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RIGID FINITE DIMENSIONAL COMPACTA WHOSE SQUARES ARE MANIFOLDS

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ABSTRACT. A space is *rigid* if its only self-homeomorphism is the identity. We answer questions of Jan van Mill by constructing for each n, $4 \leq n < \infty$, a rigid n-dimensional compactum whose square is homogeneous because it is a manifold. Moreover, for each n, $4 \leq n < \infty$, we give uncountably many topologically distinct such examples. Infinite-dimensional examples are also given.

1. Introduction. A space is *rigid* if its only self-homeomorphism is the identity. In [**vM**], Jan van Mill constructs a rigid infinite-dimensional compactum whose square is homogeneous (in fact, it is the Hilbert cube). He asks whether there exists a finite-dimensional rigid compactum whose square is homogeneous or even a topological group. The purpose of this note is to provide an affirmative answer to these questions.

As in [**vM**], the spaces constructed here are rigid because each contains a countable dense set of points whose individual point complements are topologically distinct. The authors arrived at these results independently by different methods. "The first method" due to the first author uses proper homotopy to distinguish point complements. "The second method" due to the second author uses homotopy (the fundamental group) to distinguish point complements. Since either method (or their combination) may be of some independent interest, we have treated them separately to emphasize this difference.

2. The first method.

THEOREM. *For each n, $4 \leq n < \infty$, there is a rigid n-dimensional compactum X such that $X \times X$ is homeomorphic to $S^n \times S^n$. (Hence, $X \times X$ is homogeneous.)*

PROOF. Following the strategy of [**vM**], we set $X = S^n/G$ where G is a cell-like upper semicontinuous decomposition of S^n whose nondegenerate elements form a null-sequence. Here, the nondegenerate elements of G form a null-sequence $\{M_i : i \geq 1\}$ of compact contractible n-manifolds with boundary satisfying:

(1) $\pi_1(\partial M_i) \not\approx \pi_1(\partial M_j) \not\approx \{1\}$ for $i \neq j$,

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(2) each ∂M_i is collared in $S^n - \operatorname{Int} M_i$, and

(3) $\bigcup_{i \geq 1} M_i$ is a dense subset of S^n.

We shall first argue that such an X is rigid, n-dimensional and that its square is homeomorphic to $S^n \times S^n$. Then we shall establish the existence of $\{M_i: i \geq 1\}$.

For each $i \geq 1$, the fundamental group of the end of $S^n - M_i$ is stable in the sense of **[S]** (hence, well defined) and is isomorphic to $\pi_1(\partial M_i)$. The fundamental group of the end of $S^n - \{y\}$ is stable and trivial for any y in S^n. It follows that $S^n - M_i$, $S^n - M_j$ and $S^n - \{y\}$ are proper homotopy inequivalent for $i \neq j$ and y in $S^n - (\bigcup_{i \geq 1} M_i)$.

Let $q: S^n \to X$ denote the quotient map, and let $x_i = q(M_i)$ for each $i \geq 1$. Since the singular set $\{x_i: i \geq 1\}$ of q is countable, well-known results for cell-like maps imply that q preserves dimension, and that q is a proper homotopy equivalence over each open subset of X. Hence, $\dim X = n$; and $X - \{x_i\}$, $X - \{x_j\}$ and $X - \{x\}$ are proper homotopy inequivalent for $i \neq j$ and x in $X - \{x_i: i \geq 1\}$. Thus every homeomorphism of X must fix each x_i. Since $\{x_i: i \geq 1\}$ is a dense subset of X, it follows that X is rigid.

The homeomorphism from $X \times X$ to $S^n \times S^n$ is an immediate consequence of **[B]**.

The sequence $\{M_i: i \geq 1\}$ is constructed from a single compact contractible n-manifold M_0 with nonsimply connected boundary, embedded in $\mathbf{R}^n$ so that ∂M_0 is collared in $\mathbf{R}^n - \operatorname{Int} M_0$. When $n = 4$, the example of Mazur suffices for M_0 (see **[Ma]** and **[Z]**). For each $n > 4$, **[K]** provides a compact contractible n-manifold M_0 with nonsimply connected boundary; and we embed M_0 in $\mathbf{R}^n$ with an exterior collar by observing, with the aid of **[Sm]**, that the double of M_0 is S^n.

