1. Metric Spaces

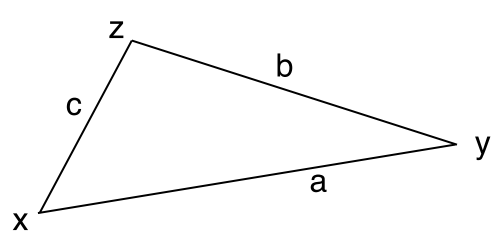
In this course, we will take the point of view that geometry is the study of *spaces* in which the distance between two elements can be measured by a distance function. The elements of such a space are called *points,* the distance function is called a *metric,* and such spaces are called *metric spaces*.

Definition. Let X be a set. A *metric* on X is a function *d* which assigns to every pair of elements *x* and *y* of the set X a real number denoted *d(x,y)*. d(x,y) is called the *distance* between x and y. Furthermore, to be a metric, the function d must satisfy the following three conditions.

1) *Positivity.* For all x, y $\in$ X, d(x,y) $\geq$ 0, and d(x,y) = 0 if and only if x = y.
2) *Symmetry.* For all x, y $\in$ X, d(x,y) = d(y,x).
3) *The Triangle Inequality.* For all x, y and z $\in$ X, d(x,z) $\leq$ d(x,y) + d(y,z).

A set X equipped with a metric d is called a *metric space*. The elements of a metric space are called *points.*

The triangle inequality is a generalization of a property of planar triangles. If a triangle has vertices x, y and z and edge lengths a = d(x,y), b = d(y,z) and c = d(x,z), then each edge length is less than or equal to the sum of the other two edge lengths. Thus, c $\leq$ a + b, b $\leq$ a + c, and a $\leq$ b + c.



Example. Every set X admits an easily defined metric called the *discrete metric* on X. We first define the *discrete distance function* on X by the formulas

$$d(x,y) = 0 \text{ if } x = y, \text{ and } d(x,y) = 1 \text{ if } x \neq y.$$

We will prove that the discrete distance function is a metric on X, after which we will call this function the *discrete metric* on X.

In-Class Exercise 1.A. Verify that the discrete distance function is actually a metric on the set X. In other words, prove that d satisfies the three defining properties of a metric: positivity, symmetry and the triangle inequality.

The discrete metric is interesting primarily because it is a very simple example of a metric that can be defined on any set. A set with the discrete metric is too simple to have interesting geometry, but it does provide a *toy example* in which we can easily test new concepts before trying to analyze these concepts in situations that are more significant geometrically but are also more complex.

Perhaps the simplest example of a geometrically interesting metric space is the real line $\mathbb{R}$ with its *standard metric.* To define the standard metric on $\mathbb{R}$ we first recall some basic terminology and concepts.

Notation. Let $a < b$ be real numbers. We recall the standard notation for the subsets of $\mathbb{R}$ known as *intervals:*

The following sets are called *open intervals:*

$$(a,b) = \{ x \in \mathbb{R} : a < x < b \}, \quad (a,\infty) = \{ x \in \mathbb{R} : a < x \}, \quad (-\infty,b) = \{ x \in \mathbb{R} : x < b \}.$$

The following sets are called *closed intervals:*

$$[a,b] = \{ x \in \mathbb{R} : a \le x \le b \}, \quad [a,\infty) = \{ x \in \mathbb{R} : a \le x \}, \quad (-\infty,b] = \{ x \in \mathbb{R} : x \le b \}.$$

The following sets are called *half open intervals:*

$$[a,b) = \{ x \in \mathbb{R} : a \le x < b \}, \quad (a,b] = \{ x \in \mathbb{R} : a < x \le b \}.$$

Definition. The *squaring* function $x \mapsto x^2 : [0,\infty) \to [0,\infty)$ with domain and range $[0,\infty)$ is a bijection. Hence, it has an inverse (by Theorem 0.5). The inverse is called the *square root* function and is denoted $x \mapsto \sqrt{x} : [0,\infty) \to [0,\infty)$. Observe that the fact that the squaring function and the square root function are inverses of each other has the following consequence: for every $x \in [0,\infty)$, $\sqrt{x^2} = x$ and $(\sqrt{x})^2 = x$.

