0. Background Material: Properties of Functions

Definition. f is a *function from* a set X *to* a set Y if f assigns to each element $x \in X$ an element $f(x) \in Y$. We write "$f : X \to Y$" as an abbreviation for the statement "f is a function from the set X to the set Y". For each $x \in X$, f(x) is called the *value of* f *at* x. The set X is called the *domain* of f, and the set Y is called the *range* of Y. Two functions f and g are *equal,* denoted f = g, if f and g have the same domain and if f(x) = g(x) for each element x of this common domain.

Notation. Suppose $f : X \to Y$. We may also express this by writing "$x \mapsto f(x) : X \to Y$". For example, we may denote the *squaring function* with domain and range the real numbers $\mathbb{R}$ by "$x \mapsto x^2 : \mathbb{R} \to \mathbb{R}$".

Definition. Suppose $f : X \to Y$. If $A \subset X$, the *image of* A *under* f is defined to be the set

$$f(A) = \{ f(x) : x \in A \}.$$

Thus, the statement "$y \in f(A)$" is equivalent to the statement "there is an $x \in A$ such that f(x) = y". This equivalence is useful for proving results about the set f(A). The set f(X) is also denoted Im(f) and is called the *image of* f. Thus,

$$\mathrm{Im}(f) = \{ f(x) : x \in X \}.$$

If $B \subset Y$, the *preimage of* B *under* f is defined to be the set

$$f^{-1}(B) = \{ x : f(x) \in B \}.$$

Definition. If X is a set, then the *identity function* of X is the function $id_X : X \to X$ defined by the formula

$$id_X(x) = x$$

for every $x \in X$.

Definition. If $f : X \to Y$ and $g : Y \to Z$ are functions, then a function $g \circ f : X \to Z$, called the *composition of* f *and* g, is defined by the formula

$$g \circ f(x) = g(f(x))$$

for every $x \in X$.

Definition. Suppose $f : X \to Y$ and suppose $A \subset X$. The *restriction of* f *to* A is the function with domain A and range Y which is denoted $f \mid A : A \to Y$ and is defined by $f \mid A(x) = f(x)$ for $x \in A$.

Theorem 0.1. a) If $f : X \to Y$ is a function, then $f \circ id_X = f = id_Y \circ f$.

b) If $f : W \to X$, $g : X \to Y$ and $h : Y \to Z$ are functions, then $h \circ (g \circ f) = (h \circ g) \circ f$.

Homework Problem 0.1. Prove Theorem 0.1.

Definition. Suppose $f : X \to Y$ is a function. A function $g : Y \to X$ is an *inverse* of $f : X \to Y$ if the following two equations hold: $g \circ f = id_X$ and $f \circ g = id_Y$.

Observation. The definition of inverse is symmetric. In other words, if $g : Y \to X$ is an inverse of $f : X \to Y$, then $f : X \to Y$ is also an inverse of $g : Y \to X$.

Theorem 0.2. If a function $f : X \to Y$ has an inverse, then that inverse is unique. In other words, if $g : Y \to X$ and $h : Y \to X$ are both inverses of $f : X \to Y$, then $g = h$.

Proof. Assume $f : X \to Y$ is a function, and assume that the functions $g : Y \to X$ and $h : Y \to X$ are both inverses of $f : X \to Y$. Then by Theorem 0.1,

$$g = id_X \circ g = (h \circ f) \circ g = h \circ (f \circ g) = h \circ id_Y = h. \ \square$$

Definition. Suppose $f : X \to Y$ is a function. If $f : X \to Y$ has an inverse, then that inverse is unique by Theorem 0.2 and we denote it by $f^{-1} : Y \to X$. Thus, $f^{-1} \circ f = id_X$ and $f \circ f^{-1} = id_Y$.

Observation. Suppose $f^{-1} : Y \to X$ is the inverse of the function $f : X \to Y$. Then, as we observed above, $f : X \to Y$ is also the inverse of $f^{-1} : Y \to X$. Observe that this assertion is also expressed by the equation $(f^{-1})^{-1} = f$.

Remark. Suppose $f : X \to Y$ is a function and suppose $B \subset Y$. Observe that the set $f^{-1}(B)$ is well defined and exists regardless of whether the function $f : X \to Y$ has a inverse.

Definition. Suppose $f : X \to Y$ is a function. f is *injective* (or *one-to-one*) if for all $x, x' \in X$, $x \neq x'$ implies $f(x) \neq f(x')$. f is *surjective* (or *onto*) if for every $y \in Y$, there is an $x \in X$ such that $f(x) = y$. $f : X \to Y$ is a *bijection* (or a *one-to-one correspondence*) if it is both injective and surjective.

Theorem 0.3. Suppose $f : X \to Y$ is a function.

a) The following three statements are equivalent:
- $f : X \to Y$ is injective.
- For all $x, x' \in X$, if $f(x) = f(x')$, then $x = x'$.
- For every $y \in Y$, the set $f^{-1}(\{ y \})$ contains at most one element.

b) The following three statements are equivalent:
- $f : X \to Y$ is surjective.
- $\mathrm{Im}(f) = Y$.
- For every $y \in Y$, the set $f^{-1}(\{ y \})$ contains at least one element.

Homework Problem 0.2. Prove Theorem 0.3.

Theorem 0.4. Suppose $f : X \to Y$ and $g : Y \to Z$ are functions.
a) If $f : X \to Y$ and $g : Y \to Z$ are injective, then $g \circ f : X \to Z$ is injective.
b) If $f : X \to Y$ and $g : Y \to Z$ are surjective, then $g \circ f : X \to Z$ is surjective.
c) If $g \circ f : X \to Z$ is injective, then $f : X \to Y$ is injective.
d) If $g \circ f : X \to Z$ is surjective, then $g : Y \to Z$ is surjective.

