6. Isometries of Euclidean Spaces

In this chapter we will investigate distance preserving functions between Euclidean spaces. We first review results we obtained in Chapters 2 and 3 concerning distance preserving functions from $\mathbb{R}$ to itself because the pattern of these results is foreshadows the theorems about distance preserving functions between Euclidean spaces. Theorem 2.6 tells us that if two distance preserving functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ agree at two distinct points, then f = g. Theorems 2.5 and 3.2 say that every isometry $g : S \to T$ between two subsets S and T of $\mathbb{R}$ extends to a rigid motion $f : \mathbb{R} \to \mathbb{R}$ and f is either a reflection or a composition of two reflections. (Theorem 2.5 actually says that f is either a reflection or a translation, but every translation is a composition of two reflections by Theorem 2.4.) These results are then applied to prove the central theorem of Chapter 2: Theorem 2.7 which asserts that every distance preserving function from $\mathbb{R}$ to itself is either a reflection or a composition of two reflections. Theorem 2.7 has two immediate corollaries which say that every distance preserving function from $\mathbb{R}$ to itself is a rigid motion of $\mathbb{R}$, and every rigid motion of $\mathbb{R}$ is either a reflection or a composition of two reflections. Each of these theorems generalizes to a theorem about distance preserving functions between Euclidean spaces. The theorems that result from this generalization process form the content of this chapter.

We beginning by formulating the appropriate analogue of Theorem 2.6 for Euclidean spaces. Theorem 2.6 says that if two distance preserving functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ agree at *two distinct points,* then f = g. If f and g are instead distance preserving functions from Euclidean n-space $\mathbb{E}^n$ to itself that agree on a finite set of points $\{ \mathbf{x}_1, \mathbf{x}_2, \dots , \mathbf{x}_k \}$, we must answer the following question. What property must the set $\{ \mathbf{x}_1, \mathbf{x}_2, \dots , \mathbf{x}_k \}$ have to force f and g to be equal? If $n > 1$, it is not sufficient to require that the points $\mathbf{x}_1, \mathbf{x}_2, \dots , \mathbf{x}_k$ simply be *distinct.* Indeed, in Euclidean 2-space $\mathbb{E}^2$, if k distinct points $\mathbf{x}_1, \mathbf{x}_2, \dots , \mathbf{x}_k$ all lie in the same line L, then $id_{\mathbb{E}^n}$ and the reflection of $\mathbb{E}^2$ in the line L are two unequal rigid motions of $\mathbb{E}^2$ that agree on the k points $\mathbf{x}_1, \mathbf{x}_2, \dots , \mathbf{x}_k$. The key property possessed by *two distinct points* in $\mathbb{R}$ that makes it possible to prove Theorem 2.6 is expressed in Theorem 1.5: every point in $\mathbb{R}$ is uniquely determined by its distances from *two distinct points.* The analogous property in $\mathbb{E}^n$ is expressed by Theorem 5.17: if $\mathbf{x}_1, \mathbf{x}_2, \dots , \mathbf{x}_k$ are *non-coplanar points* in $\mathbb{E}^n$, then every point of $\mathbb{E}^n$ is uniquely determined by its distances from $\mathbf{x}_1, \mathbf{x}_2, \dots , \mathbf{x}_k$. *Non-coplanarity* is the key property needed to generalize Theorem 2.6 to Euclidean n-space. Theorem 6.1 is the generalization of Theorem 2.6 to $\mathbb{E}^n$. Theorem 6.1 says that if $\mathbf{x}_1, \mathbf{x}_2, \dots , \mathbf{x}_k$ are non-coplanar points in $\mathbb{E}^n$ and if $f : \mathbb{E}^n \to \mathbb{E}^n$ and $g : \mathbb{E}^n \to \mathbb{E}^n$ are distance preserving functions that agree on $\mathbf{x}_1, \mathbf{x}_2, \dots , \mathbf{x}_k$, then f = g. For technical reasons, we must also assume that one of the two functions f and g is both distance preserving and onto. In other words, we must assume that one of f and g is a rigid motion of $\mathbb{E}^n$. Theorem 6.1 remains true if we omit this extra hypothesis, but we don't have the tools needed to prove it at this point.

Theorem 6.1. Suppose $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are non-coplanar points in $\mathbb{E}^n$. If $f : \mathbb{E}^n \to \mathbb{E}^n$ is a distance preserving function, $g : \mathbb{E}^n \to \mathbb{E}^n$ is a rigid motion of $\mathbb{E}^n$ and $f(\mathbf{x}_1) = g(\mathbf{x}_1), f(\mathbf{x}_2) = g(\mathbf{x}_2), \ldots, f(\mathbf{x}_k) = g(\mathbf{x}_k)$, then $f = g$.

Proof. Assume $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are non-coplanar points in $\mathbb{E}^n$, and assume that $f : \mathbb{E}^n \to \mathbb{E}^n$ is a distance preserving function and $g : \mathbb{E}^n \to \mathbb{E}^n$ is a rigid motion of $\mathbb{E}^n$ such that $f(\mathbf{x}_1) = g(\mathbf{x}_1), f(\mathbf{x}_2) = g(\mathbf{x}_2), \ldots, f(\mathbf{x}_k) = g(\mathbf{x}_k)$. Let $\mathbf{y} \in \mathbb{E}^n$. We must prove $f(\mathbf{y}) = g(\mathbf{y})$.

Since $g : \mathbb{E}^n \to \mathbb{E}^n$ is a rigid motion, then Theorem 2.1 implies g has an inverse $g^{-1} : \mathbb{E}^n \to \mathbb{E}^n$ that is also a rigid motion. Hence, according to Theorem 2.1, the composition $g^{-1} \circ f : \mathbb{E}^n \to \mathbb{E}^n$ is distance preserving. Observe that for $1 \leq i \leq k$,

$$g^{-1} \circ f(\mathbf{x}_i) = g^{-1}(f(\mathbf{x}_i)) = g^{-1}(g(\mathbf{x}_i)) = g^{-1} \circ g(\mathbf{x}_i) = id_{\mathbb{E}^n}(\mathbf{x}_i) = \mathbf{x}_i.$$

Since $g^{-1} \circ f$ is distance preserving, then it follows that

$$d(\mathbf{y},\mathbf{x}_i) = d(g^{-1} \circ f(\mathbf{y}), g^{-1} \circ f(\mathbf{x}_i)) = d(g^{-1} \circ f(\mathbf{y}), \mathbf{x}_i) \ .$$

for $1 \leq i \leq k$. Thus, the points $\mathbf{y}$ and $g^{-1} \circ f(\mathbf{y})$ have equal distances from $\mathbf{x}_i$ for $1 \leq i \leq k$. Since $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are non-coplanar, then Theorem 5.17 tells us that each point of $\mathbb{E}^n$ is uniquely determined by its distances from $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$. It follows that $\mathbf{y} = g^{-1} \circ f(\mathbf{y})$. Hence,

$$g(\mathbf{y}) = g(g^{-1} \circ f(\mathbf{y})) = g \circ (g^{-1} \circ f)(\mathbf{y}) = (g \circ g^{-1}) \circ f(\mathbf{y}) = id_{\mathbb{E}^n} \circ f(\mathbf{y}) = f(\mathbf{y}).$$

We conclude that $f = g$. ☐

Lemma 5.16 tells us that the $n + 1$ points $\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are non-coplanar in $\mathbb{E}^n$. Combining this fact with the preceding theorem, we have:

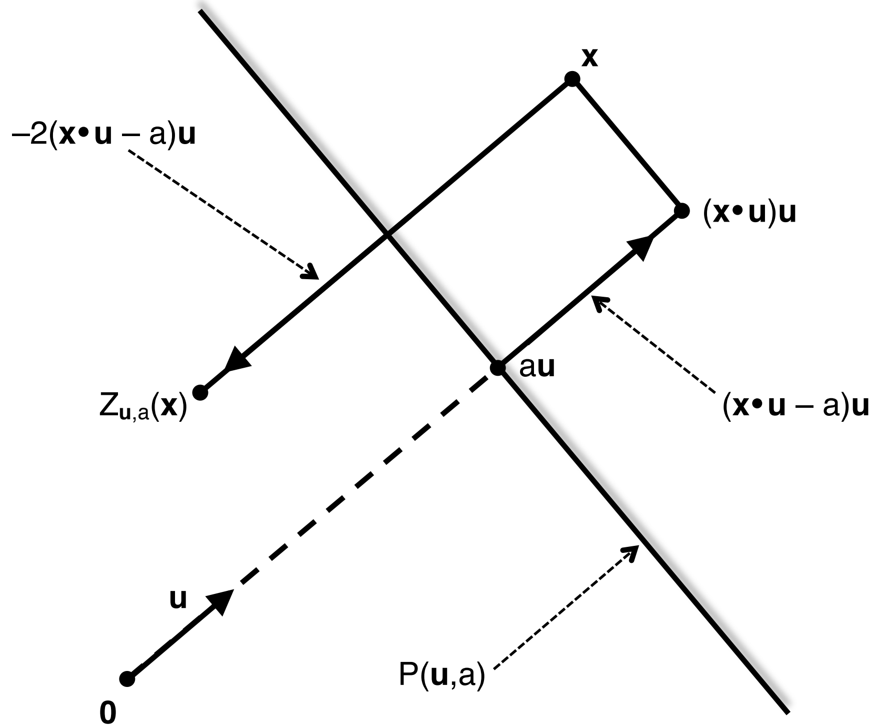
Corollary 6.2. If $f : \mathbb{E}^n \to \mathbb{E}^n$ is a distance preserving function, $g : \mathbb{E}^n \to \mathbb{E}^n$ is a rigid motion of $\mathbb{E}^n$ and $f(\mathbf{0}) = g(\mathbf{0}), f(\mathbf{e}_1) = g(\mathbf{e}_1), f(\mathbf{e}_2) = g(\mathbf{e}_2), \ldots, f(\mathbf{e}_n) = g(\mathbf{e}_n)$, then $f = g$. ☐

Our next goal is to establish an analogue of Theorems 2.5 and 3.2 for $\mathbb{E}^n$. We will prove that if S and T are finite subsets of $\mathbb{E}^n$ and $g : S \to T$ is an isometry, then there is a rigid motion $f : \mathbb{E}^n \to \mathbb{E}^n$ that extends g, and f is a composition of finitely many reflections. Before we can state and prove this theorem, we must define reflections in $\mathbb{E}^n$ and establish some of their properties.

Definition. Let $\mathbf{u} \in \mathbb{E}^n$ such that $\| \mathbf{u} \| = 1$ and let $a \in \mathbb{R}$. Define the function $Z_{\mathbf{u},a} : \mathbb{E}^n \to \mathbb{E}^n$ by the formula

$$Z_{\mathbf{u},a}(\mathbf{x}) = \mathbf{x} - 2(\mathbf{x}\bullet\mathbf{u} - a)\mathbf{u}$$

for each $\mathbf{x} \in \mathbb{E}^n$. The function $Z_{\mathbf{u},a}$ is called *reflection in the hyperplane P(**u**,a).*



Theorem 6.3. If $\mathbf{u} \in \mathbb{E}^n$ such that $\| \mathbf{u} \| = 1$ and $a \in \mathbb{R}$, then the reflection $Z_{\mathbf{u},a} : \mathbb{E}^n \to \mathbb{E}^n$ has the following properties.

a) $Z_{\mathbf{u},a} : \mathbb{E}^n \to \mathbb{E}^n$ is a rigid motion of $\mathbb{E}^n$.

b) $Z_{\mathbf{u},a}^{-1} = Z_{\mathbf{u},a}$.

c) For $\mathbf{x} \in \mathbb{E}^n$, $Z_{\mathbf{u},a}(\mathbf{x}) = \mathbf{x}$ if and only if $\mathbf{x} \in P(\mathbf{u},a)$.

d) If U and V are the opposite sides of $P(\mathbf{u},a)$, then $Z_{\mathbf{u},a}(U) = V$ and $Z_{\mathbf{u},a}(V) = U$.

Much of the proof of this theorem involves calculations.

Proof of a) and b). First we prove $Z_{\mathbf{u},a} : \mathbb{E}^n \to \mathbb{E}^n$ is distance preserving. Let $\mathbf{x}$ and $\mathbf{y} \in \mathbb{E}^n$. Then

$$(d(Z_{\mathbf{u},a}(\mathbf{x}),Z_{\mathbf{u},a}(\mathbf{y})))^2 = \| Z_{\mathbf{u},a}(\mathbf{x}) - Z_{\mathbf{u},a}(\mathbf{y}) \|^2 =$$

$$\| (\mathbf{x} - 2(\mathbf{x}\bullet\mathbf{u} - a)\mathbf{u}) - (\mathbf{y} - 2(\mathbf{y}\bullet\mathbf{u} - a)\mathbf{u}) \|^2 =$$

$\| \mathbf{x} - 2(\mathbf{x}\bullet\mathbf{u})\mathbf{u} + 2a\mathbf{u} - \mathbf{y} + 2(\mathbf{y}\bullet\mathbf{u})\mathbf{u} - 2a\mathbf{u} \|^2 =$

$\| (\mathbf{x} - \mathbf{y}) - 2((\mathbf{x} - \mathbf{y})\bullet\mathbf{u})\mathbf{u} \|^2 =$

$\| \mathbf{x} - \mathbf{y} \|^2 - 4((\mathbf{x} - \mathbf{y})\bullet\mathbf{u})^2 + \| 2((\mathbf{x} - \mathbf{y})\bullet\mathbf{u})\mathbf{u} \|^2$ (by Lemma 4.7.a) $=$

$\| \mathbf{x} - \mathbf{y} \|^2 - 4((\mathbf{x} - \mathbf{y})\bullet\mathbf{u})^2 + 4((\mathbf{x} - \mathbf{y})\bullet\mathbf{u})^2 \| \mathbf{u} \|^2 =$

$\| \mathbf{x} - \mathbf{y} \|^2 - 4((\mathbf{x} - \mathbf{y})\bullet\mathbf{u})^2 + 4((\mathbf{x} - \mathbf{y})\bullet\mathbf{u})^2 \|$ (because $\| \mathbf{u} \| = 1$) $=$

$\| \mathbf{x} - \mathbf{y} \|^2 = (d(\mathbf{x},\mathbf{y}))^2$.

Hence, $d(Z_{\mathbf{u},a}(\mathbf{x}),Z_{\mathbf{u},a}(\mathbf{y})) = d(\mathbf{x},\mathbf{y})$, proving $Z_{\mathbf{u},a}$ is distance preserving.

Second we prove $Z_{\mathbf{u},a} \circ Z_{\mathbf{u},a} = id_{\mathbb{E}^n}$. Let $\mathbf{x} \in \mathbb{E}^n$. To begin observe that

$Z_{\mathbf{u},a} \circ Z_{\mathbf{u},a}(\mathbf{x}) = Z_{\mathbf{u},a}(Z_{\mathbf{u},a}(\mathbf{x})) = Z_{\mathbf{u},a}(\mathbf{x}) - 2((Z_{\mathbf{u},a}(\mathbf{x}))\bullet\mathbf{u} - a)\mathbf{u}.$

To continue this calculation, we compute $(Z_{\mathbf{u},a}(\mathbf{x}))\bullet\mathbf{u} - a$.