Definition. For every $x \in \mathbb{R}$, since $x^2 \in [0,\infty)$ and $[0,\infty)$ is the domain (and range) of the square root function, then $\sqrt{x^2}$ is well defined. Hence, we can define the *absolute value* function $x \mapsto |x| : \mathbb{R} \to [0,\infty)$ by the formula $|x| = \sqrt{x^2}$. (Observe that the absolute value function $x \mapsto |x| : \mathbb{R} \to [0,\infty)$ is the composition of the squaring function $x \mapsto x^2 : \mathbb{R} \to [0,\infty)$ and the square root function $x \mapsto \sqrt{x} : [0,\infty) \to [0,\infty)$.)

Next we list some basic properties of the absolute value function.

Theorem 1.1. The absolute value function has the following properties.
a) For all $x \in \mathbb{R}$, $|x| = x$ if $x \ge 0$, and $|x| = -x$ if $x < 0$.
b) For all $x \in \mathbb{R}$, $-|x| \le x \le |x|$.
c) For all $x \in \mathbb{R}$, $|x|^2 = x^2$.
d) For all $x \in \mathbb{R}$, $|x| \ge 0$, and $|x| = 0$ if and only if $x = 0$.
e) For all $x, y \in \mathbb{R}$, $|xy| = |x|\,|y|$.
f) For all $x, y \in \mathbb{R}$, $|x + y| \le |x| + |y|$.

Homework Problem 1.1. Prove Theorem 1.1 using the definition of the absolute value function given above together with properties of the squaring function and the square root function.

We now use the absolute value function to define the standard metric on $\mathbb{R}$.

Definition. The *standard distance function* d on $\mathbb{R}$ is defined by the formula

$$d(x,y) = | x - y |.$$

We will prove that the standard distance function is a metric on $\mathbb{R}$, after which we will call this function the *standard metric* on $\mathbb{R}$. To prove that the standard distance function is a metric, we must verify that it satisfies the three defining conditions for a metric: positivity, symmetry and the triangle inequality.

Theorem 1.2. The standard distance function is a metric on $\mathbb{R}$.

Proof. We must prove:

a) *Positivity.* For all $x, y \in \mathbb{R}$, $d(x,y) \geq 0$, and $d(x,y) = 0$ if and only if $x = y$.
b) *Symmetry.* For all $x, y \in \mathbb{R}$, $d(x,y) = d(y,x)$.
c) *The Triangle Inequality.* For all x, y and $z \in \mathbb{R}$, $d(x,z) \leq d(x,y) + d(y,z)$.

Homework Problem 1.2. Prove a) and b).

Proof of c) *The Triangle Inequality.* Let x, y and $z \in \mathbb{R}$. Then Theorem 1.1 f) implies: $d(x,z) = | x - z | = | (x - y) + (y - z) | \leq | x - y | + | y - z | = d(x,y) + d(y,z)$.
□

Homework Problem 1.3. In this problem, we define a non-standard metric on $\mathbb{R}$ called the *square root metric* and observe that it has a curious property. Define the *square root distance function* σ on $\mathbb{R}$ by the formula

$$\sigma(x,y) = \sqrt{| x - y |}.$$

a) Prove that the square root distance function is a metric on $\mathbb{R}$.

Now we call σ the *square root metric* on $\mathbb{R}$.

b) Show that σ has the following strange property. Let $n \geq 1$ and for $0 \leq i \leq n$, let $x_i^n = {}^i/_n$. Let $L_n = \sigma(x_0^n,x_1^n) + \sigma(x_1^n,x_2^n) + \ldots + \sigma(x_{n-1}^n,x_n^n)$. Then L_n represents the length of a walk from 0 to 1 in $\mathbb{R}$ with the square root metric σ in which you step from 0 to ${}^1/_n$ to ${}^2/_n$ to … to ${}^{n-1}/_n$ to 1. Prove $\lim_{n \to \infty} L_n = \infty$. Thus, in $\mathbb{R}$ with the square root metric, if you walk from 0 to 1 taking small steps, the length of your walk converges to ∞ as the length of your steps approaches 0. Said another way, as you walk from 0 to 1 in $\mathbb{R}$ with the square root metric, as your legs grow shorter, the length of your walk converges to ∞.

