COLLECTIONS OF PATHS AND RAYS IN THE PLANE WHICH FIX ITS TOPOLOGY

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A collection $\mathcal{F}$ of proper maps into a locally compact Hausdorff space X *fixes the topology* of X if the only locally compact Hausdorff topology on X which makes each element of $\mathcal{F}$ continuous and proper is the given topology. In $I^2 = [-1, 1] \times [-1, 1]$, neither the collection of analytic paths nor the collection of regular twice differentiable paths fixes the topology. However, in I^2, both the collection of C^∞ arcs and the collection of regular C^1 arcs fix the topology. In $\mathbb{R}^2$, the collection of polynomial rays together with any collection of paths does not fix the topology. However, in $\mathbb{R}^2$, the collection of regular injective entire rays together with either the collection of C^∞ arcs or the collection of regular C^1 arcs fixes the topology.

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fixes the topology, passes through a sequence, analytic path, regular arc, C^∞ arc, polynomial ray, entire ray

1. Introduction

The results stated in the Abstract extend work of Rubin [1]. In responding to questions raised by Diestel, Sorenson and Stone, Rubin showed that in I^2 the collection of horizontal and vertical straight line segments does not fix the topology, and in $\mathbb{R}^2$ the collection of straight lines does not fix the topology. In his proofs, Rubin endowed I^2 and $\mathbb{R}^2$ with nonstandard topologies which make them locally compact, connected but not locally connected metric spaces. We shall see that these topologies can be 'improved'. In achieving our generalizations of Rubin's results, we equip I^2 and $\mathbb{R}^2$ with 'nicer' nonstandard topologies which make them homeomorphic to specific 2-dimensional polyhedra. I^2 becomes homeomorphic to the space obtained from I^2 by identifying the points $(t, 0)$ and $(-t, 0)$ for $0 \leq t \leq \frac{1}{2}$; and $\mathbb{R}^2$ becomes homeomorphic to $Q \times \mathbb{R}$ where Q is the space obtained from $(-\infty, \infty]$ by identifying the points 0 and ∞.

The terms used in the abstract are defined in Section 2. The results stated in the Abstract are contained in Corollaries 4, 7, 13, 14, 18, 22, and 23. The proofs of these results yield the following additional information. A map $f: J \to X$ *passes through* a sequence $\{x_n\}$ in X if $x_n \in f(J)$ for each $n \geq 1$. No analytic path in $\mathbb{R}^2$ passes through a subsequence of $\{(1/n, 1/e^n): n \geq 1\}$; and no regular twice differentiable path in $\mathbb{R}^2$ passes through a subsequence of $\{(1/n^3, 1/n^4): n \geq 1\}$. If $\{z_n\}$ is a bounded sequence in $\mathbb{R}^2$, then a C^∞ arc and a regular C^1 arc pass through a subsequence of $\{z_n\}$. No polynomial ray passes through a subsequence of $\{(n, 1/(n+1)): n \geq 1\}$. If $\{z_n\}$ is an unbounded sequence in $\mathbb{R}^2$, then a regular injective entire ray passes through a subsequence of $\{z_n\}$. The results just stated are found in Corollaries 5, 8 and 19 and Theorems 11, 12 and 21.

2. Definitions

The following definitions bring precise meaning to the terms used in the Abstract, such as 'analytic path' and 'regular injective entire ray'.

The term 'map' is reserved for continuous functions. A map is *proper* if under the map the inverse image of each compactum is compact. A *path* is a map whose domain is $[0, 1]$. An *arc* is an injective path. A *ray* is a proper map whose domain is $[0, \infty)$.

Let $J \subset \mathbb{R}$ and let $f: J \to \mathbb{R}$ be a map. If f extends to a map $F: L \to \mathbb{R}$ where L is an open subset of $\mathbb{R}$ containing J, and if F is n-times differentiable/C^n/C^∞, then we say that f is *n-times differentiable/C^n/C^∞*. Also if the extension F of f has an nth derivative, denoted $F^{(n)}$, then the *nth derivative* of f, denoted $f^{(n)}$, is defined by $f^{(n)} = F^{(n)}|J$.