$(Z_{\mathbf{u},a}(\mathbf{x}))\bullet\mathbf{u} - a = (\mathbf{x} - 2(\mathbf{x}\bullet\mathbf{u} - a)\mathbf{u})\bullet\mathbf{u} - a =$

$(\mathbf{x}\bullet\mathbf{u}) - 2(\mathbf{x}\bullet\mathbf{u})(\mathbf{u}\bullet\mathbf{u}) + 2a(\mathbf{u}\bullet\mathbf{u}) - a =$

$(\mathbf{x}\bullet\mathbf{u}) - 2(\mathbf{x}\bullet\mathbf{u}) + 2a - a$ (because $\mathbf{u}\bullet\mathbf{u} = \| \mathbf{u} \|^2 = 1$) $= -(\mathbf{x}\bullet\mathbf{u}) + a$.

Returning to the calculation of $Z_{\mathbf{u},a} \circ Z_{\mathbf{u},a}(\mathbf{x})$, we substitute $-(\mathbf{x}\bullet\mathbf{u}) + a$ for $(Z_{\mathbf{u},a}(\mathbf{x}))\bullet\mathbf{u} - a$ to obtain:

$Z_{\mathbf{u},a} \circ Z_{\mathbf{u},a}(\mathbf{x}) = (\mathbf{x} - 2(\mathbf{x}\bullet\mathbf{u} - a)\mathbf{u}) - 2(-(\mathbf{x}\bullet\mathbf{u}) + a)\mathbf{u} =$

$(\mathbf{x} - 2(\mathbf{x}\bullet\mathbf{u})\mathbf{u} + 2a\mathbf{u}) + (2(\mathbf{x}\bullet\mathbf{u})\mathbf{u} - 2a\mathbf{u}) = \mathbf{x} = id_{\mathbb{E}^n}(\mathbf{x})$.

Hence, $Z_{\mathbf{u},a} \circ Z_{\mathbf{u},a} = id_{\mathbb{E}^n}$. Since $id_{\mathbb{E}^n} : \mathbb{E}^n \to \mathbb{E}^n$ is onto, then $Z_{\mathbf{u},a} : \mathbb{E}^n \to \mathbb{E}^n$ must be onto by Theorem 0.4.d. Therefore, $Z_{\mathbf{u},a}$ is a rigid motion of $\mathbb{E}^n$. Also, $Z_{\mathbf{u},a} \circ Z_{\mathbf{u},a} = id_{\mathbb{E}^n}$ implies $Z_{\mathbf{u},a}^{-1} = Z_{\mathbf{u},a}$. □

Proof of c). Let $\mathbf{x} \in \mathbb{E}^n$. Statement c) follows from the observation that the following statements are equivalent.

$Z_{\mathbf{u},a}(\mathbf{x}) = \mathbf{x}$.	$\mathbf{x} - 2(\mathbf{x}\bullet\mathbf{u} - a)\mathbf{u} = \mathbf{x}$.	$2(\mathbf{x}\bullet\mathbf{u} - a)\mathbf{u} = \mathbf{0}$
$\mathbf{x}\bullet\mathbf{u} - a = 0$.	$\mathbf{x}\bullet\mathbf{u} = a$.	$\mathbf{x} \in P(\mathbf{u},a)$. □

Proof of d). Assume U and V are the opposite sides of $P(\mathbf{u},a)$. Then according to Theorems 5.9 and 5.10, we may further assume that $U = \{ \mathbf{x} \in \mathbb{E}^n : \mathbf{x}\bullet\mathbf{u} > a \}$ and $V = \{ \mathbf{x} \in \mathbb{E}^n : \mathbf{x}\bullet\mathbf{u} < a \}$. We observe that for each $\mathbf{x} \in \mathbb{E}^n$, $Z_{\mathbf{u},a}(\mathbf{x})\bullet\mathbf{u} - a = -(\mathbf{x}\bullet\mathbf{u} - a)$. Here is the proof:

$Z_{\mathbf{u},a}(\mathbf{x})\bullet\mathbf{u} - a = (\mathbf{x} - 2(\mathbf{x}\bullet\mathbf{u} - a)\mathbf{u})\bullet\mathbf{u} - a = \mathbf{x}\bullet\mathbf{u} - 2(\mathbf{x}\bullet\mathbf{u})(\mathbf{u}\bullet\mathbf{u}) + 2a(\mathbf{u}\bullet\mathbf{u}) - a =$

$\mathbf{x}\bullet\mathbf{u} - 2(\mathbf{x}\bullet\mathbf{u}) + 2a - a$ (because $\mathbf{u}\bullet\mathbf{u} = \| \mathbf{u} \|^2 = 1$) $= -(\mathbf{x}\bullet\mathbf{u}) + a = -(\mathbf{x}\bullet\mathbf{u} - a)$.

The equation $Z_{\mathbf{u},a}(\mathbf{x})\bullet\mathbf{u} - a = -(\mathbf{x}\bullet\mathbf{u} - a)$ implies that if $\mathbf{x}\bullet\mathbf{u} - a > 0$ then $Z_{\mathbf{u},a}(\mathbf{x})\bullet\mathbf{u} - a < 0$, and if $\mathbf{x}\bullet\mathbf{u} - a < 0$ then $Z_{\mathbf{u},a}(\mathbf{x})\bullet\mathbf{u} - a > 0$. These observations help us see that each statement in the following chain of statements implies the statement that follows it.

$\mathbf{x} \in U$. $\mathbf{x}\bullet\mathbf{u} > a$. $\mathbf{x}\bullet\mathbf{u} - a > 0$. $Z_{\mathbf{u},a}(\mathbf{x})\bullet\mathbf{u} - a < 0$. $Z_{\mathbf{u},a}(\mathbf{x})\bullet\mathbf{u} < a$. $Z_{\mathbf{u},a}(\mathbf{x}) \in V$.

Similarly each statement in the following chain of statements implies the statement that follows it.

$\mathbf{x} \in V$. $\mathbf{x}\bullet\mathbf{u} < a$. $\mathbf{x}\bullet\mathbf{u} - a < 0$. $Z_{\mathbf{u},a}(\mathbf{x})\bullet\mathbf{u} - a > 0$. $Z_{\mathbf{u},a}(\mathbf{x})\bullet\mathbf{u} > a$. $Z_{\mathbf{u},a}(\mathbf{x}) \in U$.

From the first chain of statements, we conclude: if $\mathbf{x} \in U$, then $Z_{\mathbf{u},a}(\mathbf{x}) \in V$. Hence, $Z_{\mathbf{u},a}(U) \subset V$. From the second chain of statements, we conclude: if $\mathbf{x} \in V$, then $Z_{\mathbf{u},a}(\mathbf{x}) \in U$. Hence, $Z_{\mathbf{u},a}(V) \subset U$. From the inclusions $Z_{\mathbf{u},a}(U) \subset V$ and $Z_{\mathbf{u},a}(V) \subset U$ together with the fact (proved above) that $Z_{\mathbf{u},a} \circ Z_{\mathbf{u},a} = id_{\mathbb{E}^n}$, we derive the following strings of equalities and inclusions:

$$U = id_{\mathbb{E}^n}(U) = Z_{\mathbf{u},a} \circ Z_{\mathbf{u},a}(U) = Z_{\mathbf{u},a}(Z_{\mathbf{u},a}(U)) \subset Z_{\mathbf{u},a}(V),$$

and

$$V = id_{\mathbb{E}^n}(V) = Z_{\mathbf{u},a} \circ Z_{\mathbf{u},a}(V) = Z_{\mathbf{u},a}(Z_{\mathbf{u},a}(V)) \subset Z_{\mathbf{u},a}(U).$$

Hence, we have established that $Z_{\mathbf{u},a}(U) \subset V$, $V \subset Z_{\mathbf{u},a}(U)$, $Z_{\mathbf{u},a}(V) \subset U$ and $U \subset Z_{\mathbf{u},a}(V)$. We conclude that $Z_{\mathbf{u},a}(U) = V$ and $Z_{\mathbf{u},a}(V) = U$. □

Definition. If $f : X \to X$ is a function from a set X to itself, and if x is an element of X such that $f(x) = x$, then we say that f *fixes* x and we call x a *fixed point* of f. Furthermore, we call the set $\{ x \in X : f(x) = x \}$ the *fixed point set of f*. Hence, statement c) of Theorem 6.3 says that $P(\mathbf{u},a)$ is the fixed point set of $Z_{\mathbf{u},a}$.