From now on, whenever we consider the real line $\mathbb{R}$ as a metric space, we assume that the metric on $\mathbb{R}$ is the standard metric.

Next we introduce three geometrically useful concepts: the *sphere of a given positive radius centered at a given point,* a point being *equidistant from* two points, and a *midpoint between two points.*

Definition. Let X be a metric space with metric d. Let x be a point of X and let r > 0. The *sphere of radius r centered at x* is the set

$$S(x,r) = \{ y \in X : d(x,y) = r \}.$$

Definition. Let X be a metric space with metric d. Let x, y and z be points of X. z is *equidistant from* x and y if d(x,z) = d(y,z). z is a *midpoint* between x and y if $d(x,z) = d(y,z) = (1/2)d(x,y)$.

In-Class Exercise 1.B. Let X be a set with the discrete metric.
a) If $x \in X$ and $r > 0$, describe the set S(x,r). (The answer depends on the value of r.)
b) If x and $y \in X$, describe the set of all points of X that are equidistant from x and y.
c) If x and $y \in X$, describe the set of all midpoints between x and y.

The following theorem characterizes midpoints between x and y in terms of spheres centered at x and y.

Theorem 1.3. Let X be a metric space X with metric d. Let x and y be points of X and let $r = (1/2)d(x,y)$. Then a point z is a midpoint between x and y if and only if z is an element of the set $S(x,r) \cap S(y,r)$.

Homework Problem 1.4. Prove Theorem 1.3.

Remark. The definition of midpoint leaves open the possibility that there might be *more than one* midpoint (or *no* midpoints) between two given points. Theorem 1.3 tells us that the set of all midpoints between the points x and y equals the set $S(x,r) \cap S(y,r)$. This theorem does not rule out the possibility that this set has more than one element or that the set is empty.

We now examine the concepts of spheres and midpoints in the real line $\mathbb{R}$ (with the standard metric).

In-Class Exercise 1.C. Let $x \in \mathbb{R}$ and let $r > 0$. What are the elements of the set S(x,r)? Prove your answer.

In-Class Exercise 1.D. Let x and y be distinct points of $\mathbb{R}$. Use Theorem 1.3 and Class Exercise 1.C to prove that there is one and only one midpoint between x and y, and that midpoint is $(1/2)(x + y)$. **Hint:** We may assume $x < y$. Why?

We can actually prove a theorem that is a little stronger than the result stated in Class Exercise 1.D.

Theorem 1.4. Let x and y be distinct points of $\mathbb{R}$. Then a point $z \in \mathbb{R}$ is equidistant from x and y if and only if $z = (1/2)(x + y)$, the midpoint between x and y.

Remark. Theorem 1.4 says that $(1/2)(x + y)$ is the only point in $\mathbb{R}$ that is equidistant from x and y. Theorem 1.4 also tells us that to find a midpoint z between x and y in $\mathbb{R}$, it is sufficient to find a point that is equidistant from x and y. It is not necessary to impose the additional restriction that d(x,z) and d(y,z) are equal to $(1/2)$d(x,y). In other words, if we can establish the equation d(x,z) = d(y,z), then the addition restriction that these numbers equal $(1/2)$d(x,y) will follow automatically.

The property of $\mathbb{R}$ revealed by Theorem 1.4 – any point which is *equidistant* from x and y is a *midpoint* between x and y – fails to hold in most geometrically interesting spaces including circles and all the higher dimensional metric spaces that we will study subsequently.

Proof of Theorem 1.4. Let x and y be distinct points of $\mathbb{R}$. First we assume z is a midpoint between x and y, and we prove z is equidistant from x and y. Since z is a midpoint between x and y, then by definition d(x,z) = d(y,z) = $(1/2)$d(y,x). Therefore, d(x,z) = d(y,z). Hence, z is equidistant from x and y. That's all there is to it.