Homework Problem 0.3. Prove Theorem 0.4.

Homework Problem 0.4. Find an example of functions $f : X \to Y$ and $g : Y \to Z$ such that $g \circ f : X \to Z$ is a bijection, but $f : X \to Y$ is *not* surjective and $g : Y \to Z$ is *not* injective.

Theorem 0.5. Suppose $f : X \to Y$ is a function. Then $f : X \to Y$ has an inverse if and only if $f : X \to Y$ is a bijection.

Proof. First assume $f : X \to Y$ has an inverse $f^{-1} : Y \to X$. Thus, $f^{-1} \circ f = id_X$ and $f \circ f^{-1} = id_Y$. Since identity functions are injective and surjective, then $f^{-1} \circ f$ is injective and $f \circ f^{-1}$ is surjective. Therefore, parts c) and d) of Theorem 0.4 implies that f is both injective and surjective. Hence, f is a bijection.

Now assume that $f : X \to Y$ is a bijection. Then according to Theorem 0.3, for every $y \in Y$, the set $f^{-1}(\{ y \})$ contains exactly one element. For each $y \in Y$, let $g(y)$ denote this unique element of $f^{-1}(\{y\})$. For each $y \in Y$, since $f^{-1}(\{ y \}) \subset X$, then $g(y) \in X$. Hence, we have defined a function $g : Y \to X$ with the property that for each $y \in Y$, $f^{-1}(\{ y \}) = \{ g(y) \}$.

Let $y \in Y$. Since $f^{-1}(\{ y \}) = \{ g(y) \}$, then $g(y) \in f^{-1}(\{ y \})$. Hence, $f(g(y)) \in \{ y \}$. Therefore, $f(g(y)) = y$. Thus, $f \circ g(y) = id_Y(y)$ for every $y \in Y$. It follows that $f \circ g = id_Y$.

Let $x \in X$. Clearly $f(x) \in \{ f(x) \}$. Hence, $x \in f^{-1}(\{ f(x) \})$. On the other hand, the definition of the function g implies that $f^{-1}(\{ f(x) \}) = \{ g(f(x)) \}$. Thus, $x \in \{ g(f(x)) \}$. Hence, $g(f(x)) = x$. Thus, $g \circ f(x) = id_X(x)$ for every $x \in X$. It follows that $g \circ f = id_X$.

Since $g \circ f = id_X$ and $f \circ g = id_Y$, then $g : Y \to X$ is an inverse of $f : X \to Y$. We conclude that $f : X \to Y$ has an inverse. □

Homework Problem 0.5. Suppose $f : X \to Y$ is a function.
a) Prove that if $A \subset B \subset X$, then $f(A) \subset f(B)$.
b) Prove that if $C \subset D \subset Y$, then $f^{-1}(C) \subset f^{-1}(D)$.

Homework Problem 0.6. Suppose $f : X \to Y$ is a function, and suppose A and B are subsets of X, and C and D are subsets of Y. In each part of this problem, decide whether either of the two given sets must be a subset of or equal to the other. If a subset or equality relation must hold, prove it. If no such relation must hold, exhibit an example of a function $f : X \to Y$ and sets A and $B \subset X$ or C and $D \subset Y$ that illustrates this failure.
a) $f(A \cup B)$ and $f(A) \cup f(B)$.
b) $f(A \cap B)$ and $f(A) \cap f(B)$.
c) $f(A - B)$ and $f(A) - f(B)$.
d) A and $f^{-1}(f(A))$.
e) $f^{-1}(C \cup D)$ and $f^{-1}(C) \cup f^{-1}(D)$.
f) $f^{-1}(C \cap D)$ and $f^{-1}(C) \cap f^{-1}(D)$.
g) $f^{-1}(C - D)$ and $f^{-1}(C) - f^{-1}(D)$.
h) C and $f(f^{-1}(C))$.

Homework Problem 0.7. As in Exercise 0.6, suppose $f : X \to Y$ is a function, and suppose A and B are subsets of X, and C and D are subsets of Y. Consider one of the parts a) through h) of Exercise 0.6 in which equality fails to hold between the two given sets. Does the addition of either of the hypotheses "$f : X \to Y$ is injective" or "$f : X \to Y$ is surjective" change this situation and allow equality between the two given sets to be proved? If so, state and prove this result. In other words, in any part of Exercise 0.6 where equality between the two given sets doesn't hold but addition of one of the hypotheses "f is injective" or "f is surjective" makes it possible to prove equality between the given sets, then state and prove this result.

Definition. For $n \geq 1$, an *ordered n-tuple* $(x_1, x_2, \dots, x_n)$ is an object with the following property:

$$(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_n) \text{ if and only if } x_1 = y_1, x_2 = y_2, \dots \text{ and } x_n = y_n.$$

If $X_1, X_2, \dots, X_n$ are sets, then the *Cartesian product of* $X_1, X_2, \dots, X_n$ is the set

$$X_1 \times X_2 \times \dots \times X_n = \{ (x_1, x_2, \dots, x_n) : x_1 \in X_1, x_2 \in X_2, \dots \text{ and } x_n \in X_n \}.$$

When n = 2 in the preceding definition, we have the special case of the *ordered pair* (x,y). Thus, $(x,y) = (x',y')$ if and only if $x = x'$ and $y = y'$. Also, if X and Y are sets, then the Cartesian product of X and Y is the set $X \times Y = \{ (x,y) : x \in X \text{ and } y \in Y \}$.

Observe that the statements “$z \in X \times Y$” and “there is an $x \in X$ and a $y \in Y$ such that $(x,y) = z$” are equivalent. This equivalence is useful for proving results about the set $X \times Y$.