Let $J \subset \mathbb{R}$ and let $f, g: J \to \mathbb{R}$ be maps. Define the map $h: J \to \mathbb{R}^2$ by $h(t) = (f(t), g(t))$. h is *n-times differentiable/C^n/C^∞* if both its components f and g are n-times differentiable/C^n/C^∞. If f and g are n-times differentiable, the *nth derivative* of h, denoted $h^{(n)}$, is defined by $h^{(n)}(t) = (f^{(n)}(t), g^{(n)}(t))$. A differentiable map $h: J \to \mathbb{R}^2$ is *regular* if $h'(t) \neq (0, 0)$ for every $t \in J$.

Throughout this paper $\mathbb{R}^2$ has two identities: the Cartesian product of two real lines, and the field of complex numbers. Hence we identify $\mathbb{R}$ with the subset $\mathbb{R} \times \{0\}$ of $\mathbb{R}^2$. We reserve the symbol 'i' for one of the complex square roots of -1. For $z = (x, y) \in \mathbb{R}^2$, let $|z| = (x^2 + y^2)^{1/2}$.

If U is an open subset of $\mathbb{R}^2$, a map $f: U \to \mathbb{R}^2$ is *analytic* if for every $z_0 \in U$ there is an $\varepsilon > 0$ and a sequence $\{a_n\}$ of complex numbers such that $\sum_{n=0}^{\infty} |a_n| \varepsilon^n < \infty$ and $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ whenever $z \in U$ and $|z - z_0| < \varepsilon$. In this case, $a_n = f^{(n)}(z_0)/n!$ for each $n \geq 0$. An analytic map whose domain is all of $\mathbb{R}^2$ is called an *entire* map. A *polynomial* is an entire map of the form $\sum_{k=0}^{n} a_k z^k$ where $a_0, a_1, \ldots, a_n$ are complex numbers.

Let $J \subset \mathbb{R}$ and let $f: J \to \mathbb{R}$ or $\mathbb{R}^2$ be a map. Keeping in mind that $\mathbb{R}$ is identified with $\mathbb{R} \times \{0\}$, we define f to be *analytic* if f extends to an analytic map $F: U \to \mathbb{R}^2$ where U is an open subset of $\mathbb{R}^2$ containing J. f is *entire*/a *polynomial* if f is the restriction to J of an entire map/a polynomial.

3. Collections of paths in I^2 which fail to fix its topology

The next proposition reveals the technique by which we endow I^2 with a nonstandard topology that preserves the continuity of every element in an appropriate collection of maps into I^2.

Proposition 1. *Let $\alpha, \beta : [0, 1] \to \mathbb{R}$ be maps such that $\alpha(0) = \beta(0) = 0$ and $\alpha(x) < \beta(x)$ for each $x > 0$. Let*

$$W = \{(x, y) \in I^2 : x > 0 \text{ and } \alpha(x) < y < \beta(x)\}.$$

Then there is a space Y and a bijective function $\Phi : I^2 \to Y$ with the following properties. Y is homeomorphic to the space obtained from I^2 by identifying the points $(t, 0)$ and $(-t, 0)$ for $0 < t \leq \frac{1}{2}$. For every open neighborhood N of $(0, 0)$ in I^2, $\Phi | I^2 - (N \cap W)$ is continuous.