Next we will assume z is a point of $\mathbb{R}$ that is equidistant from x and y, and we will prove that z is the midpoint between x and y. (This part of the proof is a little harder than the first part.) So, assume $z \in \mathbb{R}$ and d(x,z) = d(y,z). We will prove $z = (1/2)(x + y)$. Since d(x,z) = d(y,z), then

$$|x - z| = |y - z|.$$

Hence,

$$(|x - z|)^2 = (|y - z|)^2.$$

Therefore,

$$(x - z)^2 = (y - z)^2.$$

So

$$x^2 - 2xz + z^2 = y^2 - 2yz + z^2.$$

Hence,

$$x^2 - y^2 = 2xz - 2yz.$$

Thus,

$$(x - y)(x + y) = 2(x - y)z.$$

Since $x \neq y$, then $x - y \neq 0$. Therefore,

$$x + y = 2z.$$

So

$$z = (^1/_2)(x + y).$$

Thus, z is the midpoint between x and y by In-Class Exercise 1.D. ☐

Theorem 1.4 has a useful corollary which we now state and prove.

Theorem 1.5. Let x and y be distinct points of $\mathbb{R}$. Then every point z of $\mathbb{R}$ is uniquely determined by the two numbers d(x,z) and d(y,z). In other words, every point of $\mathbb{R}$ is uniquely determined by its distances from x and y.

Remark. Like Theorem 1.4, Theorem 1.5 is states a property that holds in $\mathbb{R}$ but not in other geometrically interesting spaces we will study. This property holds with some important exceptions in circles and it fails to hold in higher dimensional spaces. With the help of Theorem 1.4, we can give a surprisingly simple proof of Theorem 1.5. The proof is by contradiction.

Proof of Theorem 1.5. Let x and y be distinct points of $\mathbb{R}$. Assume that z and z´ are two points of $\mathbb{R}$ such that d(x,z) = d(x,z´) and d(y,z) = d(y,z´). We must prove that z = z´.

Assume $z \neq z´$. Since d(x,z) = d(x,z´), then Theorem 1.4 implies that $x = (^1/_2)(z + z´)$. Also since d(y,z) = d(y,z´), then Theorem 1.4 implies that $y = (^1/_2)(z + z´)$. Hence, x = y. This contradicts our hypothesis that x and y are distinct. We must conclude that the assumption $z \neq z´$ is false. Thus, z = z´. We have proved that the point z is *uniquely determined* by the two numbers d(x,z) and d(y,z). ☐

Homework Problem 1.5. This problem explores an interesting example in which midpoints between two given points may not exist. Recall the square root metric σ on $\mathbb{R}$ from Homework Problem 1.3.

a) Prove that if x and y are distinct points in $\mathbb{R}$ with the metric σ, then there is no midpoint between x and y.

b) Also prove that if x and y are distinct points of $\mathbb{R}$, then there is a point of $\mathbb{R}$ that is equidistant from x and y with respect to the square root metric.

Since this point can't be a midpoint, then we observe that the analogue of Theorem 1.4 fails to hold for $\mathbb{R}$ with the square root metric: a point which is equidistant from x and y can't be a midpoint.

We introduce another geometrically useful concept: the *diameter* of a space.

Definition. Let X be a metric space with metric d. Let A be a subset of X. Define the *diameter* of A with respect to d to be the element diam(A) $\in [0,\infty]$ determined by the formula

$$\text{diam}(A) = \sup \{ d(x,y) : x \in A \text{ and } y \in A \}.$$

If diam(A) $< \infty$, then we call A a *bounded* subset or X; whereas if diam(A) $= \infty$, then we call A an *unbounded* subset of X. If diam(X) $< \infty$, we call X a *bounded* space; while if diam(X) $= \infty$, we call X an *unbounded* space.

In-Class Exercise 1.E. a) If X is a set with the discrete metric, what is diam(X)?

b) Let x, y $\in \mathbb{R}$ such that $x < y$. What is the diameter of the subset { x, y }? What are the diameters of the intervals [x,y], (x,y), [x,y) and (x,y]?

c) Let $\mathbb{N} = \{ 0, 1, 2, \dots \}$ be the subset of $\mathbb{R}$ consisting of all non-negative integers, and let $\mathbb{Z} = \{ \dots -2, -1, 0, 1, 2, \dots \}$ be the subset of $\mathbb{R}$ consisting the set of all integers. What are the diameters of the sets $\mathbb{N}$, $\mathbb{Z}$, $[0,\infty)$ and $\mathbb{R}$?

