RIGID 3-DIMENSIONAL COMPACTA WHOSE SQUARES ARE MANIFOLDS

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ABSTRACT. A space is *rigid* if its only self-homeomorphism is the identity. In response to a question of Jan van Mill, Ancel and Singh have given examples of rigid *n*-dimensional compacta, for each $n \ge 4$, whose squares are manifolds. We construct a rigid 3-dimensional compactum whose square is the manifold $S^3 \times S^3$. In fact, we construct uncountably many topologically distinct compacta with these properties.

1. Introduction. A space is *rigid* if it does not have any self-homeomorphism other than the identity. In response to a question of Arhangel'skii [2], Jan van Mill constructed a rigid *infinite dimensional* compactum X whose square $X \times X$ is homogeneous [7]. This result led van Mill to raise the following question (see [7]): Does there exist a rigid finite dimensional compactum whose square is homogeneous or perhaps even a (Lie) group? An affirmative answer to this question is provided in [1], where the following result is proved. For each integer $n \ge 4$, there exists a rigid n-dimensional compactum X is a closed 2n-manifold (and hence homogeneous). Moreover $X \times X$ can be chosen to be a Lie group. There are uncountably many topologically distinct compacta X with these properties.

The purpose of this note is to prove a similar result in the dimension, n = 3, which eluded [1]. As in [7] and [1], each of the spaces constructed here is rigid because it contains a countable dense set of points whose individual point complements are topologically distinct. Moreover, the point complements are distinguished by their proper homotopy type.

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2. The main result.

THEOREM. There is a null cell-like decomposition G of S^3 such that S^3/G is a rigid 3-dimensional compactum and $S^3/G \times S^3/G$ is homeomorphic to (the Lie group) $S^3 \times S^3$.

PROOF. We begin by recalling the construction in [6]. P is an uncountable index set whose elements are certain infinite sequences of primes. For each $p \in P$, W_p is a

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contractible open subset of $R^3 = S^3 - \{\infty\}$ such that for distinct elements p and q of P, W_p , W_q and R^3 are topologically distinct. Furthermore, for each $p \in P$, $W_p = \bigcup_{i=0}^{\infty} T_i$ such that for each $i \ge 0$:

- (1) T_i and $U_i = cl(S^3 T_i)$ are solid tori in S^3 ,
- (2) $T_i \subset int(T_{i+1})$, and
- (3) T_i contracts to a point in int (T_{i+1}) .

 $({T_i})$ and $\{U_i\}$ vary with the choice of $p \in P$.) We shall actually need to know that for distinct elements p and q of P, W_p , W_q and R^3 have distinct proper homotopy types; this follows from Corollary (2.6) of [4].

For each $p \in P$, let $C_p = S^3 - W_p$. The nondegenerate elements of the decomposition G are chosen from among the C_p 's. So we must verify that each C_p is cell-like. Let $p \in P$. Since $C_p = \bigcap_{i=0}^{\infty} U_i$, it suffices to show that U_{i+1} contracts to a point in U_i for each $i \ge 0$. Let $i \ge 0$. Since T_i contracts to a point in $\operatorname{int}(T_{i+1})$, the simple closed curve core J of T_i is null-homologous in the complement of the simple closed curve core K of U_{i+1} . From the symmetry of linking number, it follows that K is null-homologous in $S^3 - J$. Since $\pi_1(S^3 - J) \approx \pi_1(S^3 - U_i) = \pi_1(\operatorname{int} T_i) \approx \mathbb{Z}$, then the abelianization homomorphism $\pi_1(S^3 - J) \to H_1(S^3 - J)$ is an isomorphism. Therefore K is null-homotopic in $S^3 - J$. It follows that U_{i+1} contracts to a point in $S^3 - T_i = \operatorname{int}(U_i)$.

To specify the nondegenerate elements of G, choose a sequence p(1), p(2),... of distinct elements of P. Reposition $C_{p(1)}, C_{p(2)}, \ldots$ in S^3 (by homeomorphisms of S^3) so that

- (1) they are disjoint,
- (2) their union is dense in S^3 , and
- (3) diam $C_{p(i)} < 1/i$.

Let G be the null cell-like decomposition of S^3 whose nondegenerate elements are $C_{p(1)}, C_{p(2)}, \ldots$

A homeomorphism from the square $S^3/G \times S^3/G$ to $S^3 \times S^3$ is provided by [3].

It remains to show that S^3/G is 3 dimensional and rigid. Let $\pi: S^3 \to S^3/G$ denote the quotient map, and let $x_i = \pi(C_{p(i)})$ for each $i \ge 1$. Since the singular set $\{x_i: i \ge 1\}$ of π is countable, then π preserves dimension, and $\pi \mid \pi^{-1}V: \pi^{-1}V \to V$ is a proper homotopy equivalence for each open subset V of S^3/G (cf. [5]). Hence dim $S^3/G = 3$. Since for $i \ne j$ and $y \in S^3 - \bigcup_{k=1}^{\infty} C_{p(k)}, W_{p(i)} = S^3 - C_{p(i)}, W_{p(j)}$ $= S^3 - C_{p(j)}$ and $S^3 - \{y\}$ have distinct proper homotopy types, then so do their images under π . Thus for $i \ne j$ and $z \in S^3/G - \{x_i: i \ge 1\}, S^3/G - \{x_i\}, S^3/G - \{x_j\}$ and $S^3/G - \{z\}$ have distinct proper homotopy types. Therefore, no x_i can be moved by any homeomorphism of S^3/G . Since $\{x_i: i \ge 1\}$ is a dense subset of S^3/G , we conclude that S^3/G is rigid. \Box

COROLLARY. There are uncountably many distinct rigid 3-dimensional compacta whose squares are homeomorphic to (the Lie group) $S^3 \times S^3$.

The corollary follows immediately from the proof of the theorem.

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