5. Lines and Hyperplanes

In $\mathbb{E}^n$, lines and hyperplanes ((n – 1)-dimensional planes) are subsets of special geometric significance. We now explore their properties.

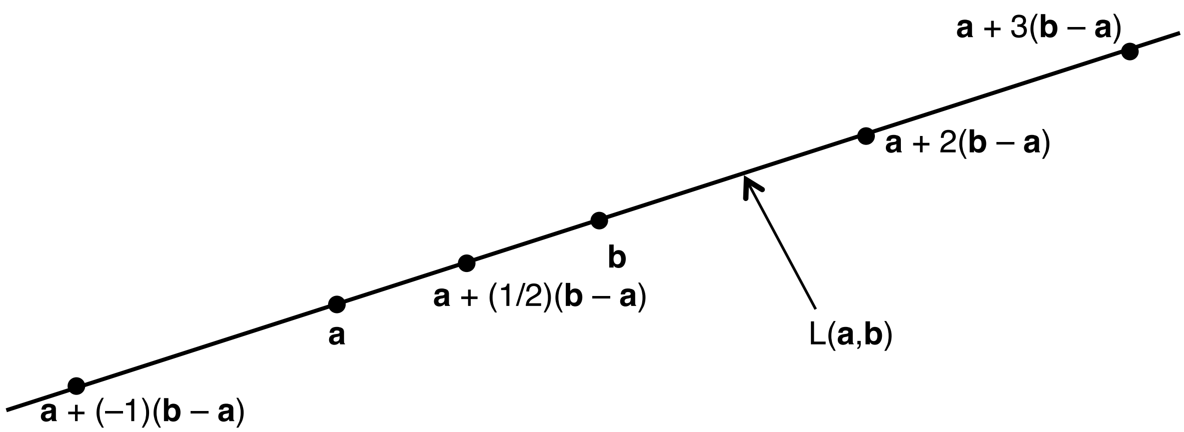
Definition. Let **a** and **b** be *distinct* points of $\mathbb{E}^n$ (i.e., $\mathbf{a} \neq \mathbf{b}$). Define the *line in* $\mathbb{E}^n$ *determined by* **a** *and* **b** to be the set

$$L(\mathbf{a},\mathbf{b}) = \{ (1-t)\mathbf{a} + t\mathbf{b} : t \in \mathbb{R} \}.$$

A subset of $\mathbb{E}^n$ is called a *line* if and only if it is a set of the form L(**a**,**b**) where **a** and $\mathbf{b} \in \mathbb{E}^n$ and $\mathbf{a} \neq \mathbf{b}$.

Observe that if **a** and **b** are distinct points of $\mathbb{E}^n$, the the following statements are equivalent:

- $\mathbf{x} \in L(\mathbf{a},\mathbf{b})$.
- There is a $t \in \mathbb{R}$ such that $\mathbf{x} = (1-t)\mathbf{a} + t\mathbf{b}$.
- There is a $t \in \mathbb{R}$ such that $\mathbf{x} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$.
- There exist s and $t \in \mathbb{R}$ such that $s + t = 1$ and $\mathbf{x} = s\mathbf{a} + t\mathbf{b}$.



The first important theorems about lines is:

Theorem 5.1. *The Existence and Uniqueness of Lines.* If **a** and **b** are any two distinct points of $\mathbb{E}^n$, then L(**a**,**b**) is the one and only line in $\mathbb{E}^n$ that contains **a** and **b**.

Proof. Assume **a** and **b** are distinct points of $\mathbb{E}^n$. We must prove that L(**a**,**b**) contains **a** and **b**, and that L(**a**,**b**) is the *only* line that contains **a** and **b**.

Since $\mathbf{a} = 1\mathbf{a} + 0\mathbf{b}$ and $\mathbf{b} = 0\mathbf{a} + 1\mathbf{b}$ and $1 + 0 = 1 = 0 + 1$, then $\mathbf{a}$ and $\mathbf{b} \in L(\mathbf{a},\mathbf{b})$. Hence, $L(\mathbf{a},\mathbf{b})$ contains $\mathbf{a}$ and $\mathbf{b}$.

To prove that $(\mathbf{a},\mathbf{b})$ is the only line that contains $\mathbf{a}$ and $\mathbf{b}$, assume M is a line that contains $\mathbf{a}$ and $\mathbf{b}$. We will prove that $M = L(\mathbf{a},\mathbf{b})$. Since M is a line, then there are distinct elements $\mathbf{c}$ and $\mathbf{d}$ of $\mathbb{E}^n$ such that $M = L(\mathbf{c},\mathbf{d})$. So $\mathbf{a}$ and $\mathbf{b}$ are elements of $L(\mathbf{c},\mathbf{d})$.

We now prove a lemma to help carry out the proof.

Lemma 5.2. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d} \in \mathbb{E}^n$ such that $\mathbf{a} \neq \mathbf{b}$ and $\mathbf{c} \neq \mathbf{d}$ and if $\mathbf{a}$ and $\mathbf{b} \in L(\mathbf{c},\mathbf{d})$, then $L(\mathbf{a},\mathbf{b}) \subset L(\mathbf{c},\mathbf{d})$.

Proof. We assume $\mathbf{a}$ and $\mathbf{b} \in L(\mathbf{c},\mathbf{d})$. Hence, there are real numbers r and s such that

$$\mathbf{a} = (1-r)\mathbf{c} + r\mathbf{d} \quad \text{and} \quad \mathbf{b} = (1-s)\mathbf{c} + s\mathbf{d}.$$

Let $\mathbf{x} \in L(\mathbf{a},\mathbf{b})$. Then there is a real number t such that $\mathbf{x} = (1-t)\mathbf{a} + t\mathbf{b}$. Hence,

$$\begin{aligned} \mathbf{x} &= (1-t)(\,(1-r)\mathbf{c} + r\mathbf{d}\,) + t(\,(1-s)\mathbf{c} + s\mathbf{d}\,) \\ &= (\,(1-t)(1-r) + t(1-s)\,)\mathbf{c} + (\,(1-t)r + ts\,)\mathbf{d}. \end{aligned}$$

Let $u = (\,(1-t)(1-r) + t(1-s)\,)$ and let $v = (\,(1-t)r + ts\,)$. Then u and v are real numbers such that $\mathbf{x} = u\mathbf{c} + v\mathbf{b}$. Furthermore,

$$\begin{aligned} u + v &= (\,(1-t)(1-r) + t(1-s)\,) + (\,(1-t)r + ts\,) \\ &= (1-t)(1-r) + (1-t)r + t(1-s) + ts \\ &= (1-t)(1-r+r) + t(1-s+s) \\ &= (1-t) + t = 1. \end{aligned}$$

Since $\mathbf{x} = u\mathbf{c} + v\mathbf{b}$ and $u + v = 1$, then $\mathbf{x} \in L(\mathbf{c},\mathbf{d})$. This proves $L(\mathbf{a},\mathbf{b}) \subset L(\mathbf{c},\mathbf{d})$. □

Returning to the proof of Theorem 5.1, we have distinct points $\mathbf{a}$ and $\mathbf{b}$ of $\mathbb{E}^n$, and we have assumed that $\mathbf{a}$ and $\mathbf{b} \in M = L(\mathbf{c},\mathbf{d})$ where $\mathbf{c}$ and $\mathbf{d}$ are distinct points of $\mathbb{E}^n$. Hence, Lemma 5.2 implies $L(\mathbf{a},\mathbf{b}) \subset L(\mathbf{c},\mathbf{d})$.