Next we consider three different metrics on the 2-dimensional Cartesian plane $\mathbb{R}^2$, and we explore the differences in the geometries that are determined by these three metrics. The *Cartesian plane* is the set

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{ (x_1,x_2) : x_1 \in \mathbb{R} \text{ and } x_2 \in \mathbb{R} \}.$$

Thus, $\mathbb{R}^2$ is the set of all ordered pairs of real numbers. We will follow the practice of denoting a point of $\mathbb{R}^2$ by $\mathbf{x} = (x_1,x_2)$. Thus, a point of $\mathbb{R}^2$ is denoted by a boldface letter while the coordinates of the point are denoted by the same letter – not in boldface – with subscripts 1 and 2. Also we let $\mathbf{0} = (0,0)$ and call this point the *origin* of $\mathbb{R}^2$.

The *Euclidean distance function* d_E on $\mathbb{R}^2$ is defined by the formula

$$d_E(\mathbf{x},\mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

for $\mathbf{x} = (x_1,x_2)$ and $\mathbf{y} = (y_1,y_2) \in \mathbb{R}^2$.

The *taxicab distance function* d_T on $\mathbb{R}^2$ is defined by the formula

$$d_T(\mathbf{x},\mathbf{y}) = | x_1 - y_1 | + | x_2 - y_2 |$$

for $\mathbf{x} = (x_1,x_2)$ and $\mathbf{y} = (y_1,y_2) \in \mathbb{R}^2$.

The *maximum distance function* d_M on $\mathbb{R}^2$ is defined by the formula

$$d_M(\mathbf{x},\mathbf{y}) = \max \{ | x_1 - y_1 |, | x_2 - y_2 | \}$$

for $\mathbf{x} = (x_1,x_2)$ and $\mathbf{y} = (y_1,y_2) \in \mathbb{R}^2$.

We will prove that the Euclidean, taxicab and maximum distance functions are metrics on $\mathbb{R}^2$, after which we will call these function the *Euclidean, taxicab* and *maximum metrics* on $\mathbb{R}^2$.

Theorem 1.6. The Euclidean, taxicab and maximum distance functions, d_E, d_T and d_M, are metrics on $\mathbb{R}^2$.

Remark. The proof of Theorem 1.6 involves showing that d_E, d_T and d_M satisfy the three conditions that comprise the definition of a metric: positivity and symmetry and the triangle inequality.

Homework Problem 1.6. Prove Theorem 1.6 with the exception of the fact that d_E satisfies the triangle inequality. We postpone the proof that d_E satisfies the triangle inequality because we will introduce an auxiliary concept a little later that will make this proof easy.

Notation. For $\mathbf{x} \in \mathbb{R}^2$ and $r > 0$, let $S_E(\mathbf{x},r)$, $S_T(\mathbf{x},r)$ and $S_M(\mathbf{x},r)$ denote the spheres of radius r centered at $\mathbf{x}$ with respect to the Euclidean, taxicab and maximum metrics, respectively. Thus,

$$S_E(\mathbf{x},r) = \{ \mathbf{y} \in \mathbb{R}^2 : d_E(\mathbf{x},\mathbf{y}) = r \},$$

$$S_T(\mathbf{x},r) = \{ \mathbf{y} \in \mathbb{R}^2 : d_T(\mathbf{x},\mathbf{y}) = r \} \text{ and}$$

$$S_M(\mathbf{x},r) = \{ \mathbf{y} \in \mathbb{R}^2 : d_M(\mathbf{x},\mathbf{y}) = r \}.$$

Next we analyze and draw pictures of the spheres $S_E(\mathbf{x},r)$, $S_T(\mathbf{x},r)$ and $S_M(\mathbf{x},r)$ with respect to the Euclidean, taxicab and maximum metrics on $\mathbb{R}^2$. Let $\mathbf{x} = (x_1,x_2) \in \mathbb{R}^2$ and $r > 0$.