Given M_0, one builds M_i in $\mathbf{R}^n$ by connecting i disjoint copies of M_0 via $i - 1$ tubes (regular neighborhoods of arcs). Then M_i is the boundary-connected sum of i copies of M_0, ∂M_i is collared in $\mathbf{R}^n - \operatorname{Int} M_i$, and $\pi_i(\partial M_i)$ is isomorphic to the free product of i copies of $\pi_i(\partial M_0)$. Since $\pi_1(\partial M_0)$ is finitely generated, Grushko's Theorem **[M**, p. 225] implies that $\pi_1(\partial M_i) \not\approx \pi_1(\partial M_j) \not\approx \{1\}$ for $i \neq j$. To embed M_i in S^n, one identifies $\mathbf{R}^n$ with an open cell in S^n which has been suitably positioned to insure that $\{M_i: i \geq 1\}$ is a null-sequence whose elements are disjoint and whose union is dense in S^n. $\square$

We make three observations about the proof.

First, the construction can be done with $\{M_i: i \geq 1\}$ replaced by any of its subsequences, and distinct subsequences give rise to nonhomeomorphic rigid spaces. Thus, in each finite dimension $n \geq 4$, there are uncountably many distinct n-dimensional rigid spaces whose squares are homeomorphic to $S^n \times S^n$.

Second, the construction can be modified to yield another example of the type produced in **[vM]**: a rigid space whose square is a Hilbert cube. The existence in each dimension ≥ 4 of the sequence $\{M_i: i \geq 1\}$ allows us to choose a sequence $\{N_k\}$ such that

(1) each N_k is a compact contractible k-manifold with boundary,

(2) $\pi_1(\partial N_k) \not\approx \pi_1(\partial N_j) \not\approx \{1\}$ for $k \neq j$, and

(3) N_k is embedded in $\operatorname{Int} I^k$ so that ∂N_k is collared in $I^k - \operatorname{Int} N_k$.

For each $k \geq 1$, let $\tilde{N}_k = N_k \times I \times I \times \cdots$; we regard $\tilde{N}_k$ as a subset of the Hilbert cube I^∞. Each N_k can be positioned in I^k so that $\{\tilde{N}_k: k \geq 1\}$ is a null-sequence whose elements are disjoint and whose union is dense in I^∞. Now in the above proof, if we replace S^n by I^∞, $\{M_i: i \geq 1\}$ by $\{\tilde{N}_k: k \geq 1\}$, and **[B]** by **[T]**, then we obtain a rigid infinite-dimensional compactum whose square is a Hilbert cube. As before, distinct subsequences of $\{\tilde{N}_k: k \geq 1\}$ give rise to uncountably many distinct rigid spaces of this type.

Third, in **[vM]**, van Mill also asks whether there is a rigid space whose square is a topological group. Clearly the preceding construction can be done with S^n replaced by the n-torus $T^n = (S^1)^n$. This leads to a rigid space whose square is the Lie group T^{2n}.

3. The second method. All spaces considered here will be at least separable metric and all manifolds will be without boundary. We now give a construction of rigid spaces based on the results proved in **[DS]**. A space X is a *generalized n-manifold* if X is a retract of an open subset of some Euclidean space $\mathbf{R}^m$ and the integral homology groups $H_i(X, X - \{x\})$ and $H_i(\mathbf{R}^n, \mathbf{R}^n - \{0\})$ are isomorphic for all $i \geq 0$ and all x in X. We now prove the following.

THEOREM. *For each simply connected and triangulable n-manifold M^n with $n \geq 5$, there exists an uncountable collection $C(M^n)$ of topologically distinct rigid generalized n-manifolds such that $X \times X$ is homeomorphic to $M^n \times M^n$ (hence, homogeneous) for every X in $C(M^n)$.*