Next we will prove that $\mathbf{c}$ and $\mathbf{d} \in L(\mathbf{a},\mathbf{b})$. Since $\mathbf{a}$ and $\mathbf{b} \in L(\mathbf{c},\mathbf{d})$, then there are real numbers r and s such that the following equations hold:

$$\mathbf{a} = (1-r)\mathbf{c} + r\mathbf{d} \quad \text{and} \quad \mathbf{b} = (1-s)\mathbf{c} + s\mathbf{d} \qquad (*)$$

Since $\mathbf{a} \neq \mathbf{b}$, then $r \neq s$. Hence, $r - s \neq 0$ and $s - r \neq 0$. We now solve the equations in $(*)$ to express $\mathbf{c}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. Observe that

$$s\mathbf{a} = (1-r)s\mathbf{c} + rs\mathbf{d} \quad \text{and} \quad r\mathbf{b} = r(1-s)\mathbf{c} + rs\mathbf{d}.$$

Hence,

$$s\mathbf{a} - r\mathbf{b} \;=\; (\,(1-r)s - r(1-s)\,)\mathbf{c} \;=\; (s-r)\mathbf{c}.$$

We divide both sides of this equation by s – r to obtain:

$$\mathbf{c} = \left(\frac{s}{s-r}\right)\mathbf{a} + \left(\frac{-r}{s-r}\right)\mathbf{b}.$$

Since

$$\frac{s}{s-r} + \frac{-r}{s-r} = 1,$$

then $\mathbf{c} \in L(\mathbf{a},\mathbf{b})$. Similarly, we solve the equations in (∗) to express **d** in terms of **a** and **b**. Observe that

$$(1-s)\mathbf{a} \;=\; (1-r)(1-s)\mathbf{c} + r(1-s)\mathbf{d} \;\text{ and }\; (1-r)\mathbf{b} \;=\; (1-r)(1-s)\mathbf{c} + (1-r)s\mathbf{d}.$$

Hence,

$$(1-s)\mathbf{a} - (1-r)\mathbf{b} \;=\; (\,r(1-s) - (1-r)s\,)\mathbf{d} \;=\; (r-s)\mathbf{d}.$$

Dividing both sides of this equation by r – s, we obtain:

$$\mathbf{d} = \left(\frac{1-s}{r-s}\right)\mathbf{a} + \left(\frac{r-1}{r-s}\right)\mathbf{b}.$$

Since

$$\frac{1-s}{r-s} + \frac{r-1}{r-s} = 1,$$

then $\mathbf{d} \in L(\mathbf{a},\mathbf{b})$. We have now proved that **c** and $\mathbf{d} \in L(\mathbf{a},\mathbf{b})$.

Since **c** and $\mathbf{d} \in L(\mathbf{a},\mathbf{b})$, then Lemma 5.2 (with the roles of **a** and **b** interchanged with the roles of **c** and **d**) implies $L(\mathbf{c},\mathbf{d}) \subset L(\mathbf{a},\mathbf{b})$. We have now shown that $L(\mathbf{a},\mathbf{b}) \subset L(\mathbf{c},\mathbf{d})$ and $L(\mathbf{c},\mathbf{d}) \subset L(\mathbf{a},\mathbf{b})$. Hence, $L(\mathbf{a},\mathbf{b}) = L(\mathbf{c},\mathbf{d})$. Therefore, $M = L(\mathbf{a},\mathbf{b})$. It follows that $L(\mathbf{a},\mathbf{b})$ is the one and only line in $\mathbb{E}^n$ that contains **a** and **b**. This completes the proof of Theorem 5.1. □

The following theorem allows us to detect which subsets of $\mathbb{E}^n$ are lines from their metric properties.

Theorem 5.3. *The Metric Characterization of Lines.* A subset L of $\mathbb{E}^n$ is a line if and only if L is isometric to $\mathbb{R}$.

Proof. First assume L is a line. Then there are distinct points **a** and **b** in $\mathbb{E}^n$ such that $L = L(\mathbf{a},\mathbf{b})$. Let

$$\mathbf{u} \;=\; \left(\frac{1}{\|\,\mathbf{b}-\mathbf{a}\,\|}\right)(\mathbf{b}-\mathbf{a}).$$

Then $\| \mathbf{u} \| = 1$. Observe that

$$\mathbf{a} + \mathbf{u} = \mathbf{a} + \left(\frac{1}{\| \mathbf{b} - \mathbf{a} \|}\right)(\mathbf{b} - \mathbf{a}).$$

Hence, $\mathbf{a}$ and $\mathbf{a} + \mathbf{u} \in L(\mathbf{a},\mathbf{b})$. Also, since $\mathbf{u} \neq \mathbf{0}$, then $\mathbf{a} \neq \mathbf{a} + \mathbf{u}$. Hence, Theorem 5.1 implies $L(\mathbf{a},\mathbf{b}) = L(\mathbf{a},\mathbf{a} + \mathbf{u})$. Hence, $L = L(\mathbf{a},\mathbf{a} + \mathbf{u})$.

Define $f : \mathbb{R} \to \mathbb{E}^n$ by $f(t) = \mathbf{a} + t\mathbf{u}$. Then for $s, t \in \mathbb{R}$,

$$d(f(s),f(t)) = \| f(s) - f(t) \| = \| (\mathbf{a} + s\mathbf{u}) - (\mathbf{a} + t\mathbf{u}) \| =$$

$$\| (s - t)\mathbf{u} \| = | s - t | \| \mathbf{u} \| = | s - t | (1) = | s - t | = d(s,t).$$

Hence, $f : \mathbb{R} \to \mathbb{E}^n$ is distance preserving.

Next we prove $f(\mathbb{R}) = L$. If $t \in \mathbb{R}$, then

$$f(t) = \mathbf{a} + t\mathbf{u} = \mathbf{a} - t\mathbf{a} + t\mathbf{a} + t\mathbf{u} = (1 - t)\mathbf{a} + t(\mathbf{a} + \mathbf{u}) \in L(\mathbf{a},\mathbf{a} + \mathbf{u}) = L.$$

This proves $f(\mathbb{R}) \subset L$. To prove $L \subset f(\mathbb{R})$, let $\mathbf{x} \in L$. Since $L = L(\mathbf{a},\mathbf{a} + \mathbf{u})$, then there is a $t \in \mathbb{R}$ such that

$$\mathbf{x} = (1 - t)\mathbf{a} + t(\mathbf{a} + \mathbf{u}) = \mathbf{a} - t\mathbf{a} + t\mathbf{a} + t\mathbf{u} = \mathbf{a} + t\mathbf{u} = f(t) \in f(\mathbb{R}).$$

We conclude that $L \subset f(\mathbb{R})$. Since $f(\mathbb{R}) \subset L$ and $L \subset f(\mathbb{R})$, then $f(\mathbb{R}) = L$.

Since $f : \mathbb{R} \to \mathbb{E}^n$ is distance preserving and $f(\mathbb{R}) = L$, then $f : \mathbb{R} \to L$ is a distance preserving onto function. Hence, $f : \mathbb{R} \to L$ is an isometry. Therefore, L is isometric to $\mathbb{R}$. This completes the proof in one direction.

To complete the proof in the other direction, assume L is a subset of $\mathbb{E}^n$ that is isometric to $\mathbb{R}$. We must prove that L is a line. Since L is isometric to $\mathbb{R}$, then there is an isometry $f : \mathbb{R} \to L$. Therefore, $f : \mathbb{R} \to \mathbb{E}^n$ is distance preserving and $f(\mathbb{R}) = L$. Let $\mathbf{a} = f(0)$ and $\mathbf{b} = f(1)$. We will prove that $L = L(\mathbf{a},\mathbf{b})$. Since $f : \mathbb{R} \to \mathbb{E}^n$ is distance preserving, then we may invoke Theorem 4.13 to conclude that $f : \mathbb{R} \to \mathbb{E}^n$ is affine. Therefore, for every $t \in \mathbb{R}$,

$$f(t) = f((1 - t)0 + t1) = (1 - t)f(0) + tf(1) = (1 - t)\mathbf{a} + t\mathbf{b} \in L(\mathbf{a},\mathbf{b}).$$

Thus, $f(\mathbb{R}) \subset L(\mathbf{a},\mathbf{b})$. So $L \subset L(\mathbf{a},\mathbf{b})$. To prove the opposite inclusion, let $\mathbf{x} \in L(\mathbf{a},\mathbf{b})$. Then there is a $t \in \mathbb{R}$ such that $\mathbf{x} = (1 - t)\mathbf{a} + t\mathbf{b}$. We just showed that $f(t) = (1 - t)\mathbf{a} + t\mathbf{b}$. Hence, $f(t) = \mathbf{x}$. Therefore, $\mathbf{x} \in f(\mathbb{R}) = L$. We conclude that $L(\mathbf{a},\mathbf{b}) \subset L$. Since $L \subset L(\mathbf{a},\mathbf{b})$. and $L(\mathbf{a},\mathbf{b}) \subset L$, then $L = L(\mathbf{a},\mathbf{b})$. We have now prove that if L is a subset of $\mathbb{E}^n$ which is isometric to $\mathbb{R}$, then L is a line. ◻

Homework Problem 5.1. This problem asks you to show that one direction of Theorem 5.3 holds in $\mathbb{R}^2$ with the taxicab metric, but the other does not.

a) Prove that if **a** and **b** are distinct points of $\mathbb{R}^2$, then the subset L(**a**,**b**) of $\mathbb{R}^2$ with the taxicab metric is isometric to $\mathbb{R}$.

b) Find a subset of $\mathbb{R}^2$ with the taxicab metric that is isometric to $\mathbb{R}$ but is not of the form L(**a**,**b**) for any two distinct points **a** and **b** of $\mathbb{R}^2$.

Here is an application of Theorem 5.3.

Corollary 5.4. If L is a line in $\mathbb{E}^m$ and $f : \mathbb{E}^m \to \mathbb{E}^n$ is a distance preserving function, then f(L) is a line in $\mathbb{E}^n$.