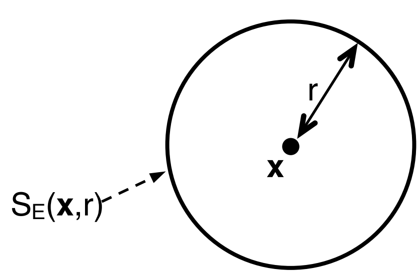
First we study $S_E(\mathbf{x},r)$. Observe that the following three statements are Equivalent.

i) $\mathbf{y} = (y_1,y_2) \in S_E(\mathbf{x},r)$.

ii) $d_E(\mathbf{x},\mathbf{y}) = r$.

iii) $(x_1 - y_1)^2 + (x_2 - y_2)^2 = r^2$.

The last statement is the equation of a circle of radius r centered at $\mathbf{x} = (x_1,x_2)$. Hence, $S_E(\mathbf{x},r)$ is this circle. Here is the picture:



Next we study $S_T(\mathbf{x},r)$. Observe that the following four statements are equivalent.

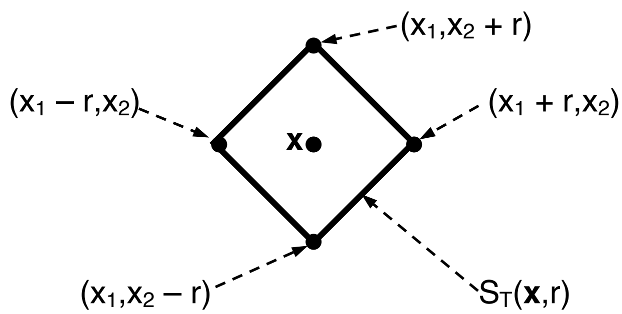
i) $\mathbf{y} = (y_1,y_2) \in S_T(\mathbf{x},r)$.

ii) $d_T(\mathbf{x},\mathbf{y}) = r$.

iii) $| x_1 - y_1 | + | x_2 - y_2 | = r$.

iv) $(x_1 \le y_1 \le x_1 + r$ and $y_2 = -y_1 + (x_1 + x_2 + r))$ or
$(x_1 - r \le y_1 \le x_1$ and $y_2 = y_1 + (- x_1 + x_2 + r))$ or
$(x_1 - r \le y_1 \le x_1$ and $y_2 = -y_1 + (x_1 + x_2 - r))$ or
$(x_1 \le y_1 \le x_1 + r$ and $y_2 = y_1 + (-x_1 + x_2 - r))$.

The last statement consists of four constrained equations which determine a square with vertices $(x_1 - r,x_2)$, $(x_1,x_2 - r)$, $(x_1 + r,x_2)$ and $(x_1,x_2 + r)$. Hence, $S_T(\mathbf{x},r)$ is this square. Here is the picture:



Finally we study $S_M(\mathbf{x},r)$. Observe that the following four statements are equivalent:

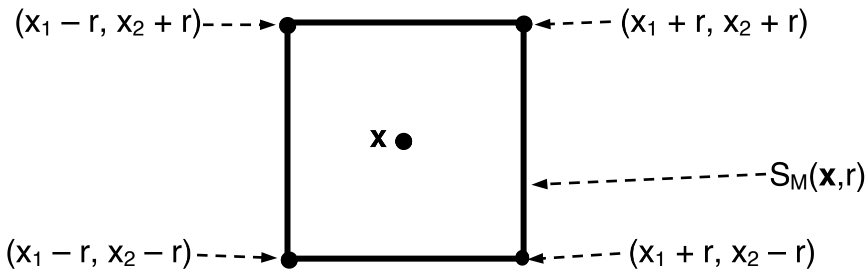
i) $\mathbf{y} = (y_1,y_2) \in S_M(\mathbf{x},r)$,

ii) $d_M(\mathbf{x},\mathbf{y}) = r$,

iii) $\max \{ | x_1 - y_1 |, | x_2 - y_2 | \} = r$,

iv) $(y_1 = x_1 + r$ and $x_2 - r \le y_2 \le x_2 + r)$ or
$(y_2 = x_2 + r$ and $x_1 - r \le y_1 \le x_1 + r)$ or
$(y_1 = x_1 - r$ and $x_2 - r \le y_2 \le x_2 + r)$ or
$(y_2 = x_2 - r$ and $x_1 - r \le y_1 \le x_1 + r)$.

