## 2. Isometries and Rigid Motions of the Real Line

Suppose two metric spaces have different names but are essentially the same geometrically. Then we need a way of relating the two spaces. Similarly, suppose two subsets of a metric space are located in different places but are geometrically identical; in other words, suppose the two subsets are *congruent.* Again in this situation we need a concept that allows us to relate the two subsets. The notion of an *isometry* is exactly the concept needed here.

**Definition.** Let X and Y be metric spaces with metrics $d_X$ and $d_Y$, respectively. Suppose $f : X \to Y$ is a function. We say that $f : X \to Y$ is *distance preserving* if for all x and $x' \in X$, $d_Y(f(x),f(x')) = d_X(x,x')$). We call $f : X \to Y$ an *isometry from* X *to* Y if $f : X \to Y$ is distance preserving and surjective. If there is an isometry from X to Y, then we say that X *is isometric to* Y.

**Theorem 2.1.** Let X, Y and Z be metric spaces. Then:

**a)** The identity function $id_X$ is an isometry from X to itself.

**b)** Every distance preserving function $f : X \to Y$ is injective. Hence, every isometry $f : X \to Y$ is a bijection. Therefore (by Theorem 0.5), every isometry $f : X \to Y$ has an inverse $f^{-1} : Y \to X$.

**c)** If $f : X \to Y$ and $g : Y \to Z$ are distance preserving functions, then so is their composition $g \circ f : X \to Z$. Hence, if $f : X \to Y$ and $g : Y \to Z$ are isometries, then so is $g \circ f : X \to Z$ (with the help of Theorem 0.4.b).

**d)** If $f : X \to Y$ is an isometry, then so is its inverse $f^{-1} : Y \to X$.

**In-Class Exercise 2.A.** Prove parts a), b) and c) of Theorem 2.1.

**Proof of part d).** Let $d_X$ and $d_Y$ be the given metrics on X and Y, respectively. Let $f : X \to Y$ be an isometry. Then by part b), $f : X \to Y$ is a bijection and has an inverse $f^{-1} : Y \to X$. We must prove $f^{-1} : Y \to X$ is an isometry. Since $f^{-1} \circ f = id_X$ and $id_X : X \to X$ is surjective, then $f^{-1} : Y \to X$ is surjective (by Theorem 0.4.d). It remains to prove that $f^{-1} : Y \to X$ is distance preserving. To this end, let y and $y' \in Y$. We must show that $d_X(f^{-1}(y),f^{-1}(y')) = d_Y(y,y')$. Let $x = f^{-1}(y)$ and $x' = f^{-1}(y')$. Then $f(x) = f(f^{-1}(y)) = f \circ f^{-1}(y)) = id_Y(y) = y$ and $f(x') = f(f^{-1}(y')) = f \circ f^{-1}(y')) = id_Y(y') = y'$. Since $f : X \to Y$ is an isometry, then

$$d_X(x,x') = d_Y(f(x),f(x')).$$

Therefore, after replacing x, x′, f(x) and f(x′) by $f^{-1}(y)$, $f^{-1}(y')$, y and y′ for x, x′, f(x), respectively, we obtain

$$d_X(f^{-1}(y),f^{-1}(y')) = d_Y(y,y').$$

This proves $f^{-1}$ is a distance preserving surjection and, hence, an isometry. □

**In-Class Exercise 2.B.** Let X and Y be sets with the discrete metric. Can you think of a purely set-theoretic (non-metric) relationship between X and Y that would make X isometric to Y?

**Definition.** Let X be a metric space. If $f : X \to X$ is an isometry from a metric space X to itself, then f is also called a *rigid motion* of X. Thus, $f : X \to X$ is a rigid motion if and only if f is distance preserving and surjective.

**In-Class Exercise 2.C.** Let X be a metric space. Must a distance preserving function from X to itself be surjective?

**In-Class Exercise 2.D. a)** Give examples of different types of rigid motions of $\mathbb{R}$. (As mentioned in Lesson 1, we assume that $\mathbb{R}$ is equipped with the standard metric, unless an alternative is specified.)