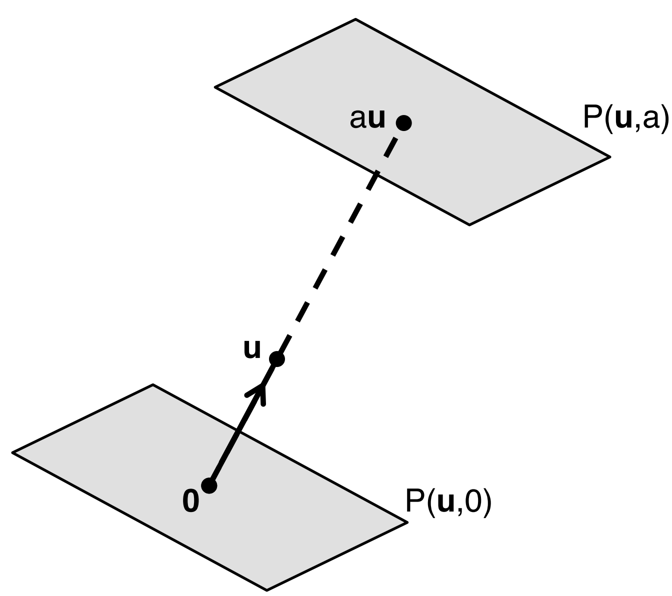
Proof. Since L is a line in $\mathbb{E}^m$, then Theorem 5.3 implies there is an isometry $g : \mathbb{R} \to L$. Since $f : \mathbb{E}^m \to \mathbb{E}^n$ is distance preserving and the restriction $f \mid L : L \to f(L)$ is onto, then $f \mid L : L \to f(L)$ is an isometry. Hence, the composition $(f \mid L) \circ g : \mathbb{R} \to f(L)$ is an isometry by Theorem 2.1.c. Thus, f(L) is isometric to $\mathbb{R}$. Now a second application of Theorem 5.3 implies f(L) is a line in $\mathbb{E}^n$. □

Definition. Let $\mathbf{u} \in \mathbb{E}^n$ such that $\| \mathbf{u} \| = 1$ and let $a \in \mathbb{R}$. Define

$$P(\mathbf{u},a) = \{ \mathbf{x} \in \mathbb{E}^n : \mathbf{x} \bullet \mathbf{u} = a \}.$$

A subset of $\mathbb{E}^n$ is called a *hyperplane* if and only if it is of the form $P(\mathbf{u},a)$ where $\mathbf{u} \in \mathbb{E}^n$, $\| \mathbf{u} \| = 1$ and $a \in \mathbb{R}$.

A hyperplane in $\mathbb{E}^n$ is, intuitively, a flat subset of $\mathbb{E}^n$ of dimension n – 1. Since the word *plane* is usually reserved for 2-dimensional objects, then the only hyperplanes that are called *planes* are the hyperplanes in $\mathbb{E}^3$.



Observe that if $\mathbf{u} \in \mathbb{E}^n$, $\| \mathbf{u} \| = 1$ and $a \in \mathbb{R}$, then $a\mathbf{u} \in P(\mathbf{u},a)$ because $(a\mathbf{u})\bullet\mathbf{u} = a\| \mathbf{u} \|^2 = a\cdot 1 = a$. Since $P(\mathbf{u},0) = \{ \mathbf{x} \in \mathbb{E}^n : \mathbf{x}\bullet\mathbf{u} = 0 \}$, then we can visualize $P(\mathbf{u},0)$ as the $(n-1)$–dimensional plane in $\mathbb{E}^n$ that passes through the origin $\mathbf{0}$ and is perpendicular to $\mathbf{u}$. Then the translation which moves $\mathbf{0}$ to $a\mathbf{u}$ moves $P(\mathbf{u},0)$ to $P(\mathbf{u},a)$.

Theorem 5.5. Suppose $\mathbf{u}$ and $\mathbf{v} \in \mathbb{E}^n$ such that $\| \mathbf{u} \| = \| \mathbf{v} \| = 1$ and let a and b $\in \mathbb{R}$. Then $P(\mathbf{u},a) = P(\mathbf{v},b)$ if and only if: either $\mathbf{u} = \mathbf{v}$ and $a = b$, or $\mathbf{u} = -\mathbf{v}$ and $a = -b$.

We break the proof of Theorem 5.5 into two lemmas which we assign as homework problems.

Lemma 5.6. Suppose $\mathbf{u}$ and $\mathbf{v} \in \mathbb{E}^n$ such that $\| \mathbf{u} \| = \| \mathbf{v} \| = 1$ and let a and b $\in \mathbb{R}$. If either $\mathbf{u} = \mathbf{v}$ and $a = b$, or $\mathbf{u} = -\mathbf{v}$ and $a = -b$, then $P(\mathbf{u},a) = P(\mathbf{v},b)$.

Homework Problem 5.2. Prove Lemma 5.6.

Lemma 5.7. Suppose $\mathbf{u}$ and $\mathbf{v} \in \mathbb{E}^n$ such that $\| \mathbf{u} \| = \| \mathbf{v} \| = 1$ and let a and b $\in \mathbb{R}$. If $P(\mathbf{u},a) \subset P(\mathbf{v},b)$, then either $\mathbf{u} = \mathbf{v}$ and $a = b$, or $\mathbf{u} = -\mathbf{v}$ and $a = -b$.

Homework Problem 5.3. Prove Lemma 5.7.

Hint of Homework Problem 5.3. Suppose $\mathbf{u}$ and $\mathbf{v} \in \mathbb{E}^n$ such that $\| \mathbf{u} \| = \| \mathbf{v} \| = 1$ and let a and b $\in \mathbb{R}$. Assume $P(\mathbf{u},a) \subset P(\mathbf{v},b)$. Let $\mathbf{x} = \mathbf{v} - (\mathbf{u}\bullet\mathbf{v})\mathbf{u}$.

Step 1: Prove $a\mathbf{u}$ and $a\mathbf{u} + \mathbf{x} \in P(\mathbf{u},a)$.

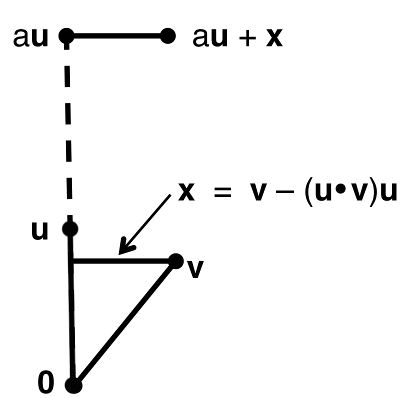
Step 2: Prove $a\mathbf{u}$ and $a\mathbf{u} + \mathbf{x} \in P(\mathbf{v},b)$.

Step 3: Prove $a(\mathbf{u}\bullet\mathbf{v}) = b$ and $\mathbf{x}\bullet\mathbf{v} = 0$.

Step 4: Prove $\mathbf{u}\bullet\mathbf{v} = \pm 1$.

Step 5: Prove $\mathbf{u} = \pm\mathbf{v}$.

Step 6: Prove $\mathbf{u} = \mathbf{v} \Rightarrow a = b$, and $\mathbf{u} = -\mathbf{v} \Rightarrow a = -b$.



We now show how to prove Theorem 5.5 from Lemmas 5.6 and 5.7.

Proof of Theorem 5.5. Suppose $\mathbf{u}$ and $\mathbf{v} \in \mathbb{E}^n$ such that $\| \mathbf{u} \| = \| \mathbf{v} \| = 1$ and let a and b $\in \mathbb{R}$.

First assume $P(\mathbf{u},a) = P(\mathbf{v},b)$. Then $P(\mathbf{u},a) \subset P(\mathbf{v},b)$. Therefore Lemma 5.7 implies either $\mathbf{u} = \mathbf{v}$ and $a = b$, or $\mathbf{u} = -\mathbf{v}$ and $a = -b$.

Second assume either $\mathbf{u} = \mathbf{v}$ and $a = b$, or $\mathbf{u} = -\mathbf{v}$ and $a = -b$. Then Lemma 5.6 implies $P(\mathbf{u},a) = P(\mathbf{v},b)$. □

Lemmas 5.6 and 5.7 yield another result which we will use later.

Corollary 5.8. If P and Q are hyperplanes in $\mathbb{E}^n$ and $P \subset Q$, then $P = Q$.