The last statement consists of four constrained equations which determine a square with vertices $(x_1 - r,x_2 + r)$, $(x_1 + r,x_2 + r)$, $(x_1 + r,x_2 - r)$ and $(x_1 - r,x_2 - r)$. Hence, $S_M(\mathbf{x},r)$ is this square. Here is the picture:



Homework Problem 1.7. Let $\mathbf{x} = (x_1,x_2) \in \mathbb{R}^2$ and $r > 0$. This problem asks you to give algebraic proofs of some of the assertions of equivalence made in the preceding paragraphs.

a) Algebraically verify the equivalence of the following two statements.

i) $| x_1 - y_1 | + | x_2 - y_2 | = r$.

ii) $(x_1 \leq y_1 \leq x_1 + r$ and $y_2 = -y_1 + (x_1 + x_2 + r))$ or
$(x_1 - r \leq y_1 \leq x_1$ and $y_2 = y_1 + (- x_1 + x_2 + r))$ or
$(x_1 - r \leq y_1 \leq x_1$ and $y_2 = -y_1 + (x_1 + x_2 - r))$ or
$(x_1 \leq y_1 \leq x_1 + r$ and $y_2 = y_1 + (-x_1 + x_2 - r))$.

b) Algebraically verify the equivalence of the following two statements.

i) $\max \{ | x_1 - y_1 |, | x_2 - y_2 | \} = r$

ii) $(y_1 = x_1 + r$ and $x_2 - r \leq y_2 \leq x_2 + r)$ or
$(y_2 = x_2 + r$ and $x_1 - r \leq y_1 \leq x_1 + r)$ or
$(y_1 = x_1 - r$ and $x_2 - r \leq y_2 \leq x_2 + r)$ or
$(y_2 = x_2 - r$ and $x_1 - r \leq y_1 \leq x_1 + r)$.

Homework Problem 1.8. Let **x** and **y** be distinct points of $\mathbb{R}^2$. For each of the metrics d_E, d_T and d_M, decide whether a midpoint between **x** and **y** exists and whether such a midpoint (if it exists) is unique. Does your answer depend on the relative positions of **x** and **y** in $\mathbb{R}^2$?

Hint. Let r equal $^1/_2$ the distance from **x** to **y**. Use the characterization of midpoints stated in Theorem 1.3 together with the shapes of the spheres of radius r centered at **x** and **y** to create pictures that answer the questions raised in Homework Problem 1.8.

Homework Problem 1.9. Let $\mathbf{x} = (x_1,x_2)$ and $\mathbf{y} = (y_1,y_2)$ be distinct points of $\mathbb{R}^2$. Define the point $\mathbf{m} \in \mathbb{R}^2$ by $\mathbf{m} = ((^1/_2)(x_1 + y_1), (^1/_2)(x_2 + y_2))$.

a) For each of the metrics d_E, d_T and d_M, prove *algebraically* that **m** is a midpoint between **x** and **y**.

b) Prove *algebraically* that for the Euclidean metric d_E, $\mathbf{m}$ is the *only* midpoint between $\mathbf{x}$ and $\mathbf{y}$. In other words, assume that $\mathbf{z} = (z_1,z_2)$ is a midpoint between $\mathbf{x}$ and $\mathbf{y}$ with respect to the Euclidean metric d_E and prove *algebraically* that $\mathbf{z} = \mathbf{m}$.

Homework Problem 1.10. Define a metric on the unit circle $S_E(\mathbf{0},1)$ called the *straight line metric* by the formula $s(\mathbf{x},\mathbf{y}) = d_E(\mathbf{x},\mathbf{y})$. In other words, the straight line metric s is the restriction to $S_E(\mathbf{0},1)$ of the Euclidean metric d_E on $\mathbb{R}^2$. Because d_E is a metric on $\mathbb{R}^2$ and s is a restriction of d_E to a subset of $\mathbb{R}^2$, the s is automatically a metric. (Verify this statement.) Prove that if $\mathbf{x}$ and $\mathbf{y}$ are distinct points of $S_E(\mathbf{0},1)$, then there is are no midpoints between $\mathbf{x}$ and $\mathbf{y}$ in $S_E(\mathbf{0},1)$ with the straight line metric s.

