3. Congruence and Symmetry

It is possible for metric spaces which are not isometric to have subsets that are isometric, and we shall have occasion to consider such possibilities. For this reason, we formulate the following definition.

Definition. Let X and Y be metric spaces with metrics d_X and d_Y, respectively; and let $S \subset X$ and $T \subset Y$. We call a function $f : S \to T$ is an *isometry from* S *to* T if f is an isometry from S with the metric d_X^S to T with the metric d_Y^T, where d_X^S is the restriction of the metric d_X to S and d_Y^T is the restriction of the metric d_Y to T. Thus, $f : S \to T$ is an isometry from S to T if and only if

- $f : S \to T$ is distance preserving: $d_Y(f(x),f(x')) = d_X(x,x')$ for all $x, x' \in S$, and
- $f : S \to T$ is surjective: $f(S) = T$.

If there is an isometry from S to T, then we say that S *is isometric to* T.

Two subsets of a metric space may be isometric, or they may satisfy a potentially stronger relationship known as *congruence* which we now define.

Definition. If S and T are subsets of a metric space X and if there is a rigid motion $f : X \to X$ of X such that $f(S) = T$, then we say that S it *congruent to* T and we write $S \cong T$.

Theorem 3.1. Let X be a metric space. Then:

a) For every subset S of X, $S \cong S$.

b) For all subsets S and T of X, if $S \cong T$, then $T \cong S$.

c) For all subsets R, S and T of X, if $R \cong S$ and $S \cong T$, then $R \cong T$.

Remark. Theorem 3.1 says that congruence is an *equivalence relation* on the collection of all subsets of a metric space X.

Homework Problem 3.1. Prove Theorem 3.1.

In-Class Exercise 3.A. Which of the following pairs of subsets of $\mathbb{R}$ are congruent? Justify your answers. (One way to establish a congruence between two subsets S and T of $\mathbb{R}$ is to exhibit a rigid motion of $\mathbb{R}$ that moves S to T.)

a) $\{ 3, 7, 12 \}$ and $\{ -3, 1, 6 \}$

b) $\{ 5, 6, 10 \}$ and $\{ 8, 12, 13 \}$

c) $\{ 4, 9, 17 \}$ and $\{ -12, -5, 1 \}$

d) $\{ 1, 3, 4, 7 \}$ and $\{ 8, 9, 12, 14 \}$

e) $\{ 10n, 10n + 1, 10n + 3 : n \in \mathbb{Z} \} = \{ \dots -10, -9, -7, 0, 1, 3, 10, 11, 13, \dots \}$ and $\{ 10n, 10n + 1, 10n + 8 : n \in \mathbb{Z} \} = \{ \dots -10, -9, -2, 0, 1, 8, 10, 11, 18, \dots \}$.
(Recall that $\mathbb{Z} = \{ -2, -1, 0, 1, 2, \dots \}$ is the set of all integers.)

Remark. Let S and T be subsets of a metric space X. If S is congruent to T, then there is a rigid motion $f : X \to X$ such that $f(S) = T$. In this situation, if we form the restriction $f \mid S$, then $f \mid S : S \to T$ is an isometry from S to T. (In other words, $f \mid S : S \to T$ is a distance preserving surjective function.) Hence, S is isometric to T. We have just proved that if S is congruent to T, then S is isometric to T. We now raise the question about whether the converse of this statement is true. In other words, is it true that if S is isometric to T, then must S be congruent to T? If S is isometric to T, then there is a distance preserving surjective function $g : S \to T$. However, g is not a rigid motion of X because its domain and range are S and T, not X. If it were the case that g extends to a rigid motion of X, then it would follow that S is congruent to T, but we don't know that such an extension of g exists. In general, isometries between subsets of a metric space X may not extend to rigid motions of X. Hence, in general, isometric subsets may not be congruent.

A metric space X may have the property that any two isometric subsets of X are congruent. In this case, we say that *in X, isometry implies congruence.* However, another metric space Y may contain two isometric subsets that are not congruent, in which case we would say that *in Y, isometry fails to imply congruence.* We will consider examples of both types of behavior.