Homework Problem 6.1. Suppose $f : \mathbb{E}^n \to \mathbb{E}^n$ is a distance preserving function and $\mathbf{a}$ and $\mathbf{b}$ are distinct points of $\mathbb{E}^n$. Prove that if $\mathbf{a}$ and $\mathbf{b}$ are fixed points of f, then every point of $L(\mathbf{a},\mathbf{b})$ is a fixed point of f.

Hint. Recall that distance preserving functions are affine (Theorem 4.13).

Recall that Theorem 5.5 tells us that two hyperplanes $P(\mathbf{u},a)$ and $P(\mathbf{v},b)$ are equal if and only if either $\mathbf{u} = \mathbf{v}$ and $a = b$ or $\mathbf{u} = -\mathbf{v}$ and $a = -b$. The next theorem extends this relation to the associated reflections $Z_{\mathbf{u},a}$ and $Z_{\mathbf{v},b}$.

Theorem 6.4. Let $\mathbf{u}$ and $\mathbf{v} \in \mathbb{E}^n$ such that $\| \mathbf{u} \| = \| \mathbf{v} \| = 1$ and let a and $b \in \mathbb{R}$. Then the following three statements are equivalent.

a) $Z_{\mathbf{u},a} = Z_{\mathbf{v},b}$. **b)** $P(\mathbf{u},a) = P(\mathbf{v},b)$. **c)** Either $\mathbf{u} = \mathbf{v}$ and $a = b$, or $\mathbf{u} = -\mathbf{v}$ and $a = -b$.

Proof. Theorem 5.5 tells us that statements b) and c) are equivalent.

Proof that a) implies b). Assume $Z_{\mathbf{u},a} = Z_{\mathbf{v},b}$. Let $\mathbf{x} \in \mathbb{E}^n$. Then with the aid of Theorem 6.3.c, we see that the following statements are equivalent:

$$\mathbf{x} \in P(\mathbf{u},a). \quad Z_{\mathbf{u},a}(\mathbf{x}) = \mathbf{x}. \quad Z_{\mathbf{v},b}(\mathbf{x}) = \mathbf{x}. \quad \mathbf{x} \in P(\mathbf{v},b).$$

Consequently, $P(\mathbf{u},a) = P(\mathbf{v},b)$. □

Proof that b) implies a). Assume $P(\mathbf{u},a) = P(\mathbf{v},b)$. Then Theorem 5.5 implies either $\mathbf{u} = \mathbf{v}$ and $a = b$, or $\mathbf{u} = -\mathbf{v}$ and $a = -b$. In the first case: if $\mathbf{u} = \mathbf{v}$ and $a = b$, then $Z_{\mathbf{u},a} = Z_{\mathbf{v},b}$ by substitution. In the second case: if $\mathbf{u} = -\mathbf{v}$ and $a = -b$, then $Z_{\mathbf{u},a} = Z_{-\mathbf{v},-b}$, and it remains to show that $Z_{-\mathbf{v},-b} = Z_{\mathbf{v},b}$. Here is a proof that $Z_{-\mathbf{v},-b} = Z_{\mathbf{v},b}$. Let $\mathbf{x} \in \mathbb{E}^n$ $\mathbb{E}^n$. Then

$$Z_{-\mathbf{v},-b}(\mathbf{x}) = \mathbf{x} - 2(\mathbf{x}\bullet(-\mathbf{v}) - (-b))(-\mathbf{v}) = \mathbf{x} - 2((-1)\mathbf{x}\bullet\mathbf{v} - (-1)b)(-1)\mathbf{v} =$$

$$\mathbf{x} - 2(-1)(\mathbf{x}\bullet\mathbf{v} - b)(-1)\mathbf{v} = \mathbf{x} - 2(-1)(-1)(\mathbf{x}\bullet\mathbf{v} - b)\mathbf{v} = \mathbf{x} - 2(\mathbf{x}\bullet\mathbf{v} - b)\mathbf{v} =$$

$$Z_{\mathbf{v},b}(\mathbf{x}).$$ □

Recall that if $\mathbf{x}$ and $\mathbf{y}$ are distinct points of $\mathbb{E}^n$, then Theorem 5.18 tells us that the set $E(\mathbf{x},\mathbf{y}) = \{ \mathbf{z} \in \mathbb{E}^n : d(\mathbf{x},\mathbf{z}) = d(\mathbf{y},\mathbf{z}) \}$ is a hyperplane. The next theorem tells us that the reflection which interchanges the points $\mathbf{x}$ and $\mathbf{y}$ is precisely the reflection in this hyperplane.

Theorem 6.5. Let $\mathbf{x}$ and $\mathbf{y}$ be distinct points of $\mathbb{E}^n$, and let $\mathbf{u} \in \mathbb{E}^n$ such that $\| \mathbf{u} \| = 1$ and let $a \in \mathbb{R}$. Then $Z_{\mathbf{u},a}(\mathbf{x}) = \mathbf{y}$ if and only if $P(\mathbf{u},a) = E(\mathbf{x},\mathbf{y})$.

Proof. First assume $Z_{\mathbf{u},a}(\mathbf{x}) = \mathbf{y}$.

We begin by proving $P(\mathbf{u},a) \subset E(\mathbf{x},\mathbf{y})$. Let $\mathbf{z} \in P(\mathbf{u},a)$. Then Theorem 6.3.c tells us that $Z_{\mathbf{u},a}(\mathbf{z}) = \mathbf{z}$. Since $Z_{\mathbf{u},a}$ is distance preseving (by Theorem 6.3.a), then

$$d(\mathbf{x},\mathbf{z}) = d(Z_{\mathbf{u},a}(\mathbf{x}),Z_{\mathbf{u},a}(\mathbf{z})) = d(\mathbf{y},\mathbf{z}).$$

Hence, $\mathbf{z} \in E(\mathbf{x},\mathbf{y})$. This proves $P(\mathbf{u},a) \subset E(\mathbf{x},\mathbf{y})$.

$P(\mathbf{u},a)$ is, by definition, a hyperplane; and $E(\mathbf{x},\mathbf{y})$ is a hyperplane by Theorem 5.18. Since $P(\mathbf{u},a) \subset E(\mathbf{x},\mathbf{y})$, then Corollary 5.8 implies $P(\mathbf{u},a) = E(\mathbf{x},\mathbf{y})$. This completes the proof of the forward direction of Theorem 6.5.