Proof. Define the map $\phi : (0, 1] \times [-1, 1] \to [0, \frac{1}{2}]$ by

$$\phi(x, y) = \begin{cases} 0 & \text{if } -1 \leq y \leq \alpha(x) \text{ or } \beta(x) \leq y \leq 1, \\ \dfrac{y - \alpha(x)}{\beta(x) - \alpha(x)} & \text{if } \alpha(x) \leq y \leq \frac{1}{2}(\alpha(x) + \beta(x)), \\ \dfrac{\beta(x) - y}{\beta(x) - \alpha(x)} & \text{if } \frac{1}{2}(\alpha(x) + \beta(x)) \leq \beta(x). \end{cases}$$

Define the function $\Phi : I^2 \to [0, 1] \times [-1, 1] \times [0, 1]$ by

$$\Phi(x, y) = \begin{cases} (0, y, -x) & \text{if } -1 \leq x \leq 0, \\ (x, y, \phi(x, y)) & \text{if } 0 < x \leq 1. \end{cases}$$

Let $Y = \Phi(I^2)$. See Fig. 1. We leave the verification of the properties of Y and Φ to the reader. □

Observe that I^2 is not homeomorphic to the space Z obtained from I^2 by identifying the points $(t, 0)$ and $(-t, 0)$ for $0 < t \leq \frac{1}{2}$, for the Jordan Curve Theorem implies that the only simple closed curve in I^2 which fails to separate I^2 is its boundary. However Z clearly contains many distinct non-separating simple closed curves.

Corollary 2. *Let α, β and W be as in Proposition 1. Suppose $\mathcal{F}$ is a collection of proper maps into I^2 with the property that for every element $f : J \to I^2$ of $\mathcal{F}$, there is an open neighborhood N of $(0, 0)$ in I^2 such that $f(J) \cap (N \cap W) = \emptyset$. Then $\mathcal{F}$ does not fix the topology of I^2.*

Proof. Let $f : J \to I^2$ be an element of $\mathcal{F}$. Then $\Phi \circ f : J \to Y$ is continuous. Since f is proper and I^2 is compact, J must be compact. So $\Phi \circ f$ is proper as well.

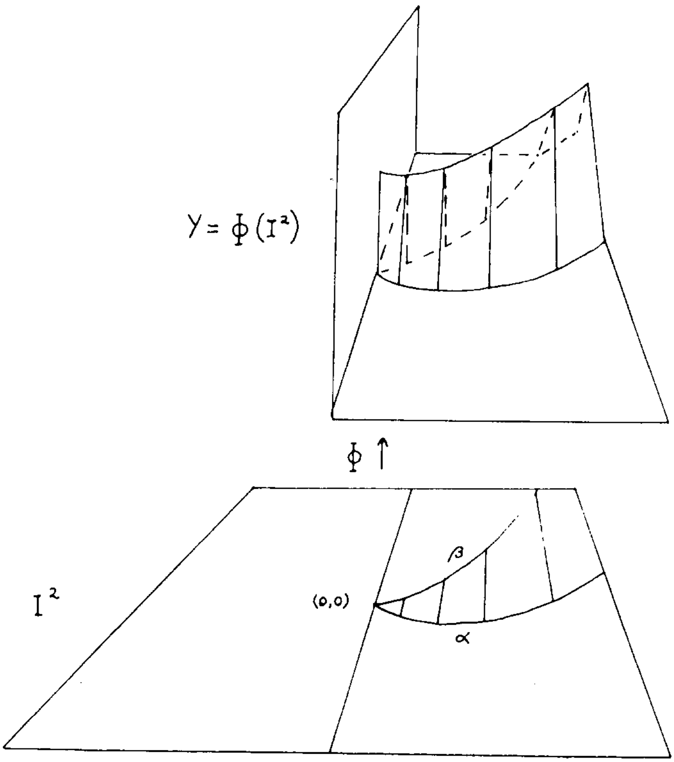

Fig. 1.

We equip I^2 with a nonstandard topology by using Φ to pull back the topology on Y. It follows that the nonstandard topology on I^2 makes each element of $\mathscr{F}$ continuous and proper. □

We now show that for appropriate choices of α and β, the collection of analytic paths in I^2 and the collection of regular twice differentiable paths in I^2 satisfy the hypothesis of Corollary 2.