We call attention to the fact that we have already established *isometry-implies-congruence theorems* for subsets of $\mathbb{R}$ and C_r that have *two elements*. We are referring here to Theorems 2.5 and 2.19. Indeed, Theorem 2.5 says that if x_1, x_2, y_1 and $y_2 \in \mathbb{R}$ and $d(x_1,x_2) = d(y_1,y_2)$, then there is a rigid motion $f : \mathbb{R} \to \mathbb{R}$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Observe that the condition $d(x_1,x_2) = d(y_1,y_2)$ is equivalent to saying that the function $g : \{ x_1, x_2 \} \to \{ y_1, y_2 \}$ defined by $g(x_1) = y_1$ and $g(x_2) = y_2$ is an isometry from $\{ x_1, x_2 \}$ to $\{ y_1, y_2 \}$, and the existence of the rigid motion $f : \mathbb{R} \to \mathbb{R}$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$ implies that $\{ x_1, x_2 \}$ is congruent to $\{ y_1, y_2 \}$. So Theorem 2.5 implies that if the two-element subsets $\{ x_1, x_2 \}$ and $\{ y_1, y_2 \}$ of $\mathbb{R}$ are isometric, then they are congruent.

In a similar vein, we observe that in each part of In-Class Exercise 3.A, if the two subsets of $\mathbb{R}$ are isometric, then they are congruent. Indeed, isometry implies congruence for all subsets of $\mathbb{R}$. We will a formulate theorem to this effect below.

You may already be familiar with an *isometry implies congruence* principle in plane geometry, namely the Side-Side-Side Congruence Principle for triangles in the plane equipped with the Euclidean metric. The Side-Side-Side Principle says that if the lengths of the three sides of one triangle Δ**xyz** equal the lengths of the three sides of a second triangle Δ**x´y´z´**, then the two triangles are congruent. Since the length of a side of a triangle equals the distance between the two endpoints of that side, then saying that the lengths of the sides of $\Delta\mathbf{xyz}$ equal the lengths of the sides of $\Delta\mathbf{x}'\mathbf{y}'\mathbf{z}'$ is equivalent to saying that there is an isometry from the set $\{ \mathbf{x}, \mathbf{y}, \mathbf{z} \}$ of vertices of the first triangle and the set $\{ \mathbf{x}', \mathbf{y}', \mathbf{z}' \}$ of vertices of the second triangle. Saying that $\Delta\mathbf{xyz}$ is congruent to $\Delta\mathbf{x}'\mathbf{y}'\mathbf{z}'$ is equivalent to saying that $\{ \mathbf{x}, \mathbf{y}, \mathbf{z} \}$ is congruent to $\{ \mathbf{x}', \mathbf{y}', \mathbf{z}' \}$, because a planar rigid motion moves $\Delta\mathbf{xyz}$ to $\Delta\mathbf{x}'\mathbf{y}'\mathbf{z}'$ if and only if it moves $\{ \mathbf{x}, \mathbf{y}, \mathbf{z} \}$ to $\{ \mathbf{x}', \mathbf{y}', \mathbf{z}' \}$. Thus, the Side-Side-Side Principle can be reinterpreted to say that if the two three-point sets $\{ \mathbf{x}, \mathbf{y}, \mathbf{z} \}$ and $\{ \mathbf{x}', \mathbf{y}', \mathbf{z}' \}$ are isometric, then they are congruent. In other words, the Side-Side-Side Congruence Principle is an *isometry implies congruence* principle for three-point subsets of the Euclidean plane.

In-Class Exercise 3.B. Consider a set X with the discrete metric. Are isometric subsets of X necessarily congruent?

Homework Problem 3.2. The goal of this exercise is to illustrate that in $\mathbb{R}^2$ with the taxicab metric, there are isometric subsets that are not congruent. Specifically, in $\mathbb{R}^2$ with the taxicab metric, consider the two subsets $S = \{ (0,0), (1,0) \}$ and $T = \{ (0,0), (^1/_2,{}^1/_2) \}$. Prove that S and T are isometric but not congruent.

Hint. First prove the following lemma. If X is a metric space, x and y are points of X such that there is a unique midpoint between x and y, and $f : X \to X$ is a rigid motion, then there is a unique midpoint between f(x) and f(y). Then use what you learned from Homework Problem 1.6 about the uniqueness of midpoints in $\mathbb{R}^2$ with the taxicab metric.

As we stated above, isometric subsets are congruent in $\mathbb{R}$. We will now state this assertion as a theorem. It is also the case that in $\mathbb{R}^2$ with the Euclidean metric (as well as in all higher dimensional Euclidean spaces), isometry implies congruence. A formally stated theorem to this effect appears in a later lesson.