Next we present the proof of the reverse direction of Theorem 6.5. Assume $P(\mathbf{u},a) = E(\mathbf{x},\mathbf{y})$. We must prove $Z_{\mathbf{u},a}(\mathbf{x}) = \mathbf{y}$. Theorem 5.18 tells us that $E(\mathbf{x},\mathbf{y})$ is equal to the hyperplane $P(\mathbf{v},b)$ where

$$\mathbf{v} = \frac{\mathbf{y} - \mathbf{x}}{\| \mathbf{y} - \mathbf{x} \|}, \quad \mathbf{m} = (1/2)(\mathbf{x} + \mathbf{y}) \quad \text{and} \quad b = \mathbf{m}\bullet\mathbf{v}.$$

Hence, $P(\mathbf{u},a) = P(\mathbf{v},b)$. Therefore, Theorem 6.4 implies $Z_{\mathbf{u},a} = Z_{\mathbf{v},b}$. Hence,

$$Z_{\mathbf{u},a}(\mathbf{x}) = Z_{\mathbf{v},b}(\mathbf{x}) = \mathbf{x} - 2(\mathbf{x}\bullet\mathbf{v} - b)\mathbf{v} = \mathbf{x} - 2(\mathbf{x}\bullet\mathbf{v} - \mathbf{m}\bullet\mathbf{v})\mathbf{v} = \mathbf{x} - 2((\mathbf{x} - \mathbf{m})\bullet\mathbf{v})\mathbf{v}.$$

Since

$$\mathbf{x} - \mathbf{m} = \mathbf{x} - (^1/_2)(\,\mathbf{x} + \mathbf{y}\,) = -(^1/_2)(\,\mathbf{y} - \mathbf{x}\,),$$

then

$$(\mathbf{x} - \mathbf{m})\bullet\mathbf{v} = -(^1/_2)(\,\mathbf{y} - \mathbf{x}\,)\bullet\left(\frac{\mathbf{y} - \mathbf{x}}{\|\,\mathbf{y} - \mathbf{x}\,\|}\right) = -(^1/_2)\left(\frac{\|\,\mathbf{y} - \mathbf{x}\,\|^2}{\|\,\mathbf{y} - \mathbf{x}\,\|}\right) = -(^1/_2)\,\|\,\mathbf{y} - \mathbf{x}\,\|.$$

Therefore,

$$2((\mathbf{x} - \mathbf{m})\bullet\mathbf{v})\mathbf{v} = 2(-(^1/_2)\,\|\,\mathbf{y} - \mathbf{x}\,\|\,)\left(\frac{\mathbf{y} - \mathbf{x}}{\|\,\mathbf{y} - \mathbf{x}\,\|}\right) = -(\,\mathbf{y} - \mathbf{x}\,) = \mathbf{x} - \mathbf{y}.$$

Hence, $Z_{\mathbf{u},a}(\mathbf{x}) = \mathbf{x} - (\,\mathbf{x} - \mathbf{y}\,) = \mathbf{y}$. This completes the proof of Theorem 6.5 in the reverse direction. ☐

We extract from Theorem 6.5 and other previous results a simple corollary that will be useful in subsequent theorems.

Corollary 6.6. If $\mathbf{x}$ and $\mathbf{y}$ be distinct points of $\mathbb{E}^n$, then there is a reflection $Z_{\mathbf{u},a}$ of $\mathbb{E}^n$ such that

a) $Z_{\mathbf{u},a}(\mathbf{x}) = \mathbf{y}$ and

b) if $\mathbf{z} \in \mathbb{E}^n$ and $d(\mathbf{x},\mathbf{z}) = d(\mathbf{y},\mathbf{z})$, then $Z_{\mathbf{u},a}(\mathbf{z}) = \mathbf{z}$.

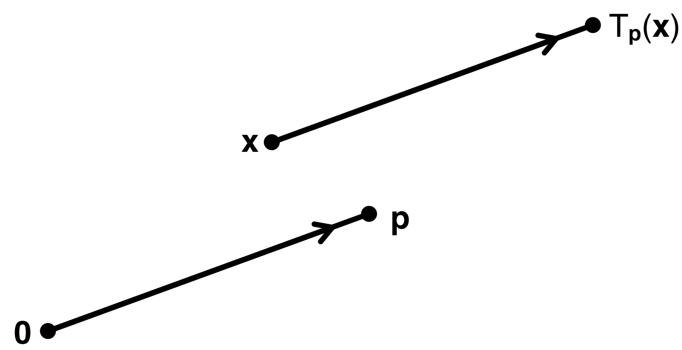
Proof. Theorem 5.18 implies that $E(\mathbf{x},\mathbf{y})$ is a hyperplane. Hence, there exist $\mathbf{u} \in \mathbb{E}^n$ such that $\|\,\mathbf{u}\,\| = 1$ and $a \in \mathbb{R}$ so that $E(\mathbf{x},\mathbf{y}) = P(\mathbf{u},a)$. Then Theorem 6.5 implies $Z_{\mathbf{u},a}(\mathbf{x}) = \mathbf{y}$. Furthermore, if $\mathbf{z} \in \mathbb{E}^n$ and $d(\mathbf{x},\mathbf{z}) = d(\mathbf{y},\mathbf{z})$, then $\mathbf{z} \in E(\mathbf{x},\mathbf{y})$. Therefore, $\mathbf{z} \in P(\mathbf{u},a)$. In this situation Theorem 6.3.c implies $Z_{\mathbf{u},a}(\mathbf{z}) = \mathbf{z}$. ☐

Reflections are the rigid motions of $\mathbb{E}^n$ that play a special role in our development of the theory of isometries of $\mathbb{E}^n$. Translations are another type of rigid motion of $\mathbb{E}^n$, but translations aren't as central to the theory as are reflections. None the less, since translations are easy to define and analyze, we do this next.

Definition. Let $\mathbf{p} \in \mathbb{E}^n$. Define the function $T_{\mathbf{p}} : \mathbb{E}^n \to \mathbb{E}^n$ by the formula

$$T_{\mathbf{p}}(\mathbf{x}) = \mathbf{x} + \mathbf{p}$$

for each $\mathbf{x} \in \mathbb{E}^n$. The function $T_{\mathbf{p}}$ is called the *translation parallel to* $\boldsymbol{p}$.



Theorem 6.7. If $\mathbf{p} \in \mathbb{E}^n$, then the translation $T_{\mathbf{p}} : \mathbb{E}^n \to \mathbb{E}^n$ has the following properties.

a) $T_{\mathbf{p}} : \mathbb{E}^n \to \mathbb{E}^n$ is a rigid motion of $\mathbb{E}^n$.

b) $T_{\mathbf{0}} = id_{\mathbb{E}^n}$, $T_{\mathbf{p}}^{-1} = T_{-\mathbf{p}}$, and if $\mathbf{q} \in \mathbb{E}^n$, then $T_{\mathbf{q}} \circ T_{\mathbf{p}} = T_{\mathbf{p}+\mathbf{q}}$.

c) If $\mathbf{p} \neq \mathbf{0}$, then $T_{\mathbf{p}}(\mathbf{x}) \neq \mathbf{x}$ for every $\mathbf{x} \in \mathbb{E}^n$. In other words, if $\mathbf{p} \neq \mathbf{0}$, then $T_{\mathbf{p}}$ has no fixed points.

d) If $\mathbf{p} \neq \mathbf{0}$, $\mathbf{u} = \dfrac{\mathbf{p}}{\|\mathbf{p}\|}$, and a and b $\in \mathbb{R}$ such that $b = a + (1/2)\|\mathbf{p}\|$, then $T_{\mathbf{p}} = Z_{\mathbf{u},b} \circ Z_{\mathbf{u},a}$.

Homework Problem 6.2. Prove Theorem 6.7.

Recall that Theorems 2.5 and 3.2 say that every isometry between two subsets of $\mathbb{R}$ extends to a rigid motion of $\mathbb{R}$ which is either a reflection or a composition of two reflections. We now prove a version of this result for $\mathbb{E}^n$. Our version of this theorem for $\mathbb{E}^n$ only applies to isometries between *finite* subsets of $\mathbb{E}^n$. There is a version of this theorem that holds for isometries between *infinite* subsets of $\mathbb{E}^n$, but we aren't prepared to prove it at this point.