**b)** Give examples of different types of rigid motions of $\mathbb{R}^2$ with the Euclidean metric.

We now begin an exploration of the rigid motions of $\mathbb{R}$. Our goal is reveal what all the different types of rigid motions of $\mathbb{R}$ are, and to uncover simple relationships between different rigid motions of $\mathbb{R}$. We begin by introducing notation for two types of rigid motions of $\mathbb{R}$.

**Definition.** Let $a \in \mathbb{R}$.. Define the function $T_a : \mathbb{R} \to \mathbb{R}$ by

$$T_a(x) = x + a$$

for $x \in \mathbb{R}$. We call $T_a : \mathbb{R} \to \mathbb{R}$ a *translation* of $\mathbb{R}$. Observe that $T_0 = id_{\mathbb{R}}$, and that if $a \neq 0$, then $T_a(x) \neq x$ for every $x \in \mathbb{R}$.

**Definition.** Let $a \in \mathbb{R}$. Define the function $Z_a : \mathbb{R} \to \mathbb{R}$ by

$$Z_a(x) = x - 2(x - a) = 2a - x$$

for $x \in \mathbb{R}$. We call $Z_a : \mathbb{R} \to \mathbb{R}$ a *reflection* of $\mathbb{R}$. Observe that $Z_a(a) = a$ and that if $x \in \mathbb{R}$ and $x \neq a$, then $Z_a(x) \neq x$.

**Definition.** Suppose $f : X \to X$ is a function. An element $x \in X$ is a *fixed point of* f if $f(x) = x$.

Observe that every point of $\mathbb{R}$ is a fixed point of $T_0$, but if $a \in \mathbb{R}$ and $a \neq 0$, then $T_a$ has no fixed points. Also observe that for every $a \in \mathbb{R}$, a is the only fixed point of $Z_a$.

**Theorem 2.2.** For every $a \in \mathbb{R}$, the translation $T_a : \mathbb{R} \to \mathbb{R}$ is a rigid motion of $\mathbb{R}$ such that $(T_a)^{-1} = T_{-a}$.

**Proof.** Let $a \in \mathbb{R}$. Then for each $x \in \mathbb{R}$, $T_{-a} \circ T_a(x) = ( x + a ) + (-a) = x = id_{\mathbb{R}}(x)$ and $T_a \circ T_{-a}(x) = ( x + (-a) ) + a = x = id_{\mathbb{R}}(x)$. Hence, $T_{-a} \circ T_a = T_a \circ T_{-a} = id_{\mathbb{R}}$. Thus, $T_{-a}$ is the inverse of $T_a$; i.e., $(T_a)^{-1} = T_{-a}$. Since $T_a \circ T_{-a} = id_{\mathbb{R}}$ and $id_{\mathbb{R}}$ is surjective, then $T_a$ is surjective (Theorem 0.4 d). It remains to prove that $T_a : \mathbb{R} \to \mathbb{R}$ is distance preserving. To this end, let $x, y \in \mathbb{R}$. Then

$$d(T_a(x),T_a(y)) = | T_a(x) - T_a(y) | = | ( x + a ) - ( y + a ) | = | x - y | = d(x,y).$$

Therefore, $T_a$ is a distance preserving surjection. Hence, $T_a$ is a rigid motion of $\mathbb{R}$. ☐

**Theorem 2.3.** Let $a \in \mathbb{R}$.

**a)** The reflection $Z_a : \mathbb{R} \to \mathbb{R}$ is a rigid motion of $\mathbb{R}$ such that $(Z_a)^{-1} = Z_a$.

**b)** For x and $y \in \mathbb{R}$, $Z_a(x) = y$ if and only if a is the midpoint between x and y.

**Homework Problem 2.1.** Prove Theorem 2.3.

For $a, b \in \mathbb{R}$, Theorem 2.1 d) implies that the four compositions $T_a \circ T_b$, $Z_a \circ Z_b$, $T_a \circ Z_b$ and $Z_a \circ T_b$ must be rigid motions of $\mathbb{R}$. We now analyze these four compositions to try to discover what types of rigid motions they are.