Theorem 3.2. Any two isometric subsets of $\mathbb{R}$ are congruent.

Proof. Suppose that S and T are isometric subsets of $\mathbb{R}$ and $g : S \to T$ is an isometry. We break the proof into three cases.

Case 1: S has one point. We may write $S = \{ a \}$. Let $b = g(a)$. Then $T = g(S) = \{ g(a) \} = \{ b \}$. The translation T_{b-a} is a rigid motion of $\mathbb{R}$ and $T_{b-a}(a) = a + (b - a) = b$. Therefore, $T_{b-a}(S) = \{ T_{b-a}(a) \} = \{ b \} = T$. Thus, S is congruent to T. (Alternatively, if $c = (^1/_2)(a + b)$, then Z_c is a rigid motion of $\mathbb{R}$ such that $Z_c(a) = b$ and, hence, $Z_c(S) = T$.)

Case 2: S has two point. We may write $S = \{ a_1, a_2 \}$ where $a_1 \neq a_2$. Let $b_1 = g(a_1)$ and $b_2 = g(a_2)$. Then $T = g(S) = \{ g(a_1), g(a_2) \} = \{ b_1, b_2 \}$. Since g is an isometry, then $d(a_1,a_2) = d(g(a_1),g(a_2)) = d(b_1,b_2)$. Therefore, Theorem 2.5 implies there is a rigid motion $f : \mathbb{R} \to \mathbb{R}$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$. Therefore, $f(S) = \{ f(a_1), f(a_2) \} =$

$\{ b_1, b_2 \} = T$. Therefore S is congruent to T.

Case 3: S has more than two points. We may choose distinct points a_1 and $a_2 \in S$. Let $b_1 = g(a_1)$ and $b_2 = g(a_2)$. Then b_1 and $b_2 \in g(S) = T$. Since g is an isometry, then $d(a_1,a_2) = d(g(a_1),g(a_2)) = d(b_1,b_2)$. Therefore, Theorem 2.5 implies there is a rigid motion $f : \mathbb{R} \to \mathbb{R}$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$.

We will now prove that $f(x) = g(x)$ for every $x \in S$. First observe that since g is distance preserving, it is injective. Therefore, since $a_1 \neq a_2$, then $g(a_1) \neq g(a_2)$. Hence, $b_1 \neq b_2$. Let $x \in S$. Since both g and f are distance preserving, then

$$d(b_1,g(x)) = d(g(a_1),g(x)) = d(a_1,x) = d(f(a_1),f(x)) = d(b_1,f(x))$$

and

$$d(b_2,g(x)) = d(g(a_2),g(x)) = d(a_2,x) = d(f(a_2),f(x)) = d(b_2,f(x)).$$

Thus, b_1 and b_2 are distinct points of $\mathbb{R}$ and $g(x)$ and $f(x)$ are equidistant from b_1 and from b_2. In this situation, Theorem 1.5 implies $f(x) = g(x)$.

Since $f(x) = g(x)$ for every $x \in S$, then $f(S) = g(S) = T$. Therefore, S is congruent to T.

Since Cases 1, 2 and 3 exhaust all possibilities, the proof is complete. □

Definition. If $S = \{ x_1, x_2, \ldots , x_n \}$ is a finite subset of $\mathbb{R}$ with n distinct elements (so $x_i \neq x_j$ for $i \neq j$), then define the *mean* or *center of gravity* of S to be

$$\mu(S) = (x_1 + x_2 + \ldots + x_n)/n.$$

Theorem 3.3. If S is a finite subset of $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is a rigid motion, then $f(\mu(S)) = \mu(f(S))$.

Homework Problem 3.3. Prove Theorem 3.3.

In-Class Exercise 3.C. The eight figures on this and the next page exhibit *symmetry.* What makes each figure symmetric? In what way is the symmetry of one figure the same as or different from that of the other figures?