Theorem 6.8. For each $k \geq 1$, if S and T are k-element subsets of $\mathbb{E}^n$ and $f : S \to T$ is an isometry, then there is a rigid motion $g : \mathbb{E}^n \to \mathbb{E}^n$ such that $g(\mathbf{x}) = f(\mathbf{x})$ for every $\mathbf{x} \in S$ and g is the composition of k or fewer reflections.

Proof. We will prove this theorem by induction on the number k of elements in the sets S and T.

Begin by assuming $k = 1$. Then S has a single element we will call $\mathbf{x}$ and T has a single element we will call $\mathbf{y}$, and the isometry $f : S \to T$ is the function determined by the equation $f(\mathbf{x}) = \mathbf{y}$. We must produce a rigid motion $g : \mathbb{E}^n \to \mathbb{E}^n$ such that $g(\mathbf{x}) = f(\mathbf{x}) = \mathbf{y}$. We consider two cases: either $\mathbf{x} = \mathbf{y}$ or $\mathbf{x} \neq \mathbf{y}$.

In the case that $\mathbf{x} = \mathbf{y}$, we let $g = id_{\mathbb{E}^n}$. Then g is a rigid motion of $\mathbb{E}^n$ (by Theorem 2.1.a) that is a composition of 0 reflections, and $g(\mathbf{x}) = \mathbf{x} = \mathbf{y} = f(\mathbf{x})$.

Now consider the case in which $\mathbf{x} \neq \mathbf{y}$. In this situation, we invoke Corollary 6.6 to obtain a reflection $Z_{\mathbf{u},a} : \mathbb{E}^n \to \mathbb{E}^n$ such that $Z_{\mathbf{u},a}(\mathbf{x}) = \mathbf{y}$, and we set $g = Z_{\mathbf{u},a}$. Then g is a rigid motion of $\mathbb{E}^n$ (by Theorem 6.3.a) which is the composition of 1 reflection, and $g(\mathbf{x}) = Z_{\mathbf{u},a}(\mathbf{x}) = \mathbf{y} = f(\mathbf{x})$. This completes the proof of the k = 1 case of this theorem.

To finish the proof, we must establish the *inductive step.* Let $k \geq 1$ and assume the *inductive hypothesis:* if S and T are k-element subsets of $\mathbb{E}^n$ and $f : S \to T$ is an isometry, then there is a rigid motion $g : \mathbb{E}^n \to \mathbb{E}^n$ such that $g(\mathbf{x}) = f(\mathbf{x})$ for every $\mathbf{x} \in S$ and g is the composition of k or fewer reflections. We must prove that this statement is true if k is replaced by k + 1. To this end, suppose S and T are (k + 1)-element subsets of $\mathbb{E}^n$ and $f : S \to T$ is an isometry. We can write $S = \{ \mathbf{x}_1, \mathbf{x}_2, \dots , \mathbf{x}_k, \mathbf{x}_{k+1} \}$. Let $\mathbf{y}_i = f(\mathbf{x}_i)$ for $1 \leq i \leq k + 1$. Since $f : S \to T$ is a bijection, then $T = \{ \mathbf{y}_1, \mathbf{y}_2, \dots , \mathbf{y}_k, \mathbf{y}_{k+1} \}$. Let $S' = S - \{ \mathbf{x}_{k+1} \} = \{ \mathbf{x}_1, \mathbf{x}_2, \dots , \mathbf{x}_k \}$, let $T' = T - \{ \mathbf{y}_{k+1} \} = \{ \mathbf{y}_1, \mathbf{y}_2, \dots , \mathbf{y}_k \}$, and let f´ be the restriction of f to S´. Then $f' : S' \to T'$ is an isometry. (f´ is distance preserving because $d(f'(\mathbf{x}_i),f'(\mathbf{x}_j)) = d(f(\mathbf{x}_i),f(\mathbf{x}_j)) = d(\mathbf{x}_i,\mathbf{x}_j)$ for $1 \leq i, j \leq k$. f´ is onto because for $1 \leq l \leq k$, $\mathbf{y}_i = f(\mathbf{x}_i) = f'(\mathbf{x}_i)$ and $\mathbf{x}_i \in S'$.) Since S´ and T´ are k-element subsets of $\mathbb{E}^n$, then the inductive hypothesis implies there is an rigid motion $g' : \mathbb{E}^n \to \mathbb{E}^n$ such that $g'(\mathbf{x}) = f'(\mathbf{x})$ for every $\mathbf{x} \in S'$ and g´ is the composition of k or fewer reflections. Thus, for $1 \leq i \leq k$, $g'(\mathbf{x}_i) = f'(\mathbf{x}_i) = \mathbf{y}_i$. We now consider two cases: either $g'(\mathbf{x}_{k+1}) = \mathbf{y}_{k+1}$ or $g'(\mathbf{x}_{k+1}) \neq \mathbf{y}_{k+1}$.

In the case that $g'(\mathbf{x}_{k+1}) = \mathbf{y}_{k+1}$, we let $g = g'$. Since g´ is a rigid motion of $\mathbb{E}^n$ that is the composition of k or fewer reflections, then so is g. Hence, g is the composition of k + 1 or fewer reflections. Also, for $1 \leq i \leq k + 1$, $g(\mathbf{x}_i) = g'(\mathbf{x}_i) = \mathbf{y}_i = f(\mathbf{x}_i)$. Thus, $g(\mathbf{x}) = f(\mathbf{x})$ for every $\mathbf{x} \in S$.

Now consider the case in which $g'(\mathbf{x}_{k+1}) \neq \mathbf{y}_{k+1}$. In this situation, we invoke Corollary 6.6 to obtain a reflection $Z_{\mathbf{u},a} : \mathbb{E}^n \to \mathbb{E}^n$ such that $Z_{\mathbf{u},a}(g'(\mathbf{x}_{k+1})) = \mathbf{y}_{k+1}$ and such that $Z_{\mathbf{u},a}(\mathbf{z}) = \mathbf{z}$ whenever $\mathbf{z} \in \mathbb{E}^n$ and $d(g'(\mathbf{x}_{k+1}),\mathbf{z}) = d(\mathbf{y}_{k+1},\mathbf{z})$. We set $g = Z_{\mathbf{u},a} \circ g'$. Since g´ and $Z_{\mathbf{u},a}$ are rigid motions of $\mathbb{E}^n$ (by Theorem 6.3.a), then g is also rigid motion of $\mathbb{E}^n$ (by Theorem 2.1.c). Since g´ is the composition of k or fewer reflections, then g is clearly the composition of k + 1 or fewer reflections. Observe that since g´ and f are distance preserving, and $g'(\mathbf{x}_i) = \mathbf{y}_i = f(\mathbf{x}_i)$ for $1 \leq i \leq k$, and $f(\mathbf{x}_{k+1}) = \mathbf{y}_{k+1}$, then for $1 \leq i \leq k$:

$$d(g'(\mathbf{x}_{k+1}),g'(\mathbf{x}_i)) = d(\mathbf{x}_{k+1},\mathbf{x}_i) = d(f(\mathbf{x}_{k+1}),f(\mathbf{x}_i)) = d(\mathbf{y}_{k+1},\mathbf{y}_i) = d(\mathbf{y}_{k+1},g'(\mathbf{x}_i)).$$

