I.13). Similarly, then space $\mathbb{R}^n$ with the standard topology described in Example I.6 is metrizable using either the taxicab metric, the Euclidean metric or the supremum metric (Example I.14).

Problem I.8(7). Decide whether the space $\mathbb{R}_{bad}$ described in Example I.7 is metrizable.

The space described in Example I.8 is not first countable. Hence, it is not metrizable by Theorem I.13.

Problem I.8(9). Decide whether the space $[\ 0, 1\ ]^2$ with the lexicographic order topology described in Example I.9 is metrizable.

Problem I.8(10). Decide whether the space Ω described in Example I.10 is metrizable.

The space Ω^+ described in Example I.11 does not have a countable basis at the point ω^+. Hence, it is not metrizable by Theorem I.13.

The spaces described in Examples I.12 through I.15 are all metric spaces by definition.