PROOF. Suppose M^n is noncompact (the compact case is similar). Choose a triangulation T of M^n (we do not require T to be a combinatorial triangulation). Let $\sigma_1, \sigma_2, \ldots$ be an enumeration of n-simplices of T without repetitions. For each $i \geq 1$, construct a null-sequence $\{A^i_j: 1 \leq j < \infty\}$ of disjoint arcs in the interior $\operatorname{Int} \sigma_i$ of σ_i such that the union of these arcs is dense in $\operatorname{Int} \sigma_i$. Furthermore, we require that for any two distinct arcs A^i_j and A^k_1, the fundamental groups $\pi_1(\sigma_i - A^i_j)$ and $\pi_1(\sigma_k - A^k_1)$ are nonisomorphic. This type of construction can be easily carried out by utilizing the arcs given in **[DS]**. Let G be a decomposition of M^n whose nondegenerate elements are the arcs A^i_j with $1 \leq i, j < \infty$. The decomposition G is clearly upper semicontinuous. Let $q: M^n \to X$ denote the quotient (projection) map onto the decomposition space $X = M^n/G$. The space X is a generalized n-manifold by **[W]**, and $X \times X$ is homeomorphic to $M^n \times M^n$ by **[B]**.

We now prove that X is rigid. Suppose $h: X \to X$ is a homeomorphism different from the identity. Since the set $\{x^i_j = q(A^i_j): 1 \leq i, j < \infty\}$ is dense in X, it follows that $y = h(x^i_j) \neq x^i_j$, for some x^i_j which will remain fixed in the following argument. Suppose $q^{-1}(y)$ is a subset of σ_k. Either $q^{-1}(y)$ equals A^k_1 for some 1, or $q^{-1}(y) = \{z\}$ for some z in σ_k. Clearly, h induces an isomorphism $\pi_1(X - \{x^i_j\}) \approx \pi_1(X - \{y\})$. Also,

$$\pi_1(X - \{x^i_j\}) \approx \pi_1(M^n - A^i_j) \quad \text{and} \quad \pi_1(X - \{y\}) \approx \pi_1(M^n - q^{-1}(y)),$$

since q is a CE map (cf. [L]). Since M^n is simply connected, it follows that $\pi_1(M^n - A_s^r) \approx \pi_1(\sigma_r - A_s^r)$ for all r and s, and $\pi_1(M^n - \{\text{point}\})$ is trivial. It follows from our discussion given above that either $\pi_1(\sigma_i - A_j^i)$ is isomorphic to $\pi_1(\sigma_k - A_1^k)$ for some 1 or it is isomorphic to the trivial group $\pi_1(M^n - \{z\})$ depending on whether $q^{-1}(y)$ equals A_1^k or $\{z\}$, respectively. In either case, we reach a contradiction. This proves that X is rigid.

Let $C(M^n)$ denote the collection of topologically distinct decomposition spaces constructed by the method given above. The results of [DS] and the following observation imply that $C(M^n)$ is uncountable: If $\{A_j^i: 1 \leq i, j < \infty\}$ and $\{B_j^i: 1 \leq i, j < \infty\}$ are collections of arcs employed to construct M^n/G and M^n/H, respectively, such that the collection $\{\pi_1(M^n - A_j^i): 1 \leq i, j < \infty\}$ contains a nontrivial group which is not isomorphic to any group in $\{\pi_1(M^n - B_j^i): 1 \leq i, j < \infty\}$, then M^n/G and M^n/H are nonhomeomorphic. This suffices to prove the theorem. □

REMARK. The preceding theorem holds for a topological n-manifold with $n \geq 5$, i.e., the hypothesis "simply connected and triangulable" can be omitted. Since the proof is rather technical and lengthy, we will not pursue it here.

COROLLARY. *For any simply connected Lie group M^n of dimension $n \geq 5$, for instance $M^6 = S^3 \times S^3$, there exists an uncountable collection $C(M^n)$ of topologically distinct rigid generalized n-manifolds such that $X \times X$ is homeomorphic to the Lie group $M^n \times M^n$.*

This answers a question of Jan van Mill [vM], see also §2. Since a Lie group is a smooth manifold and a smooth manifold is triangulable by a well-known result of J. H. C. Whitehead, our corollary is immediate from the theorem.

We conclude by stating the following result of [DS] for infinite-dimensional spaces whose proof is analogous to the discussions given above:

THEOREM. *There exists an uncountable collection $C(Q)$ of topologically distinct compacta such that $X \times X$ is homeomorphic to $Q \times Q$ (or Q since $Q \times Q \approx Q$) where Q denotes the Hilbert cube.*

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