Theorem 3. *Let* $W = \{(x, y) \in \mathbb{R}^2 : x > 0$ *and* $\mathrm{e}^{-1/x} \leq y \leq 2\,\mathrm{e}^{-1/x}\}$. *If* $f : [0, 1] \to \mathbb{R}^2$ *is an analytic path in* $\mathbb{R}^2$, *then there is an open neighborhood* N *of* $(0, 0)$ *in* $\mathbb{R}^2$ *such that* $f[0, 1] \cap (N \cap W) = \emptyset$.

Proof. It is well known that a C^∞ map $\zeta : \mathbb{R} \to [0, \infty)$ is defined by

$$\zeta(t) = \begin{cases} \mathrm{e}^{-1/t} & \text{for } t > 0, \\ 0 & \text{for } t \leq 0. \end{cases}$$

Hence $\zeta^{(n)}(0) = 0$ for each $n \geq 0$.

Assume there is an analytic path $f : [0, 1] \to \mathbb{R}^2$ for which Theorem 3 fails. Then $f(t) = (g(t), h(t))$ where $g, h : [0, 1] \to \mathbb{R}$ are analytic. (The power series coefficients for g and h are the real and imaginary parts of the power series coefficients of f.) Since f violates Theorem 3, there is a sequence $\{t_k\}$ in $[0, 1]$ such that $0 < g(t_k) < 1/k$ and $\zeta(g(t_k)) \leqslant h(t_k) \leqslant 2\zeta(g(t_k))$ for each $k \geqslant 1$. We can assume $\{t_k\}$ converges to $t_0 \in [0, 1]$. Consequently $g(t_0) = h(t_0) = 0$.

We shall now argue inductively that $h^{(n)}(t_0) = 0$ for each $n \geqslant 1$. Since h is analytic, it will then follow that $h(t) = 0$ for all $t \in [0, 1]$, contradicting the fact that $0 < \zeta(g(t_k)) \leqslant h(t_k)$ for all $k \geqslant 1$. Let $n \geqslant 1$ and assume $h^{(m)}(t_0) = 0$ for $0 \leqslant m \leqslant n-1$. Then n applications of L'Hospital's Rule yields

$$\lim_{t \to t_0} \frac{h(t)}{(t-t_0)^n} = \frac{h^{(n)}(t_0)}{n!}.$$

Since $\zeta^{(m)}(g(t_0)) = 0$ for all $m \geqslant 0$, then n applications of L'Hospital's Rule yields

$$\lim_{t \to t_0} \frac{\zeta(g(t))}{(t-t_0)^n} = (g'(t_0))^n \lim_{t \to t_0} \frac{\zeta^{(n)}(g(t))}{n!} = 0.$$

Now if we divide through the inequality $\zeta(g(t_k)) \leqslant h(t_k) \leqslant 2\zeta(g(t_k))$ by $(t_k - t_0)^n$ and take the limit as $k \to \infty$, we obtain $0 \leqslant h^{(n)}(t_0)/n! \leqslant 0$. Thus $h^{(n)}(t_0) = 0$. □

From Corollary 2 and Theorem 3, we conclude:

Corollary 4. *The collection of analytic paths in I^2 does not fix the topology.*

Corollary 5. *No analytic path in $\mathbb{R}^2$ passes through the sequence* $\{(1/n, 1/\mathrm{e}^n): n \geqslant 1\}$.

We now consider regular twice differentiable paths in I^2.