**Theorem 2.4.** For $a, b \in \mathbb{R}$, $T_a \circ T_b = T_{a+b}$, $Z_a \circ Z_b = T_{2(a-b)}$, $T_a \circ Z_b = Z_{b+(a/2)}$ and $Z_a \circ T_b = Z_{a-(b/2)}$.

**Proof.** Let $x \in \mathbb{R}$. Then:

$$T_a \circ T_b(x) = ( x + a ) + b = x + ( a + b ) = T_{a+b}(x),$$

$$Z_a \circ Z_b(x) = 2a - ( 2b - x ) = x + 2( a - b ) = T_{2(a-b)}(x),$$

$$T_a \circ Z_b(x) = ( 2b - x ) + a = 2( b + (a/2) ) - x = Z_{b+(a/2)}(x) \text{ and}$$

$$Z_a \circ T_b(x) = 2a - ( x + b ) = 2(a - (b/2)) - x = Z_{a-(b/2)}(x).$$ ☐

**In-Class Exercise 2.E. a)** Let a and $b \in \mathbb{R}$. Use the equation $Z_a \circ Z_b = T_{2(a-b)}$ to solve for c and $d \in \mathbb{R}$ such that $T_a = Z_c \circ Z_b$ and $T_a = Z_b \circ Z_d$. Once we know the values of c and d, we can give alternative proofs of the equations $T_a \circ Z_b = Z_{b+(a/2)}$ and $Z_a \circ T_b = Z_{a-(b/2)}$ as follows. $T_a \circ Z_b = (Z_c \circ Z_b) \circ Z_b = Z_c \circ (Z_b \circ Z_b) = Z_c \circ (Z_b \circ (Z_b)^{-1}) = Z_c \circ id_{\mathbb{R}} = Z_c$ and $Z_b \circ T_a = Z_b \circ (Z_b \circ Z_d) = (Z_b \circ Z_b) \circ Z_d = (Z_b \circ (Z_b)^{-1}) \circ Z_d = = id_{\mathbb{R}} \circ Z_d = Z_d$.

**b)** Let a, b and $c \in \mathbb{R}$ and identify the three-fold composition $Z_a \circ Z_b \circ Z_c$ as a specific reflection or translation.

Theorem 2.4 says that we can't generate any new types of rigid motions of $\mathbb{R}$ by composing translations and reflections of $\mathbb{R}$ in any combination or order. This suggests the conjecture that all rigid motions of $\mathbb{R}$ are either translations or reflections.

The next few theorems are devoted to answering the following questions.

**Question 1.** Are all distance preserving functions from $\mathbb{R}$ to itself surjective and, hence, rigid motions?

**Question 2.** Are all rigid motions of $\mathbb{R}$ either reflections or translations?

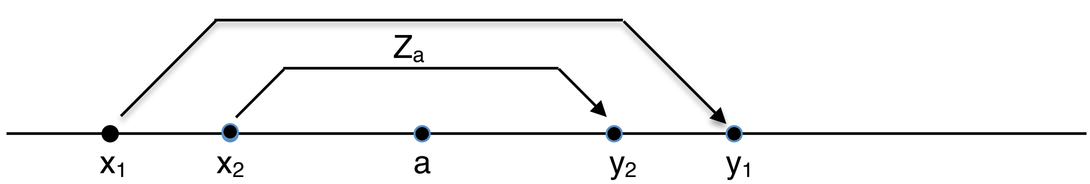
We will need the help of the following two theorems to answer these questions. The first of these theorems shows that reflections and translations are versatile at moving pairs of points around in $\mathbb{R}$.

**Theorem 2.5.** If $x_1$, $x_2$, $y_1$ and $y_2 \in \mathbb{R}$ such that $d(x_1,x_2) = d(y_1,y_2)$, then there is a rigid motion $f : \mathbb{R} \to \mathbb{R}$ such that $f(x_1) = y_1$, $f(x_2) = y_2$ and f is either a reflection or a translation.