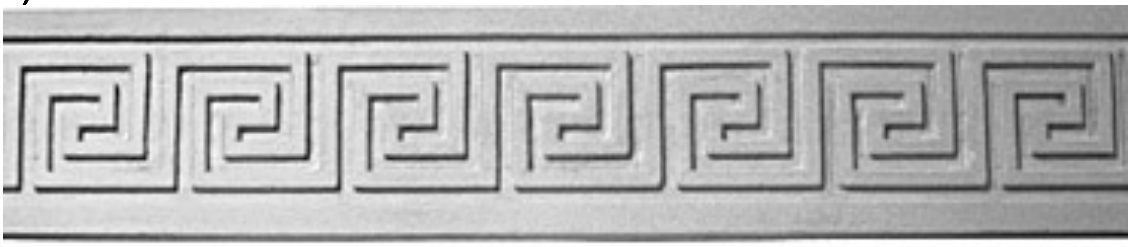
a)

b)

c)

d)

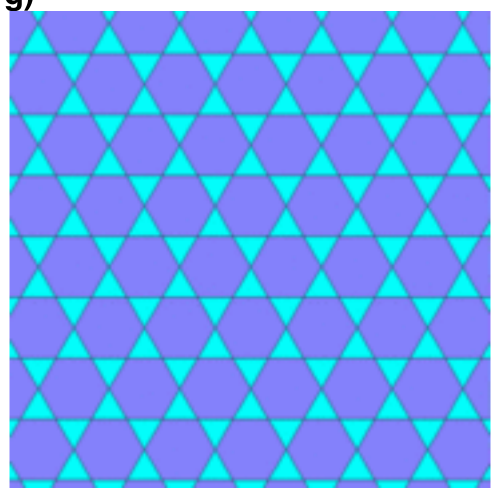
e)



(Imagine that this figure extends to the right and left forever.)

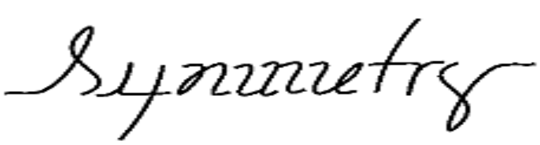
f)

g)



(Imagine that this figure fills its plane, extending right, left, up and down forever.)

h)



Mathematicians associate the symmetry of a subset S of a metric space X with the rigid motions of X that move S onto itself. For instance, if a reflection moves S to itself, then S is said to have *reflectional* or *mirror symmetry.* If a rotation moves S to itself, then S is said to have *rotational symmetry.* The collection of all rigid motions of X that move S onto itself captures all the symmetry of S. This idea motivates the following definition.

Definition. Let X be a metric space. Let S be a subset of X. A rigid motion $f : X \to X$ such that $f(S) = S$ is called a *symmetry of* S. The set of all symmetries of S is called the *symmetry group of* S and is denoted Sym(S). Observe that Sym(X) is the set of all rigid motions of X.

Theorem 3.4. Let X be a metric space and let S be a subset of X. Then:

a) $id_X \in$ Sym(S).

b) If f, g $\in$ Sym(S), then $g \circ f \in$ Sym(S).

c) If f $\in$ Sym(S), then $f^{-1} \in$ Sym(S).

Homework Problem 3.4. Prove Theorem 3.4.

Remark. Theorem 3.4 says that the symmetry group of a set S is a *group* in the mathematical sense. In other words, Sym(S) is a set that is equipped with an operation – in this case composition of functions. The operation is associative: $h \circ (g \circ f) = (h \circ g) \circ f$. The operation has an identity element – in this case id_X. Finally each element has an inverse.

In-Class Exercise 3.D. a) Let $x \in \mathbb{R}$. List all the elements of Sym({ x }).

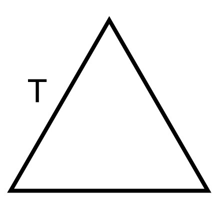
b) Let x and y be distinct points of $\mathbb{R}$. List all the elements of Sym({ x, y }).

c) List all the elements of Sym({ 1, 2, 3 }).

d) List all the elements of Sym({ 1, 2, 4 }).

e) Recall that $\mathbb{Z} = \{ -2, -1, 0, 1, 2, \dots \}$ is the set of all integers. Regard $\mathbb{Z}$ as a subset of $\mathbb{R}$ and describe all the elements of Sym($\mathbb{Z}$).

f) Let T be an equilateral triangle in the plane $\mathbb{R}^2$ equipped with the Euclidean metric. List all the elements of Sym(T).