Consequently, $Z_{\mathbf{u},a}(g'(\mathbf{x}_i)) = g'(\mathbf{x}_i)$ for $1 \leq i \leq k$. Since $g = Z_{\mathbf{u},a} \circ g'$ and $Z_{\mathbf{u},a}(g'(\mathbf{x}_i)) = g'(\mathbf{x}_i) = f'(\mathbf{x}_i) = f(\mathbf{x}_i)$ for $1 \leq l \leq k$, then we have $g(\mathbf{x}_i) = Z_{\mathbf{u},a} \circ g'(\mathbf{x}_i) = Z_{\mathbf{u},a}(g'(\mathbf{x}_i)) = f(\mathbf{x}_i)$ for $1 \leq i \leq k$. Also since $g = Z_{\mathbf{u},a} \circ g'$ and $Z_{\mathbf{u},a}(g'(\mathbf{x}_{k+1})) = \mathbf{y}_{k+1} = f(\mathbf{x}_{k+1})$, then $g(\mathbf{x}_{k+1}) = Z_{\mathbf{u},a} \circ g'(\mathbf{x}_{k+1}) =$

$Z_{\mathbf{u},a}(g'(\mathbf{x}_{k+1})) = f(\mathbf{x}_{k+1})$. Thus, $g(\mathbf{x}) = f(\mathbf{x})$ for every $\mathbf{x} \in S$. This completes the proof of the inductive step. ◻

Homework Problem 6.3. Suppose $1 \le r \le k$, S and T are k-element subsets of $\mathbb{E}^n$, and $f : S \to T$ is an isometry that fixes at least r points of S. (Thus, S and T have at least r points in common.) Prove that there is a rigid motion $g : \mathbb{E}^n \to \mathbb{E}^n$ such that $g(\mathbf{x}) = f(\mathbf{x})$ for every $\mathbf{x} \in S$ and g is the composition of $k - r$ or fewer reflections.

We now come to Theorem 6.9 which can be considered the main result of this chapter. Theorem 6.9 is the n-dimensional analogue of Theorem 2.7. Theorem 2.7 says that every distance preserving function from $\mathbb{R}$ to itself is a rigid motion which is the composition of 2 or fewer reflections. (Recall that each translation of $\mathbb{R}$ is a composition of two reflections.)

Theorem 6.9. Every distance preserving function from $\mathbb{E}^n$ to itself is a rigid motion of $\mathbb{E}^n$ which is the composition of $n + 1$ or fewer reflections.

Proof. Let $f : \mathbb{E}^n \to \mathbb{E}^n$ be a distance preserving function. Let $S = \{ \mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \dots , \mathbf{e}_n \}$, and let $T = \{ f(\mathbf{0}), f(\mathbf{e}_1), f(\mathbf{e}_2), \dots , f(\mathbf{e}_n) \}$. Let $f' : S \to T$ be the restriction of f to S. Since f is distance preserving, then $f' : S \to T$ is an isometry. Since S has $n + 1$ elements, then Theorem 6.8 implies there is a rigid motion $g : \mathbb{E}^n \to \mathbb{E}^n$ such that $g(\mathbf{x}) = f'(\mathbf{x})$ for every $\mathbf{x} \in S$ and g is the composition of $n + 1$ or fewer reflections. Since $g(\mathbf{x}) = f'(\mathbf{x}) = f(\mathbf{x})$ for every $\mathbf{x} \in S$, then $f(\mathbf{0}) = g(\mathbf{0})$, $f(\mathbf{e}_1) = g(\mathbf{e}_1)$, $f(\mathbf{e}_2) = g(\mathbf{e}_2)$, … , $f(\mathbf{e}_n) = g(\mathbf{e}_n)$. Since $f : \mathbb{E}^n \to \mathbb{E}^n$ is distance preserving and $g : \mathbb{E}^n \to \mathbb{E}^n$ is a rigid motion, then Corollary 6.2 implies $f = g$. We conclude that f is a rigid motion of $\mathbb{E}^n$ that is the composition of $n + 1$ or fewer reflections. ◻

Homework Problem 6.4. a) Prove that for every $\mathbf{p} \in \mathbb{E}^n$, the $n + 1$ points $\mathbf{p}, \mathbf{p} + \mathbf{e}_1, \mathbf{p} + \mathbf{e}_2, \dots , \mathbf{p} + \mathbf{e}_n$ are non-coplanar in $\mathbb{E}^n$.

b) Prove that if $f : \mathbb{E}^n \to \mathbb{E}^n$ is a distance preserving function with at least one fixed point, then f is a rigid motion of $\mathbb{E}^n$ which is the composition of *n or fewer* reflections.

We state two immediate corollaries of Theorem 6.9 because these corollaries are significant statements in their own right.

Corollary 6.10. Every distance preserving function from $\mathbb{E}^n$ to itself is a rigid motion of $\mathbb{E}^n$. ◻

Corollary 6.11. Every rigid motion of $\mathbb{E}^n$ is the composition of $n + 1$ or fewer reflections. ◻

Corollary 6.11 might be paraphrased by saying that reflections of $\mathbb{E}^n$ are the *atoms* from which all other rigid motions of $\mathbb{E}^n$ are built.

Corollary 6.10 resolves an issue raised by the hypotheses of Theorem 6.1 and Corollary 6.2. Theorem 6.1 says that if two distance preserving functions $f : \mathbb{E}^n \to \mathbb{E}^n$ and $g : \mathbb{E}^n \to \mathbb{E}^n$ agree on a non-coplanar set, then f = g, provided one of f or g is a rigid motion of $\mathbb{E}^n$. Since Corollary 6.10 tells that every distance preserving function from $\mathbb{E}^n$ to itself is a rigid motion, we can now drop the *rigid motion hypothesis*.

Corollary 6.12. Suppose $f : \mathbb{E}^n \to \mathbb{E}^n$ and $g : \mathbb{E}^n \to \mathbb{E}^n$ are distance preserving functions. If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are non-coplanar points in $\mathbb{E}^n$ such that $f(\mathbf{x}_i) = g(\mathbf{x}_i)$ for $1 \le i \le k$, then f = g. In particular, if $f(\mathbf{0}) = g(\mathbf{0})$ and $f(\mathbf{e}_i) = g(\mathbf{e}_i)$ for $1 \le i \le n$, then f = g. ◻

The chain of logic which led to Corollary 6.12 went from Theorem 6.1 to Corollary 6.2 to Theorem 6.9 to Corollaries 6.10 and 6.12. Hence, we must still begin this sequence of proofs by establishing the original version of Theorem 6.1 that includes the *rigid motion hypothesis*.

Theorem 6.9 and its corollaries reveal information about distance preserving functions between Euclidean spaces of different dimensions.

Corollary 6.13. No distance preserving function between Eucldiean spaces can lower dimension. In other words, if $f : \mathbb{E}^m \to \mathbb{E}^n$ is a distance preserving function, then $m \le n$.

Proof. This is a proof by contradiction. Assume $f : \mathbb{E}^m \to \mathbb{E}^n$ is a distance preserving function and $m > n$.

Define the function $g : \mathbb{E}^n \to \mathbb{E}^m$ by $g(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n, 0, 0, \ldots, 0)$.

Then g is distance preserving. Indeed, if $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{E}^n$, then

$$\mathbf{x} - \mathbf{y} = (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)$$

and

$$g(\mathbf{x}) - g(\mathbf{y}) = (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n, 0, 0, \ldots, 0).$$

Hence,

$$\begin{aligned} \| g(\mathbf{x}) - g(\mathbf{y}) \|^2 &= (x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2 + 0^2 + 0^2 + \ldots + 0^2 \\ &= (x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2 = \| \mathbf{x} - \mathbf{y} \|^2. \end{aligned}$$

Therefore $d(g(\mathbf{x}),g(\mathbf{y})) = \| g(\mathbf{x}) - g(\mathbf{y}) \| = \| \mathbf{x} - \mathbf{y} \| = d(\mathbf{x},\mathbf{y})$.