Hint. First, for $\mathbf{x} \in S_E(\mathbf{0},1)$ and $r > 0$, define $S_s(\mathbf{x},r) = \{ \mathbf{y} \in S_E(\mathbf{0},1) : s(\mathbf{x},\mathbf{y}) = r \}$. Thus, $S_s(\mathbf{x},r)$ is the sphere of radius r centered at $\mathbf{x}$ in $S_E(\mathbf{0},1)$ with the straight line metric. Then observe that for $\mathbf{x} \in S_E(\mathbf{0},1)$ and $r > 0$, $S_s(\mathbf{x},r) = S_E(\mathbf{x},r) \cap S_E(\mathbf{0},1)$. Use this fact together with Theorem 1.3 to create a picture of the set of midpoints between two distinct points in $S_E(\mathbf{0},1)$ with the straight line metric.

Recall that if X is a metric space with metric d and A is a subset of X, then the *diameter* of A with respect to d is the element of $[0,\infty]$ defined by the formula

$$\text{diam}(A) = \sup \{ d(x,y) : x \in A \text{ and } y \in A \}.$$

We call A a *bounded* subset of X if $\text{diam}(A) < \infty$, and we call A an *unbounded* subset of X if $\text{diam}(A) = \infty$. Similarly, we call X a *bounded space* if $\text{diam}(X) < \infty$, and we call X an *unbounded space* if $\text{diam}(X) = \infty$.

If A is a subset of $\mathbb{R}^2$, let $\text{diam}_E(A)$, $\text{diam}_T(A)$ and $\text{diam}_M(A)$ denote the diameters of A with respect to the Euclidean, taxicab and maximum metrics, respectively. Observe that

$$\text{diam}_E(\mathbb{R}^2) = \text{diam}_T(\mathbb{R}^2) = \text{diam}_M(\mathbb{R}^2) = \infty.$$

Recall that for $r > 0$, $S_E(\mathbf{0},r)$ is the circle of radius r centered at $\mathbf{0}$ in $\mathbb{R}^2$.

In-Class Exercise 1.F. Let $r > 0$. Evaluate $\text{diam}_E(S_E(\mathbf{0},r))$.

Homework Problem 1.11. Let $r > 0$. Evaluate $\text{diam}_T(S_E(\mathbf{0},r))$ and $\text{diam}_M(S_E(\mathbf{0},r))$.

Hint. To evaluate $\text{diam}_T(S_E(\mathbf{0},r))$ it may be helpful to find the smallest $s > 0$ such that $S_E(\mathbf{0},r)$ is inscribed in $S_T(\mathbf{0},s)$. A similar remark applies to $\text{diam}_M(S_E(\mathbf{0},r))$.

Homework Problem 1.12. For distinct points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$, let $E_T(\mathbf{x},\mathbf{y})$ denote the set of all points in $\mathbb{R}^2$ that are equidistant from $\mathbf{x}$ and $\mathbf{y}$ with respect to the taxicab metric; in other words, $E_T(\mathbf{x},\mathbf{y}) = \{ \mathbf{z} \in \mathbb{R}^2 : d_T(\mathbf{x},\mathbf{z}) = d_T(\mathbf{y},\mathbf{z}) \}$. Describe and sketch a picture of the set $E_T((0,0),(1,1))$.

Homework Problem 1.13. Suppose X is a metric space with metric d. Suppose that $\mathcal{L}$ is a collection of subsets of X called *lines.* Three points x, y and z of X are said to be *collinear* if there is a line $L \in \mathcal{L}$ such that x, y and $z \in L$. Three points of X which are not collinear are said to be *non-collinear.* For distinct points x and y of X, let E(x,y) denote the set of all points in X that are equidistant from x and y with respect to the metric d; in other words, $E(x,y) = \{ z \in X : d(x,z) = d(y,z) \}$. Assume that if x and y are distinct points of X, then $E(x,y) \in \mathcal{L}$. Let x, y and z be three distinct non-collinear points of X. Prove that every point of X is uniquely determined by its distances from x, y and z. In other words, prove that if w and $w' \in X$ and $d(x,w) = d(x,w')$, $d(y,w) = d(y,w')$ and $d(z,w) = d(z,w')$, then $w = w'$.