**Proof.** Let $a = (^1/_2)( x_1 + y_1 )$, the midpoint between $x_1$ and $y_1$. Then Theorem 2.3 b) implies that $Z_a(x_1) = y_1$.

We now break the proof into two cases.

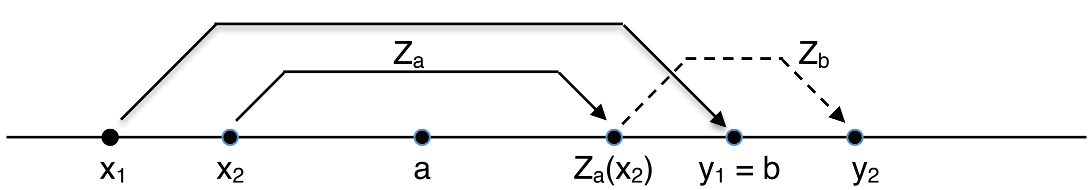
**Case 1: $Z_a(x_2) = y_2$.** In this case, we are done because $Z_a : \mathbb{R} \to \mathbb{R}$ is a reflection and, hence, a rigid motion of $\mathbb{R}$ such that $Z_a(x_1) = y_1$ and $Z_a(x_2) = y_2$.




**Case 2: $Z_a(x_2) \neq y_2$.** Let $b = y_1 = Z_a(x_1)$. Then

$$d(b,Z_a(x_2)) = d(Z_a(x_1),Z_a(x_2)) = d(x_1,x_2) = d(y_1,y_2) = d(b,y_2).$$

Hence, b is equidistant from $Z_a(x_2)$ and $y_2$. Therefore, b is the midpoint between $Z_a(x_2)$ and $y_2$ by Theorem 1.4. Hence, Theorem 2.3 b) implies $Z_b(Z_a(x_2)) = y_2$. Also



$Z_b(Z_a(x_1)) = Z_b(b) = b = y_1$. Thus, $Z_b \circ Z_a(x_1) = y_1$ and $Z_b \circ Z_a(x_2) = y_2$. Theorem 2.4 tells us that $Z_b \circ Z_a = T_{2(b-a)}$. Therefore, $T_{2(b-a)}(x_1) = y_1$ and $T_{2(b-a)}(x_2) = y_2$. This completes the proof in Case 2 because $T_{2(b-a)} : \mathbb{R} \to \mathbb{R}$ is a translation and, hence, a rigid motion of $\mathbb{R}$ such that $T_{2(b-a)}(x_1) = y_1$ and $T_{2(b-a)}(x_2) = y_2$. □

The second of our "helper" theorems shows that distance preserving functions from $\mathbb{R}$ to itself are uniquely determined by their values at two points.

**Theorem 2.6.** If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are distance preserving functions, and if there are two distinct points x and y of $\mathbb{R}$ such that $f(x) = g(x)$ and $f(y) = g(y)$, then $f = g$.

**Proof.** Assume $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are distance preserving functions and x and y are distinct points of $\mathbb{R}$ such that $f(x) = g(x)$ and $f(y) = g(y)$. Let $z \in \mathbb{R}$. We must prove $f(z) = g(z)$.

Since distance preserving functions are injective (by Theorem 2.1.b) and $x \neq y$, then $f(x) \neq f(y)$. Since $f(x) = g(x)$ and $f(y) = g(y)$, then

$$d(f(z),f(x)) = d(z,x) = d(g(z),g(x)) = d(g(z),f(x))$$

and

$$d(f(z),f(y)) = d(z,y) = d(g(z),g(y)) = d(g(z),f(y)).$$

Thus, the distances from $f(z)$ to $f(x)$ and $f(y)$ equal the distances from $g(z)$ to $f(x)$ and $f(y)$. Since $f(x)$ and $f(y)$ are distinct points of $\mathbb{R}$, then Theorem 1.5 tells us that every point of $\mathbb{R}$ is uniquely determined by its distances from $f(x)$ and $f(y)$. It follows that $f(z) = g(z)$. We conclude that $f = g$. □

We are now in a position to answer Questions 1 and 2.