Also, $g : \mathbb{E}^n \to \mathbb{E}^m$ is not onto. Indeed, the element $(0, 0, \ldots, 0, 1)$ of $\mathbb{E}^m$ is not an element of $g(\mathbb{E}^n)$, because the m^{th} coordinate of every element of $g(\mathbb{E}^n)$ is 0. Thus, $g(\mathbb{E}^n) \neq \mathbb{E}^m$.

Now consider the function $g \circ f : \mathbb{E}^m \to \mathbb{E}^m$. $g \circ f : \mathbb{E}^m \to \mathbb{E}^m$ is distance preserving by Theorem 2.1.c because f and g are distance preserving. $g \circ f : \mathbb{E}^m \to \mathbb{E}^m$ is not onto because $g \circ f(\mathbb{E}^m) = g(f(\mathbb{E}^m)) \subset g(\mathbb{E}^n) \neq \mathbb{E}^m$. Thus, $g \circ f : \mathbb{E}^m \to \mathbb{E}^m$ is a distance preserving function which is not a rigid motion of $\mathbb{E}^m$. This contradicts Corollary 6.10. We conclude that $m \leq n$. ☐

Corollary 6.14. Every isometry between Euclidean spaces preserves dimension. In other words, if $f : \mathbb{E}^m \to \mathbb{E}^n$ is an isometry, them $m = n$.

Proof. If $f : \mathbb{E}^m \to \mathbb{E}^n$ is an isometry, then so is $f^{-1} : \mathbb{E}^n \to \mathbb{E}^m$ by Theorem 2.1.d. Hence, both $f : \mathbb{E}^m \to \mathbb{E}^n$ and $f^{-1} : \mathbb{E}^n \to \mathbb{E}^m$ are distance preserving. Now two applications of Corollary 6.13 tells us that $m \leq n$ and $n \leq m$. Therefore, $m = n$. ☐

We end this chapter with a discussion of *conjugation* of isometries. Conjugation is a method of *moving* an isometry without changing its fundamental character.

Definition. Two isometries $f : \mathbb{E}^n \to \mathbb{E}^n$ and $g : \mathbb{E}^n \to \mathbb{E}^n$ of $\mathbb{E}^n$ are *conjugate* if there is an isometry $h : \mathbb{E}^n \to \mathbb{E}^n$ such that $h \circ f \circ h^{-1} = g$. If $f : \mathbb{E}^n \to \mathbb{E}^n$ and $g : \mathbb{E}^n \to \mathbb{E}^n$ are conjugate, then we say g is obtained from f by *conjugation,* we call g a *conjugate* of f and we call h the *conjugating isometry.*

Next we state a fundamental property of the conjugation relation.

Theorem 6.15. If $f : \mathbb{E}^n \to \mathbb{E}^n$, $g : \mathbb{E}^n \to \mathbb{E}^n$ and $h : \mathbb{E}^n \to \mathbb{E}^n$ are isometries of $\mathbb{E}^n$, then the following three statements hold.

a) f is conjugate to itself.

b) If f is conjugate to g, then g is conjugate to f.

c) If f is conjugate to g and g is conjugate to h, then f is conjugate to h.

Homework Problem 6.5. Prove Theorem 6.15.

Theorem 6.15 says that conjugation is an equivalence relation on the set of all isometries of $\mathbb{E}^n$.

If two isometries are conjugate, then the conjugating isometry moves sets that have a significant relation to the first isometry to sets that have the same relation to the second isometry. The following homework problem illustrates this pheomenon. For example, it asserts that the conjugating isometry moves fixed points of the first isometry to fixed points of the second isometry.

Definition. Let $f : \mathbb{E}^n \to \mathbb{E}^n$ be an isometry. The set $\{ \mathbf{x} \in \mathbb{E}^n : f(\mathbf{x}) = \mathbf{x} \}$ is called the *fixed point set* of f. For example, the fixed point set of a translation is empty, and the fixed point set of a reflection $Z_{\mathbf{u},a}$ is the hyperplane $P(\mathbf{u},a)$. If S is a subset of $\mathbb{E}^n$ such that $f(S) \subset S$, then we call S an *invariant set* of f.

Homework Problem 6.6. a) Let $\mathbf{p} \in \mathbb{E}^n$. Prove that for each $\mathbf{x} \in \mathbb{E}^n$, the line $L(\mathbf{x},\mathbf{x} + \mathbf{p})$ is an invariant set of the translation $T_{\mathbf{p}}$.

b) Let $\mathbf{u} \in \mathbb{E}^n$ such that $\| \mathbf{u} \| = 1$ and let $a \in \mathbb{R}$. Prove that for each $\mathbf{x} \in \mathbb{E}^n$, the line $L(\mathbf{x},\mathbf{x} + \mathbf{u})$ is an invariant set of the reflection $Z_{\mathbf{u},a}$.

c) Suppose $f : \mathbb{E}^n \to \mathbb{E}^n$ and $g : \mathbb{E}^n \to \mathbb{E}^n$ are conjugate isometries of $\mathbb{E}^n$ and $h : \mathbb{E}^n \to \mathbb{E}^n$ is the conjugating isometry; in other words, $h \circ f \circ h^{-1} = g$. Prove that if F is the fixed point set of f, then h(F) is the fixed point set of g.

d) Suppose $f : \mathbb{E}^n \to \mathbb{E}^n$ and $g : \mathbb{E}^n \to \mathbb{E}^n$ are conjugate isometries of $\mathbb{E}^n$ and $h : \mathbb{E}^n \to \mathbb{E}^n$ is the conjugating isometry; in other words, $h \circ f \circ h^{-1} = g$. Prove that if the subset S of $\mathbb{E}^n$ is an invariant set of f, then h(S) is an invariant set of g.

As we said previously, conjugation *moves* an isometry without changing its character. The next two theorems illustrate this assertion by revealing that each conjugate of a translation must be a translation, and each conjugate of a reflection must be a reflection.

Theorem 6.16. If $\mathbf{p} \in \mathbb{E}^n$ and $h : \mathbb{E}^n \to \mathbb{E}^n$ is an isometry of $\mathbb{E}^n$, then $h \circ T_{\mathbf{p}} \circ h^{-1} = T_{\mathbf{q}}$ where $\mathbf{q} = h(\mathbf{p}) - h(\mathbf{0})$.

Homework Problem 6.7. Prove Theorem 6.16.

Hint. Use the fact that h is affine to prove $h \circ T_{\mathbf{p}} = T_{\mathbf{q}} \circ h$.

Theorem 6.17. If $\mathbf{u} \in \mathbb{E}^n$ such that $\| \mathbf{u} \| = 1$, $a \in \mathbb{R}$ and $h : \mathbb{E}^n \to \mathbb{E}^n$ is an isometry of $\mathbb{E}^n$, then $h \circ Z_{\mathbf{u},a} \circ h^{-1} = Z_{\mathbf{v},b}$ where $\mathbf{v} = h(\mathbf{u}) - h(\mathbf{0})$ and $b = h(a\mathbf{u}) \bullet \mathbf{v}$.

Homework Problem 6.8. Prove Theorem 6.17.

Hint. Prove $\| \mathbf{v} \| = 1$, $\mathbf{x} \bullet \mathbf{u} - a = h(\mathbf{x}) \bullet \mathbf{v} - b$ for all $\mathbf{x} \in \mathbb{E}^n$, and $h \circ Z_{\mathbf{u},a} = Z_{\mathbf{v},b} \circ h$. Use the facts that h preserves dot products of differences and that h is affine.