Proof. Suppose P and Q are hyperplanes in $\mathbb{E}^n$ such that $P \subset Q$. Then there exist $\mathbf{u}$ and $\mathbf{v} \in \mathbb{E}^n$ such that $\| \mathbf{u} \| = \| \mathbf{v} \| = 1$ and there exist a and b $\in \mathbb{R}$ such that $P = P(\mathbf{u},a)$ and $Q = P(\mathbf{v},b)$. Hence $P(\mathbf{u},a) \subset P(\mathbf{v},b)$. Therefore, Lemma 5.7 implies either $\mathbf{u} = \mathbf{v}$ and $a = b$, or $\mathbf{u} = -\mathbf{v}$ and $a = -b$. Consequently, Lemma 5.6 implies $P(\mathbf{u},a) = P(\mathbf{v},b)$. Thus, $P = Q$. □

Definition. Let $\mathbf{a}$ and $\mathbf{b}$ be *distinct* points of $\mathbb{E}^n$ (i.e., $\mathbf{a} \neq \mathbf{b}$). Define the *line segment in* $\mathbb{E}^n$ *joining* $\mathbf{a}$ *to* $\mathbf{b}$ to be the set

$$J(\mathbf{a},\mathbf{b}) \ = \ \{ (1-t)\mathbf{a} + t\mathbf{b} : t \in [0,1] \}.$$

Thus, $\mathbf{x} \in J(\mathbf{a},\mathbf{b})$ if and only if there is a t such that $0 \le t \le 1$ and $\mathbf{x} = (1-t)\mathbf{a} + t\mathbf{b}$. Since $t \in [0,1] \Rightarrow 1 - t \in [0,1]$ and $(1-t)\mathbf{a} + t\mathbf{b} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$, then it is also true that:

$\mathbf{x} \in J(\mathbf{a},\mathbf{b})$ if and only if there exist s and $t \in [0,1]$ such that $s + t = 1$ and $\mathbf{x} = s\mathbf{a} + t\mathbf{b}$

and

$\mathbf{x} \in J(\mathbf{a},\mathbf{b})$ if and only if there is a $t \in [0,1]$ such that $\mathbf{x} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$.

The points $\mathbf{a}$ and $\mathbf{b}$ are called the *endpoints* of $J(\mathbf{a},\mathbf{b})$, and $J(\mathbf{a},\mathbf{b})$ is also called the *line segment in* $\mathbb{E}^n$ *with endpoints* $\mathbf{a}$ *and* $\mathbf{b}$. A subset of $\mathbb{E}^n$ is called a *line segment* if and only if it is a set of the form $J(\mathbf{a},\mathbf{b})$ where $\mathbf{a}$ and $\mathbf{b} \in \mathbb{E}^n$ and $\mathbf{a} \neq \mathbf{b}$.

Definition. A subset S of $\mathbb{E}^n$ is *convex* if $\mathbf{a}$ and $\mathbf{b} \in S$ and $\mathbf{a} \neq \mathbf{b}$ implies $J(\mathbf{a},\mathbf{b}) \subset S$. In other words, S is convex if and only if S contains $J(\mathbf{a},\mathbf{b})$ whenever S contains $\mathbf{a}$ and $\mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$.

Theorem 5.9. *The Hyperplane Separation Theorem.* Suppose $P = P(\mathbf{u},c)$ is a hyperplane in $\mathbb{E}^n$ where $\mathbf{u} \in \mathbb{E}^n$, $\| \mathbf{u} \| = 1$ and $c \in \mathbb{R}$. Let $U = \{ \mathbf{x} \in \mathbb{E}^n : \mathbf{x} \bullet \mathbf{u} > c \}$ and $V = \{ \mathbf{x} \in \mathbb{E}^n : \mathbf{x} \bullet \mathbf{u} < c \}$. Then U and V are non-empty disjoint convex subsets of $\mathbb{E}^n$ such that $\mathbb{E}^n - P = U \cup V$ and if $\mathbf{a} \in U$ and $\mathbf{b} \in V$, then $J(\mathbf{a},\mathbf{b}) \cap P \neq \emptyset$.

Proof. Suppose $P = P(\mathbf{u},c)$ is a hyperplane in $\mathbb{E}^n$ where $\mathbf{u} \in \mathbb{E}^n$, $\| \mathbf{u} \| = 1$ and $c \in \mathbb{R}$. Let $U = \{ \mathbf{x} \in \mathbb{E}^n : \mathbf{x}\bullet\mathbf{u} > c \}$ and $V = \{ \mathbf{x} \in \mathbb{E}^n : \mathbf{x}\bullet\mathbf{u} < c \}$. The definition of $P(\mathbf{u},c)$ implies $P = \{ \mathbf{x} \in \mathbb{E}^n : \mathbf{x}\bullet\mathbf{u} = c \}$.

Since $((c + 1)\mathbf{u})\bullet\mathbf{u} = (c + 1)\| \mathbf{u} \|^2 = c + 1$ and $((c - 1)\mathbf{u})\bullet\mathbf{u} = (c - 1)\| \mathbf{u} \|^2 = c - 1$, then $(c + 1)\mathbf{u} \in U$ and $(c - 1)\mathbf{u} \in V$. Therefore, U and V are non-empty.

Since for each $\mathbf{x} \in \mathbb{E}^n$, $\mathbf{x}$ must satisfy exactly one of the conditions $\mathbf{x}\bullet\mathbf{u} = c$, $\mathbf{x}\bullet\mathbf{u} > c$ and $\mathbf{x}\bullet\mathbf{u} < c$, then $\mathbf{x}$ belongs to exactly one of the sets P, U and V. Hence, $\mathbb{E}^n = P \cup U \cup V$ and $P \cap U = P \cap V = U \cap V = \varnothing$. Therefore, $U \cap V = \varnothing$ and $\mathbb{E}^n - P = U \cup V$.

Next we prove that U is a convex subset of $\mathbb{E}^n$. Assume $\mathbf{a}$ and $\mathbf{b} \in U$ and $\mathbf{a} \neq \mathbf{b}$. We must prove $J(\mathbf{a},\mathbf{b}) \subset U$. To this end let $\mathbf{x} \in J(\mathbf{a},\mathbf{b})$. We must prove $\mathbf{x} \in U$. Since $\mathbf{x} \in J(\mathbf{a},\mathbf{b})$, then there is a $t \in [0,1]$ such that $\mathbf{x} = (1 - t)\mathbf{a} + t\mathbf{b}$. If $t = 0$, then $\mathbf{x} = \mathbf{a} \in U$, and if $t = 1$, then $\mathbf{x} = \mathbf{b} \in U$. So we may assume $0 < t < 1$. Therefore $t > 0$ and $1 - t > 0$. Since $\mathbf{a}$ and $\mathbf{b} \in U$, then $\mathbf{a}\bullet\mathbf{u} > c$ and $\mathbf{b}\bullet\mathbf{u} > c$. It follows that $(1 - t)\mathbf{a}\bullet\mathbf{u} > (1 - t)c$ and $t\mathbf{b}\bullet\mathbf{u} > tc$. Hence,

$$\mathbf{x}\bullet\mathbf{u} = ((1 - t)\mathbf{a} + t\mathbf{b})\bullet\mathbf{u} = (1 - t)\mathbf{a}\bullet\mathbf{u} + t\mathbf{b}\bullet\mathbf{u} > (1 - t)c + tc = c.$$

Thus, $\mathbf{x} \in U$. This completes the proof that $J(\mathbf{a},\mathbf{b}) \subset U$. We conclude that U is convex.

The proof that V is a convex subset of $\mathbb{E}^n$ is similar. Assume $\mathbf{a}$ and $\mathbf{b} \in V$ and $\mathbf{a} \neq \mathbf{b}$. Let $\mathbf{x} \in J(\mathbf{a},\mathbf{b})$. Then there is a $t \in [0,1]$ such that $\mathbf{x} = (1 - t)\mathbf{a} + t\mathbf{b}$. We may assume $0 < t < 1$, because if $t = 0$, then $\mathbf{x} = \mathbf{a} \in V$, and if $t = 1$, then $\mathbf{x} = \mathbf{b} \in V$. Therefore $t > 0$ and $1 - t > 0$. Since $\mathbf{a}$ and $\mathbf{b} \in V$, then $\mathbf{a}\bullet\mathbf{u} < c$ and $\mathbf{b}\bullet\mathbf{u} < c$. It follows that $(1 - t)\mathbf{a}\bullet\mathbf{u} < (1 - t)c$ and $t\mathbf{b}\bullet\mathbf{u} < tc$. Hence,

$$\mathbf{x}\bullet\mathbf{u} = ((1 - t)\mathbf{a} + t\mathbf{b})\bullet\mathbf{u} = (1 - t)\mathbf{a}\bullet\mathbf{u} + t\mathbf{b}\bullet\mathbf{u} < (1 - t)c + tc = c.$$

Thus, $\mathbf{x} \in V$. It follows that $J(\mathbf{a},\mathbf{b}) \subset V$, thereby proving V is convex.