Homework Problem 3.5. Let S be a bounded subset of $\mathbb{R}$. (Recall that S is bounded if diam(S) < ∞.) Consider the symmetry group Sym(S).

a) For different choices of S, how many elements can Sym(S) have?

b) For different choices of S, what kinds of elements can Sym(S) have? Can Sym(S) contain reflections? Can Sym(S) contain translations that are not the identity?

c) List all the different possible *types* of groups that can occur as the symmetry groups of various bounded sets S. (This question is somewhat vague because we haven't defined what different types of groups are. As a hint about what this might mean, we remark that two groups with different numbers of elements are of different types. Also if one group contains reflections and another does not, then these groups are of different types.) Prove that your list is correct.

Homework Problem 3.6. Let S be an unbounded subset of $\mathbb{R}$ (i.e., diam(S) = ∞). Consider the symmetry group Sym(S).

a) Is there an unbounded set S for which Sym(S) is a finite group? If so, how many different elements can Sym(S) contain?

b) If for some unbounded set S, Sym(S) contains a non-identity translation, must Sym(S) be infinite?

c) If for some unbounded set S, Sym(S) contains two different reflections, must Sym(S) contain a non-identity translation?

d) If for some unbounded set S, Sym(S) contains a non-identity translation and a reflection, then must Sym(S) contain more than one reflection? Must Sym(S) contain infinitely many different reflections?

e) Is there an unbounded set S for which Sym(S) contains non-identity translations but no reflections?

f) Is there an unbounded set S such that Sym(S) contains a sequence of translations $T_{a(n)}$, $n \geq 1$, for which each $a(n) > 0$ and the sequence { a(n) } converges to 0?

g) List all the different possible *types* of groups that can occur as the symmetry groups of various unbounded sets S. (The parenthetical remark in Homework Problem 3.5.c applies here.) Prove that your list is correct.

Definition. Let S be a subset of a metric space X with metric d. Define the *distance spectrum* of S to be the set

$$\text{D-Spec}(S) = \{ d(x,y) : x \text{ and } y \in S \text{ and } x \neq y \}.$$

Observe that D-Spec(S) is a finite or infinite subset of (0,∞). For example, D-Spec({ 1, 3, 8 }) = { 2, 5, 7 }.

Homework Problem 3.7. Let S and T be subsets of $\mathbb{R}$. Here we investigate the following conjecture.

If D-Spec(S) = D-Spec(T), then S must be congruent to T.

Recall that according to Theorem 3.2, isometric subsets of $\mathbb{R}$ are congruent. On the other hand, congruent subsets of $\mathbb{R}$ are clearly isometric. (Why?) Thus, the preceding conjecture is equivalent to the following conjecture.

If D-Spec(S) = D-Spec(T), then S must be isometric to T?

We break this conjecture into two parts.

i) If D-Spec(S) = D-Spec(T), then S and T must have the same number of elements.

ii) If D-Spec(S) = D-Spec(T) and S and T have the same number of elements, then S must be isometric to T.

Approach conjectures **i)** and **ii)** by solving the following problems.

a) Prove that conjectures **i)** and **ii)** are true if S has two elements.

b) Is conjecture **i)** true if S has 3 elements? Is conjecture **ii)** true if S has 3 elements? Prove your answers.

c) Is conjecture **i)** true if S has 4 elements? Is conjecture **ii)** true if S has 4 elements? Prove your answers.

d) Let $4 < n < \infty$. Is conjecture **i)** true if S has n elements? Is conjecture **ii)** true if S has n elements? Prove your answers.

e) Is conjecture **i)** true if S has infinitely many elements? Is conjecture **ii)** true if S has infinitely many elements? Prove your answers.

Next we formulate a more delicate version of the distance spectrum.

Definition. Let S be a finite subset of a metric space X with metric d. For $r > 0$, define the *distance multiplicity* of r to be number of unordered pairs { x, y } such that x and y $\in$ S and d(x,y) = r, and denote this number by $\mu_S(r)$. Define the *distance census* of S to be the set

$$\text{D-Census}(S) = \{ (d(x,y), \mu_S(d(x,y))) : x \text{ and } y \in S \text{ and } x \neq y \}.$$

Hence, D-Census(S) is a finite subset of $(0,\infty) \times \{ 1, 2, 3, \dots \}$. For example, D-Census({ 1, 2, 3, 5 }) = { (1,2), (2,2), (3,1), (4,1) }.

Homework Problem 3.8. Is the following conjecture true or false?

If S and T are finite subsets of $\mathbb{R}$ and D-Census(S) = D-Census(T), then S is congruent to T.