**Theorem 2.7.** Every distance preserving function from $\mathbb{R}$ to itself is either a reflection or a translation.

**Proof.** Let $f : \mathbb{R} \to \mathbb{R}$ be a distance preserving function. Let $x_1$ and $x_2$ be distinct points of $\mathbb{R}$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then $d(x_1,x_2) = d(f(x_1),f(x_2)) = d(y_1,y_2)$. Hence, Theorem 2.5 implies there is a rigid motion $g : \mathbb{R} \to \mathbb{R}$ such that $g(x_1) = y_1$, $g(x_2) = y_2$ and g is either a reflection or a translation. Observe that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are distance preserving functions such that $f(x_1) = g(x_1)$ and $f(x_2) = g(x_2)$. Therefore, Theorem 2.6 implies $f = g$. We conclude that f is either a reflection or a translation. □

Since all reflections and translations of $\mathbb{R}$ are rigid motions, then Theorem 2.7 yields an answer to Question 1.

**Corollary 2.8.** Every distance preserving function from $\mathbb{R}$ to itself is a rigid motion.

Since all rigid motions of $\mathbb{R}$ are distance preserving, then Theorem 2.7 also yields an answer to Question 2.

**Corollary 2.9.** Every rigid motion of $\mathbb{R}$ is either a reflection or a translation.

Since the translation $T_a$ obeys the formula $T_a(x) = x + a$, and since the reflection $Z_a$ obeys the formula $Z_a(x) = -x + 2a$, Corollary 2.9 yields the following conclusion.

**Corollary 2.10.** Every rigid motion $f : \mathbb{R} \to \mathbb{R}$ is of the form

$$f(x) = \varepsilon x + a$$

where $\varepsilon \in \{ -1, 1 \}$ and $a \in \mathbb{R}$.

We can use Corollary 2.10 to give an easy proof of the following result.

**Corollary 2.11.** Every rigid motion $f : \mathbb{R} \to \mathbb{R}$ satisfies the equation

$$f(x) = ( 1 - x )f(0) + xf(1)$$

for every $x \in \mathbb{R}$.

**Homework Problem 2.2.** Prove Corollary 2.11.

This completes our exploration of the rigid motions of $\mathbb{R}$. We end this portion of Chapter 2 with two interesting homework problems.

**Homework Problem 2.3.** Suppose X and Y are metric spaces with metrics $d_X$ and $d_Y$, respectively. For $x \in X$, $y \in Y$ and $r > 0$, let $S_X(x,r) = \{ x' \in X : d_X(x,x') = r \}$ and let $S_Y(y,r) = \{ y' \in Y : d_Y(y,y') = r \}$.

**a)** Prove that if $f : X \to Y$ is an isometry, then for every $x \in X$ and every $r > 0$, $f(S_X(x,r)) = S_Y(f(x),r)$.

**b)** Prove that if $f : X \to Y$ is a function with the property that $f(S_X(x,r)) = S_Y(f(x),r)$ for every $x \in X$ and every $r > 0$, then $f : X \to Y$ is an isometry.

**c)** Suppose $f : X \to Y$ is an isometry. Let $x_0$, $x_1$ and $x_2 \in X$. Prove that $x_2$ is a midpoint between $x_0$ and $x_1$ (in X) if and only if $f(x_2)$ is a midpoint between $f(x_0)$ and $f(x_1)$ (in Y).

**Homework Problem 2.4. a)** Is $\mathbb{R}^2$ with the Euclidean metric $d_E$ isometric to $\mathbb{R}^2$ with the taxicab metric $d_T$?

**b)** Is $\mathbb{R}^2$ with the taxicab metric $d_T$ isometric to $\mathbb{R}^2$ with the maximum metric $d_M$?

**c)** Is $\mathbb{R}^2$ with the Euclidean metric $d_E$ isometric to $\mathbb{R}^2$ with the maximum metric $d_M$?