Finally, assume $\mathbf{a} \in U$ and $\mathbf{b} \in V$. We must prove $J(\mathbf{a},\mathbf{b}) \cap P \neq \varnothing$. Since $\mathbf{a} \in U$ and $\mathbf{b} \in V$, then $\mathbf{a}\bullet\mathbf{u} > c > \mathbf{b}\bullet\mathbf{u}$. Therefore, $0 < \mathbf{a}\bullet\mathbf{u} - c < \mathbf{a}\bullet\mathbf{u} - \mathbf{b}\bullet\mathbf{u}$. Let $t = \dfrac{\mathbf{a}\bullet\mathbf{u} - c}{\mathbf{a}\bullet\mathbf{u} - \mathbf{b}\bullet\mathbf{u}}$. Then $0 < t < 1$ and $t(\mathbf{a}\bullet\mathbf{u} - \mathbf{b}\bullet\mathbf{u}) = \mathbf{a}\bullet\mathbf{u} - c$. Therefore, $t(\mathbf{b}\bullet\mathbf{u} - \mathbf{a}\bullet\mathbf{u}) = c - \mathbf{a}\bullet\mathbf{u}$. Let $\mathbf{x} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$. Since $0 < t < 1$, then $\mathbf{x} \in J(\mathbf{a},\mathbf{b})$. We will now prove $\mathbf{x} \in P$.

$$\mathbf{x}\bullet\mathbf{u} = (\mathbf{a} + t(\mathbf{b} - \mathbf{a}))\bullet\mathbf{u} = \mathbf{a}\bullet\mathbf{u} + t(\mathbf{b}\bullet\mathbf{u} - \mathbf{a}\bullet\mathbf{u}) = \mathbf{a}\bullet\mathbf{u} + (c - \mathbf{a}\bullet\mathbf{u}) = c.$$

Hence, $\mathbf{x} \in P$. Since $\mathbf{x} \in J(\mathbf{a},\mathbf{b})$ and $\mathbf{x} \in P$, then $J(\mathbf{a},\mathbf{b}) \cap P \neq \varnothing$. □

Definition. If P is a hyperplane in $\mathbb{E}^n$ and if U and V are non-empty disjoint convex subsets of $\mathbb{E}^n$ such that $\mathbb{E}^n - P = U \cup V$ and every line segment joining a point of U to a point of V intersects P, then we call U and V *opposite sides of* P.

Theorem 5.9 tells us that if P = P($\mathbf{u}$,c) where $\mathbf{u} \in \mathbb{E}^n$, $\| \mathbf{u} \| = 1$ and $c \in \mathbb{R}$, then the sets $U = \{ \mathbf{x} \in \mathbb{E}^n : \mathbf{x} \bullet \mathbf{u} > c \}$ and $V = \{ \mathbf{x} \in \mathbb{E}^n : \mathbf{x} \bullet \mathbf{u} < c \}$ are opposite sides of P. Our next result tells us that these two sets are the only possible opposite sides of P.

Theorem 5.10. If P is a hyperplane in $\mathbb{E}^n$ and U and V are opposite sides of P, then U and V are unique in the following sense. If U´ and V´ are also opposite sides of P, then either U = U´ and V = V´, or U = V´ and V = U´.

Remark. The second sentence of Theorem 5.10 is equivalent to: If U´ and V´ are also opposite sides of P, then { U, V } = { U´, V´ }.

We base our proof of Theorem 5.10 on the following lemma.

Lemma 5.11. Suppose P is a hyperplane in $\mathbb{E}^n$ and U and V are opposite sides of P. If $\mathbf{a} \in U$, $\mathbf{b} \in \mathbb{E}^n$, $\mathbf{a} \neq \mathbf{b}$ and $J(\mathbf{a},\mathbf{b}) \cap P = \varnothing$, then $\mathbf{b} \in U$.

Proof. Since $\mathbb{E}^n - P = U \cup V$, then **b** is an element of either P, U or V. However, since $\mathbf{b} \in J(\mathbf{a},\mathbf{b})$ and $J(\mathbf{a},\mathbf{b}) \cap P = \varnothing$, then $\mathbf{b} \notin P$. Hence, either $\mathbf{b} \in U$ or $\mathbf{b} \in V$. If $\mathbf{b} \in V$, then $J(\mathbf{a},\mathbf{b}) \cap P \neq \varnothing$, by the definition of *opposite sides.* However, $J(\mathbf{a},\mathbf{b}) \cap P = \varnothing$ by hypothesis. We conclude that $\mathbf{b} \in U$. ☐

Proof of Theorem 5.10. Assume that both U and V and U´ and V´ are opposite sides of P. Let $\mathbf{a} \in U$. Then $\mathbf{a} \in \mathbb{E}^n - P = U' \cup V'$. Hence, either $\mathbf{a} \in U'$ or $\mathbf{a} \in V'$.

Case 1: $\mathbf{a} \in U'$.

First we prove $U \subset U'$. Let $\mathbf{b} \in U$ such that $\mathbf{a} \neq \mathbf{b}$. Since U is convex, then $J(\mathbf{a},\mathbf{b}) \subset U$. Hence, $J(\mathbf{a},\mathbf{b}) \cap P = \varnothing$. Since $\mathbf{a} \in U'$, then Lemma 5.11 implies $\mathbf{b} \in U'$. This proves $U \subset U'$.

Second we prove $U' \subset U$ by essentially the same argument with U and U´ switched. Let $\mathbf{b} \in U'$ such that $\mathbf{a} \neq \mathbf{b}$. Since U´ is convex, then $J(\mathbf{a},\mathbf{b}) \subset U'$. Hence, $J(\mathbf{a},\mathbf{b}) \cap P = \varnothing$. Since $\mathbf{a} \in U$, then Lemma 5.11 implies $\mathbf{b} \in U$. This proves $U' \subset U$.

Since $U \subset U'$ and $U' \subset U$, then $U = U'$. Consequently,

$$V = (\mathbb{E}^n - P) - U = (\mathbb{E}^n - P) - U' = V'.$$

We conclude that U = U´ and V = V´.

Case 2: $\mathbf{a} \in V'$.

The proof of Case 2 is essentially the same as the proof of Case 1 with the roles of U´ and V´ interchanged. The conclusion of Case 2 is: U = V´ and V = U´.

Homework Problem 5.4. Write out the proof of Case 2.

Since Cases 1 and 2 exhaust all possibilities, we conclude that either U = U´ and V = V´ or U = V´ and V = U´. The proof of Theorem 5.10 is now complete. ☐

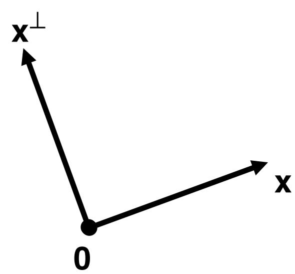
Remark. If P is a hyperplane in $\mathbb{E}^n$ and U and V are opposite sides P, then Theorem 5.10 justifies our calling U and V *the* opposite sides of P.

Recall that there is a metric characterization of lines in $\mathbb{E}^n$ as subsets of $\mathbb{E}^n$ that are isometric to $\mathbb{R}$. There is a similar metric characterization of hyperplanes in $\mathbb{E}^n$ which says that a subset of $\mathbb{E}^n$ is a hyperplane if and only if it is isometric to $\mathbb{E}^{n-1}$. We can't prove this theorem yet because the proof requires results from the next chapter.

In $\mathbb{E}^2$, lines and hyperplanes are the same objects. We will now prove this. Our proof requires us to develop some special techniques that apply only in $\mathbb{E}^2$. These techniques have many applications in 2-dimensional geometry.

Definition. For every $\mathbf{x} = (x_1,x_2) \in \mathbb{E}^2$, define $\mathbf{x}^\perp \in \mathbb{E}^2$ by the formula

$$\mathbf{x}^\perp = (-x_2,x_1).$$



Lemma 5.12. a) For each $\mathbf{x} \in \mathbb{E}^2$, $\| \mathbf{x}^\perp \| = \| \mathbf{x} \|$ and $\mathbf{x}\bullet\mathbf{x}^\perp = 0$.
b) For each $\mathbf{x} \in \mathbb{E}^2$, $(\mathbf{x}^\perp)^\perp = -\mathbf{x}$ and $(((\mathbf{x}^\perp)^\perp)^\perp)^\perp = \mathbf{x}$.

In-Class Exercise 5.A. Prove Lemma 5.12.

Lemma 5.13. a) The function $\mathbf{x} \mapsto \mathbf{x}^\perp : \mathbb{E}^2 \to \mathbb{E}^2$ is *linear;* in other words $(a\mathbf{x} + b\mathbf{y})^\perp = a(\mathbf{x}^\perp) + b(\mathbf{y}^\perp)$ for all $\mathbf{x}$ and $\mathbf{y} \in \mathbb{E}^2$ and all a and b $\in \mathbb{R}$.

b) $\mathbf{0}^\perp = \mathbf{0}$. Also the function $\mathbf{x} \mapsto \mathbf{x}^\perp : \mathbb{E}^2 \to \mathbb{E}^2$ preserves dot product; in other words, $(\mathbf{x}^\perp)\bullet(\mathbf{y}^\perp) = \mathbf{x}\bullet\mathbf{y}$ for all $\mathbf{x}$ and $\mathbf{y} \in \mathbb{E}^2$. Hence, $(\mathbf{x}^\perp)\bullet\mathbf{y} = -\mathbf{x}\bullet(\mathbf{y}^\perp)$ for all $\mathbf{x}$ and $\mathbf{y} \in \mathbb{E}^2$.

c) The function $\mathbf{x} \mapsto \mathbf{x}^\perp : \mathbb{E}^2 \to \mathbb{E}^2$ is a rigid motion of $\mathbb{E}^2$, and its inverse is the function $\mathbf{x} \mapsto ((\mathbf{x}^\perp)^\perp)^\perp : \mathbb{E}^2 \to \mathbb{E}^2$.

Homework Problem 5.5. Prove Lemma 5.13.

Theorem 5.14. If $\mathbf{u} \in \mathbb{E}^2$ and $\| \mathbf{u} \| = 1$, then for every $\mathbf{x} \in \mathbb{E}^2$,

$$\mathbf{x} = (\mathbf{x} \bullet \mathbf{u})\mathbf{u} + (\mathbf{x} \bullet \mathbf{u}^\perp)\mathbf{u}^\perp.$$

Proof. Let $\mathbf{x} = (x_1,x_2)$ and let $\mathbf{u} = (u_1,u_2)$. Then $\mathbf{u}^\perp = (-u_2,u_1)$ and $u_1^2 + u_2^2 = \| \mathbf{u} \|^2 = 1$. Hence,

$$(\mathbf{x} \bullet \mathbf{u})\mathbf{u} + (\mathbf{x} \bullet \mathbf{u}^\perp)\mathbf{u}^\perp = ((x_1,x_2) \bullet (u_1,u_2))(u_1,u_2) + ((x_1,x_2) \bullet (-u_2,u_1))(-u_2,u_1)$$
$$= (x_1u_1 + x_2u_2)(u_1,u_2) + (-x_1u_2 + x_2u_1)(-u_2,u_1) =$$
$$(x_1u_1^2 + x_2u_1u_2, x_1u_1u_2 + x_2u_2^2) + (x_1u_2^2 - x_2u_1u_2, -x_1u_1u_2 + x_2u_1^2) =$$
$$(x_1u_1^2 + x_1u_2^2 + x_2u_1u_2 - x_2u_1u_2, x_1u_1u_2 - x_1u_1u_2 + x_2u_2^2 + x_2u_1^2) =$$
$$(x_1(u_1^2 + u_2^2) + 0, 0 + x_2(u_2^2 + u_1^2)) = (x_1 \cdot 1, x_2 \cdot 1) = (x_1, x_2) = \mathbf{x}. \ \square$$

In-Class Exercise 5.B. Let $\mathbf{u} = (^3/_5, ^4/_5)$. Observe that $\| \mathbf{u} \| = 1$. Fill in the following blanks with real numbers.

$$(2, 3) = \_\_\_\_\_\_ \ \mathbf{u} + \_\_\_\_\_\_ \ \mathbf{u}^\perp.$$

We are now ready to prove that in $\mathbb{E}^2$, lines and hyperplanes coincide.

Theorem 5.15. In $\mathbb{E}^2$, every line is a hyperplane and every hyperplane is a line. More precisely:

a) if **a** and **b** are distinct points in $\mathbb{E}^2$, then $L(\mathbf{a},\mathbf{b}) = P(\mathbf{u}^\perp,c)$ where $\mathbf{u} = \left(\frac{1}{\| \mathbf{b} - \mathbf{a} \|}\right)(\mathbf{b} - \mathbf{a})$ and $c = \mathbf{a} \bullet \mathbf{u}^\perp$; and

b) if $\mathbf{u} \in \mathbb{E}^2$ such that $\| \mathbf{u} \| = 1$ and $a \in \mathbb{R}$, then $P(\mathbf{u},a) = L(a\mathbf{u},a\mathbf{u} + \mathbf{u}^\perp)$.

Proof of a). Let **a** and **b** be distinct points in $\mathbb{E}^2$. We will prove that the line $L(\mathbf{a},\mathbf{b})$ is a hyperplane. Specifically, let $\mathbf{u} = \left(\frac{1}{\| \mathbf{b} - \mathbf{a} \|}\right)(\mathbf{b} - \mathbf{a})$ and $c = \mathbf{a} \bullet \mathbf{u}^\perp$. We will prove $L(\mathbf{a},\mathbf{b}) = P(\mathbf{u}^\perp,c)$.

First we will prove $L(\mathbf{a},\mathbf{b}) \subset P(\mathbf{u}^\perp,c)$. Let $\mathbf{x} \in L(\mathbf{a},\mathbf{b})$. Then there is a $t \in \mathbb{R}$ such that $\mathbf{x} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$. Observe that $\mathbf{b} - \mathbf{a} = \| \mathbf{b} - \mathbf{a} \| \mathbf{u}$. Hence, $\mathbf{x} = \mathbf{a} + (t \| \mathbf{b} - \mathbf{a} \|) \mathbf{u}$. Therefore,

$$\mathbf{x} \bullet \mathbf{u}^\perp = \mathbf{a} \bullet \mathbf{u}^\perp + (t \| \mathbf{b} - \mathbf{a} \|) \mathbf{u} \bullet \mathbf{u}^\perp = c + (t \| \mathbf{b} - \mathbf{a} \|) 0 = c.$$

Thus, $\mathbf{x} \in P(\mathbf{u}^\perp,c)$. This proves $L(\mathbf{a},\mathbf{b}) \subset P(\mathbf{u}^\perp,c)$.

Second we prove $P(\mathbf{u}^{\perp},c) \subset L(\mathbf{a},\mathbf{b})$. Let $\mathbf{x} \in P(\mathbf{u}^{\perp},c)$. Therefore, $\mathbf{x}\bullet\mathbf{u}^{\perp} = c$. Hence, $\mathbf{x}\bullet\mathbf{u}^{\perp} = \mathbf{a}\bullet\mathbf{u}^{\perp}$. Theorem 5.11 allows us to express $\mathbf{x}$ in the form $\mathbf{x} = (\mathbf{x}\bullet\mathbf{u})\mathbf{u} + (\mathbf{x}\bullet\mathbf{u}^{\perp})\mathbf{u}^{\perp}$. Therefore, $\mathbf{x} = (\mathbf{x}\bullet\mathbf{u})\mathbf{u} + (\mathbf{a}\bullet\mathbf{u}^{\perp})\mathbf{u}^{\perp}$. Also Theorem 5.11 allows us to express $\mathbf{a}$ in the form $\mathbf{a} = (\mathbf{a}\bullet\mathbf{u})\mathbf{u} + (\mathbf{a}\bullet\mathbf{u}^{\perp})\mathbf{u}^{\perp}$. Thus, $(\mathbf{a}\bullet\mathbf{u}^{\perp})\mathbf{u}^{\perp} = \mathbf{a} - (\mathbf{a}\bullet\mathbf{u})\mathbf{u}$. We substitute this expression for $(\mathbf{a}\bullet\mathbf{u}^{\perp})\mathbf{u}^{\perp}$ into the preceding equation for $\mathbf{x}$ to obtain $\mathbf{x} = (\mathbf{x}\bullet\mathbf{u})\mathbf{u} + \mathbf{a} - (\mathbf{a}\bullet\mathbf{u})\mathbf{u} = \mathbf{a} + (\mathbf{x}\bullet\mathbf{u} - \mathbf{a}\bullet\mathbf{u})\mathbf{u} = \mathbf{a} + ((\mathbf{x} - \mathbf{a})\bullet\mathbf{u})\mathbf{u}$.

Since $\mathbf{u} = \left(\dfrac{1}{\| \mathbf{b} - \mathbf{a} \|}\right)(\mathbf{b} - \mathbf{a})$, then

$$\mathbf{x} = \mathbf{a} + ((\mathbf{x} - \mathbf{a})\bullet\mathbf{u})\left(\frac{1}{\| \mathbf{b} - \mathbf{a} \|}\right)(\mathbf{b} - \mathbf{a}) = \mathbf{a} + \left(\frac{(\mathbf{x} - \mathbf{a})\bullet\mathbf{u}}{\| \mathbf{b} - \mathbf{a} \|}\right)(\mathbf{b} - \mathbf{a}).$$

Since $\dfrac{(\mathbf{x} - \mathbf{a})\bullet\mathbf{u}}{\| \mathbf{b} - \mathbf{a} \|} \in \mathbb{R}$, then it follows that $\mathbf{x} \in L(\mathbf{a},\mathbf{b})$. This proves $P(\mathbf{u}^{\perp},c) \subset L(\mathbf{a},\mathbf{b})$.

Since $L(\mathbf{a},\mathbf{b}) \subset P(\mathbf{u}^{\perp},c)$ and $P(\mathbf{u}^{\perp},c) \subset L(\mathbf{a},\mathbf{b})$, then $L(\mathbf{a},\mathbf{b}) = P(\mathbf{u}^{\perp},c)$. □

Proof of b). Assume $\mathbf{u} \in \mathbb{E}^2$ such that $\| \mathbf{u} \| = 1$ and $a \in \mathbb{R}$. We will prove that the hyperplane $P(\mathbf{u},a)$ is a line. Specifically, we will prove that $P(\mathbf{u},a) = L(a\mathbf{u},a\mathbf{u} + \mathbf{u}^{\perp})$.

First we will prove $P(\mathbf{u},a) \subset L(a\mathbf{u},a\mathbf{u} + \mathbf{u}^{\perp})$. Let $\mathbf{x} \in P(\mathbf{u},a)$. Then $\mathbf{x}\bullet\mathbf{u} = a$. Theorem 5.11 allows us to express $\mathbf{x}$ in the form $\mathbf{x} = (\mathbf{x}\bullet\mathbf{u})\mathbf{u} + (\mathbf{x}\bullet\mathbf{u}^{\perp})\mathbf{u}^{\perp}$. Therefore,

$$\mathbf{x} = a\mathbf{u} + (\mathbf{x}\bullet\mathbf{u}^{\perp})\mathbf{u}^{\perp} = a\mathbf{u} + (\mathbf{x}\bullet\mathbf{u}^{\perp})((a\mathbf{u} + \mathbf{u}^{\perp}) - a\mathbf{u}).$$

It follows that $\mathbf{x} \in L(a\mathbf{u},a\mathbf{u} + \mathbf{u}^{\perp})$. This proves $P(\mathbf{u},a) \subset L(a\mathbf{u},a\mathbf{u} + \mathbf{u}^{\perp})$.

Second we will prove $L(a\mathbf{u},a\mathbf{u} + \mathbf{u}^{\perp}) \subset P(\mathbf{u},a)$. Let $\mathbf{x} \in L(a\mathbf{u},a\mathbf{u} + \mathbf{u}^{\perp})$. Then there is a $t \in \mathbb{R}$ such that $\mathbf{x} = a\mathbf{u} + t((a\mathbf{u} + \mathbf{u}^{\perp}) - a\mathbf{u})$. Hence, $\mathbf{x} = a\mathbf{u} + t\mathbf{u}^{\perp}$. Therefore,

$$\mathbf{x}\bullet\mathbf{u} = a\mathbf{u}\bullet\mathbf{u} + t\mathbf{u}^{\perp}\bullet\mathbf{u} = a\| \mathbf{u} \|^2 + t(0) = a(1) = a.$$

Consequently, $\mathbf{x} \in P(\mathbf{u},a)$. This proves $L(a\mathbf{u},a\mathbf{u} + \mathbf{u}^{\perp}) \subset P(\mathbf{u},a)$.

Since $P(\mathbf{u},a) \subset L(a\mathbf{u},a\mathbf{u} + \mathbf{u}^{\perp})$ and $L(a\mathbf{u},a\mathbf{u} + \mathbf{u}^{\perp}) \subset P(\mathbf{u},a)$, then $P(\mathbf{u},a) = L(a\mathbf{u},a\mathbf{u} + \mathbf{u}^{\perp})$. □

One consequence of Theorem 5.15 is that in $\mathbb{E}^2$, hyperplanes may be replaced by lines in the statements of Theorems 5.9 and 5.10. In this way we obtain the following two results.

Corollary 5.16. If L is a line in $\mathbb{E}^2$, then the complement $\mathbb{E}^2 - L$ is the union of two non-empty disjoint convex subsets U and V of $\mathbb{E}^2$ and every line segment joining a point of U to a point of V intersects L. ☐

Definition. If L is a line in $\mathbb{E}^2$ and if U and V are non-empty disjoint convex subsets of $\mathbb{E}^2$ such that $\mathbb{E}^2 - L = U \cup V$ and every line segment joining a point of U to a point of V intersects L, then we call U and V *opposite sides of* L.

Corollary 5.17. If L is a line in $\mathbb{E}^2$ and U and V are opposite sides of L, then U and V are unique in the following sense. If U´ and V´ are also opposite sides of L, then either U = U´ and V = V´, or U = V´ and V = U´.

Homework Problem 5.6. In this problem we ask whether a converse to Corollary 5.16 is true. Before we formulate this converse, we need a definition. If $\mathbf{x} \in \mathbb{E}^n$ and r > 0, then the set $N(x,r) = \{ \mathbf{y} \in \mathbb{E}^n : d(\mathbf{x},\mathbf{y}) < r \}$ is called an *open n-ball.*

Conjecture. Suppose S, U and V are subsets of $\mathbb{E}^2$ that satisfy the following four conditions.

a) U and V are non-empty disjoint convex subsets of $\mathbb{E}^n$,

b) $\mathbb{E}^2 - S = U \cup V$,

c) every line segment joining a point of U to a point of V intersects S, and

d) S contains no open 2-balls.

Then S is a line.

Either prove this conjecture or find a counterexample to it.

We remark that if hypothesis **d)** is omitted, then there are simple counterexamples to the resulting conjecture, even if we strengthen hypothesis **c)** to the statement: every line segment joining a point of U to a point of V intersects S *in a one-point set.* Try to find these counterexamples.

If you succeed in deciding whether or not this conjecture is true, then consider the truth of the extension of this conjecture to Euclidean n-space. In the conjecture, change $\mathbb{E}^2$ to $\mathbb{E}^n$, replace hypothesis **d)** by the statement *"S contains no open n-balls",* and replace the conclusion by the statement *"Then S is a hyperplane".* Is this extension of the conjecture to Euclidean n-space true?

The last topic of this chapter is a generalization of Theorem 1.5. That theorem said that if two points **x** and **y** in the real line $\mathbb{R}$ are *distinct,* then every other point of $\mathbb{R}$ is uniquely determined by its distances from **x** and **y**. We now establish an analogue of Theorem 1.5 that works in dimensions greater than 1. For $n \geq 2$, the *distinctness* of a set S of points in $\mathbb{E}^n$ is not a strong enough condition to insure that every other point of $\mathbb{E}^n$ is uniquely determined by its distances from the points in S. In Theorem 5.20 we will state a condition on a set S of points in $\mathbb{E}^n$ which insures that any other point of $\mathbb{E}^n$ is uniquely determined by its distances from the points of S. An essential step in proving Theorem 1.5 was identifying the set of points that are equidistant from two given distinct points. Our proof of Theorem 5.20 will require us to perform a similar identification of the set of all points equidistant from two given distinct points in $\mathbb{E}^n$. Theorem 1.5 played a crucial role in characterizing the isometries of the real line. In a similar fashion Theorem 5.20 will help us characterize the isometries of $\mathbb{E}^n$ in the next chapter.

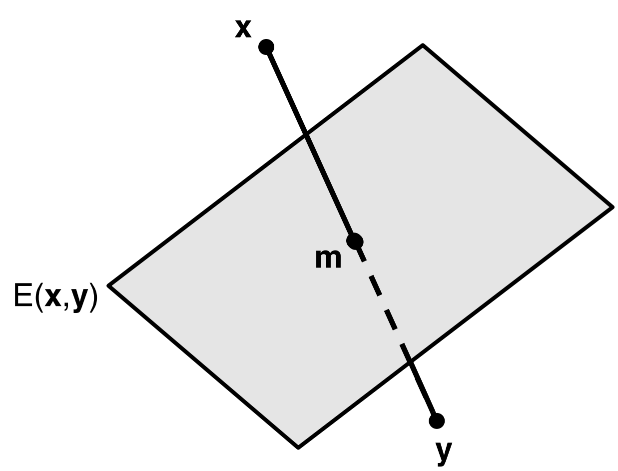
Definition. If $\mathbf{x}$ and $\mathbf{y}$ are distinct points of $\mathbb{E}^n$, let E($\mathbf{x}$,$\mathbf{y}$) denote the set of all points that are equidistant from $\mathbf{x}$ and $\mathbf{y}$. In other words,

$$E(\mathbf{x},\mathbf{y}) = \{ \mathbf{z} \in \mathbb{E}^n : d(\mathbf{x},\mathbf{z}) = d(\mathbf{y},\mathbf{z}) \}.$$

We now prove that E($\mathbf{x}$,$\mathbf{y}$) is a hyperplane.

Theorem 5.18. If $\mathbf{x}$ and $\mathbf{y}$ are distinct points of $\mathbb{E}^n$, then E($\mathbf{x}$,$\mathbf{y}$) = P($\mathbf{u}$,$\mathbf{m}\bullet\mathbf{u}$) where

$$\mathbf{u} = \left(\frac{1}{\| \mathbf{y} - \mathbf{x} \|}\right)(\mathbf{y} - \mathbf{x}) \text{ and } \mathbf{m} = (1/2)(\mathbf{x} + \mathbf{y}).$$



Proof. The proof consists of the observation that each statement in the following sequence is equivalent to the statements that precede and follow it.

- $\mathbf{z} \in E(\mathbf{x},\mathbf{y})$
- $d(\mathbf{x},\mathbf{z}) = d(\mathbf{y},\mathbf{z})$
- $\| \mathbf{x} - \mathbf{z} \| = \| \mathbf{y} - \mathbf{z} \|$
- $\| \mathbf{x} - \mathbf{z} \|^2 = \| \mathbf{y} - \mathbf{z} \|^2$

- $\| \mathbf{x} - \mathbf{z} \|^2 - \| \mathbf{y} - \mathbf{z} \|^2 = 0$.
- $((\mathbf{x} - \mathbf{z}) + (\mathbf{y} - \mathbf{z}))\bullet((\mathbf{x} - \mathbf{z}) - (\mathbf{y} - \mathbf{z})) = 0$. (See Lemma 4.4.b.)
- $((\mathbf{x} + \mathbf{y}) - 2\mathbf{z})\bullet(\mathbf{x} - \mathbf{y}) = 0$.
- $(^1/_2)\left(\frac{1}{\| \mathbf{y} - \mathbf{x} \|}\right)\big(((\mathbf{x} + \mathbf{y}) - 2\mathbf{z})\bullet(\mathbf{x} - \mathbf{y}) \big) = (^1/_2)\left(\frac{1}{\| \mathbf{y} - \mathbf{x} \|}\right)(0)$.
- $\big((^1/_2)(\mathbf{x} + \mathbf{y}) - \mathbf{z} \big)\bullet\left(\left(\frac{1}{\| \mathbf{y} - \mathbf{x} \|}\right)(\mathbf{x} - \mathbf{y})\right) = 0$.
- $(\mathbf{m} - \mathbf{z})\bullet\mathbf{u} = 0$.
- $\mathbf{m}\bullet\mathbf{u} = \mathbf{z}\bullet\mathbf{u}$.
- $\mathbf{z} \in P(\mathbf{u},\mathbf{m}\bullet\mathbf{u})$. □

Theorem 5.18 says that E($\mathbf{x}$,$\mathbf{y}$) is the hyperplane that passes through the midpoint $\mathbf{m}$ between $\mathbf{x}$ and $\mathbf{y}$ and is perpendicular to the line L($\mathbf{x}$,$\mathbf{y}$). Hence, we introduce the following terminology.

Definition. If $\mathbf{x}$ and $\mathbf{y}$ are distinct points of $\mathbb{E}^n$, then we call E($\mathbf{x}$,$\mathbf{y}$) the *perpendicular bisector* of the line segment J($\mathbf{x}$,$\mathbf{y}$).

Definition. Let $\mathbf{x}_1, \mathbf{x}_2, \dots , \mathbf{x}_k$ be points in $\mathbb{E}^n$. The points $\mathbf{x}_1, \mathbf{x}_2, \dots , \mathbf{x}_k$ are *coplanar* if there is a hyperplane P that contains $\mathbf{x}_1, \mathbf{x}_2, \dots , \mathbf{x}_k$. If there is no hyperplane that contains $\mathbf{x}_1, \mathbf{x}_2, \dots , \mathbf{x}_k$, we say that $\mathbf{x}_1, \mathbf{x}_2, \dots , \mathbf{x}_k$ are *non-coplanar.*

Notation. For $n \geq 1$, define the elements $\mathbf{e}_1, \mathbf{e}_2, \dots , \mathbf{e}_n$ of $\mathbb{E}^n$ by

$$\mathbf{e}_1 = (1, 0, \dots , 0, 0), \quad \mathbf{e}_2 = (0, 1, \dots , 0, 0), \quad \dots \quad , \quad \mathbf{e}_n = (0, 0, \dots , 0, 1).$$

Lemma 5.19. The $n + 1$ points $\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \dots , \mathbf{e}_n$ in $\mathbb{E}^n$ are non-coplanar.

Homework Problem 5.7. Prove Lemma 5.19.

Hint. Assume $\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \dots , \mathbf{e}_n$ are coplanar and lie in a hyperplane P($\mathbf{u}$,a). Prove $a = 0$ by examining $\mathbf{0}\bullet\mathbf{u}$. Then prove $\mathbf{u} = \mathbf{0}$ by examining $\mathbf{e}_i\bullet\mathbf{u}$ for $1 \leq i \leq n$. Why is this a contradiction?

Theorem 5.20. Let $\mathbf{x}_1, \mathbf{x}_2, \dots , \mathbf{x}_k$ be non-coplanar points in $\mathbb{E}^n$. Then every point of $\mathbb{E}^n$ is uniquely determined by its distances from $\mathbf{x}_1, \mathbf{x}_2, \dots , \mathbf{x}_k$. In other words, each point $\mathbf{y}$ of $\mathbb{E}^n$ is uniquely determined by the numbers $d(\mathbf{x}_1,\mathbf{y}), d(\mathbf{x}_2,\mathbf{y}), \dots , d(\mathbf{x}_k,\mathbf{y})$.

Proof. Assume $\mathbf{y}$ and $\mathbf{z} \in \mathbb{E}^n$ such that $d(\mathbf{x}_1,\mathbf{y}) = d(\mathbf{x}_1,\mathbf{z})$, $d(\mathbf{x}_2,\mathbf{y}) = d(\mathbf{x}_2,\mathbf{z})$, … , $d(\mathbf{x}_k,\mathbf{y}) = d(\mathbf{x}_k,\mathbf{z})$. We must prove $\mathbf{y} = \mathbf{z}$.

Assume $\mathbf{y} \neq \mathbf{z}$. Then $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are each elements of $E(\mathbf{y},\mathbf{z})$. Theorem 5.18 implies $E(\mathbf{y},\mathbf{z})$ is a hyperplane. Hence, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are all contained in a hyperplane. Therefore, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are coplanar. This contradicts our hypothese that $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are non-coplanar. We conclude that $\mathbf{y} = \mathbf{z}$. □

Definition. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be points in $\mathbb{E}^n$. The points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are *collinear* if there is a line L that contains $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$. If there is no line that contains $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$, we say that $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are *non-collinear.*

Since lines are hyperplanes and hyperplanes are lines in $\mathbb{E}^2$, then *coplanar* and *collinear* are equivalent in $\mathbb{E}^2$. Hence, in $\mathbb{E}^2$, Theorem 5.20 takes the following form.

Corollary 5.21. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be non-collinear points in $\mathbb{E}^2$. Then every point of $\mathbb{E}^2$ is uniquely determined by its distances from $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$. In other words, each point $\mathbf{y}$ of $\mathbb{E}^2$ is uniquely determined by the numbers $d(\mathbf{x}_1,\mathbf{y}), d(\mathbf{x}_2,\mathbf{y}), \ldots, d(\mathbf{x}_k,\mathbf{y})$. □

In $\mathbb{E}^2$, the *three* points $\mathbf{0}$, $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ are non-collinear, but any *two* distinct points $\mathbf{x}$ and $\mathbf{y}$ are collinear because they lie in the line $L(\mathbf{x},\mathbf{y})$. In general, in $\mathbb{E}^n$, the *n + 1* points $\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are non-coplanar but any *n* points are coplanar. We won't prove this now because we haven't discussed the techniques needed to prove it.

Homework Problem 5.8. Use your knowledge about vectors in $\mathbb{E}^3$ to prove that any three points in $\mathbb{E}^3$ are coplanar.

Hint. Recall that for $\mathbf{x}$ and $\mathbf{y} \in \mathbb{E}^3$, the cross product $\mathbf{x} \times \mathbf{y}$ is an element of $\mathbb{E}^3$ with the following properties:

- $\mathbf{x} \bullet (\mathbf{x} \times \mathbf{y}) = 0$ and $\mathbf{y} \bullet (\mathbf{x} \times \mathbf{y}) = 0$, and
- $\mathbf{x} \times \mathbf{y} = 0$ if and only if one of $\mathbf{x}$ and $\mathbf{y}$ is a scalar multiple of the other.