Theorem 6. *Let $W = \{(x, y) \in \mathbb{R}^2: x > 0$ and $x^{4/3} \leqslant y \leqslant 2x^{4/3}\}$. If $f: [0, 1] \to \mathbb{R}^2$ is a regular twice differentiable path in $\mathbb{R}^2$, then there is an open neighborhood N of $(0, 0)$ in $\mathbb{R}^2$ such that $f[0, 1] \cap (N \cap W) = \emptyset$.*

Proof. Assume there is a regular twice differentiable path $f: [0, 1] \to \mathbb{R}^2$ for which Theorem 6 fails. Let $f(t) = (g(t), h(t))$. Then there is a sequence $\{t_n\}$ in $[0, 1]$ such that $0 < g(t_n) < 1/n$ and $(g(t_n))^{4/3} \leqslant h(t_n) \leqslant 2(g(t_n))^{4/3}$ for each $n \geqslant 1$. We can assume $\{t_n\}$ converges to $t_0 \in [0, 1]$. Hence $g(t_0) = h(t_0) = 0$. Therefore

$$\lim_{t \to t_0} \frac{(g(t))^{4/3}}{t - t_0} = \lim_{t \to t_0} \left[\left(\frac{g(t)}{t - t_0} \right)^{4/3} (t - t_0)^{1/3} \right]$$
$$= (g'(t_0))^{4/3} \cdot 0 = 0$$

and $\lim_{t \to t_0} h(t)/(t - t_0) = h'(t_0)$. Now if we divide through the inequality $(g(t_n))^{4/3} \leqslant h(t_n) \leqslant 2(g(t_n))^{4/3}$ by $t_n - t_0$ and take the limit as $n \to \infty$, we conclude that $h'(t_0) = 0$. Since f is regular and $f'(t_0) = (g'(t_0), h'(t_0))$, then necessarily $g'(t_0) \neq 0$. Consequently $g'(t) \neq 0$ for t sufficiently close to t_0. (g' is continuous because g is twice differentiable.) Now an application of L'Hospital's Rule yields

$$\lim_{t\to t_0} \frac{h(t)}{(g(t))^{4/3}} = \left(\frac{3}{4g'(t_0)}\right) \lim_{t\to t_0} \frac{h'(t)}{(g(t))^{1/3}}$$
$$= \left(\frac{3}{4g'(t_0)}\right) \lim_{t\to t_0} \left[\left(\frac{h'(t)}{t-t_0}\right)\left(\frac{t-t_0}{g(t)}\right)^{1/3} (t-t_0)^{2/3}\right]$$
$$= \frac{3}{4g'(t_0)} \cdot h''(t_0) \cdot \left(\frac{1}{g'(t_0)}\right)^{1/3} \cdot 0 = 0.$$

We have reached a contradiction, because the inequality $0 < (g(t_n))^{4/3} \leq h(t_n)$ implies $h(t_n)/(g(t_n))^{4/3} \geq 1$ for all $n \geq 1$. □

From Corollary 2 and Theorem 6, we conclude:

Corollary 7. *The collection of regular twice differentiable paths in I^2 does not fix the topology.*

Corollary 8. *No regular twice differentiable path in $\mathbb{R}^2$ passes through a subsequence of $\{(1/n^3, 1/n^4): n \geq 1\}$.*

4. Collections of paths in I^2 which fix its topology

We begin this section with a criterion which guarantees that a collection of proper maps into a space fixes the topology.

Proposition 9. *Let $\mathcal{F}$ be a collection of proper maps from metric spaces into a locally compact metric space X. If for each sequence $\{x_n\}$ in X, some element of $\mathcal{F}$ passes through a subsequence of $\{x_n\}$, then $\mathcal{F}$ fixes the topology.*

Proof. Let Y denote X equipped with a locally compact Hausdorff topology that makes each element of $\mathcal{F}$ continuous and proper. Let $e: X \to Y$ denote the identity function. Then for each element $f: J \to X$ of $\mathcal{F}$, $e \circ f: J \to Y$ is continuous and proper. We must prove that e is a homeomorphism.

We first argue that e is continuous. For suppose not. Then there is a sequence $\{w_n\}$ in X which converges to a point x in X such that no subsequence of $e(w_n)$ converges to $e(x)$. By passing to a subsequence, we can assume there is an element $f: J \to X$ of $\mathcal{F}$ which passes through $\{w_n\}$. Hence, for each $n \geq 1$ we can select a point s_n in $f^{-1}(w_n)$. Since the set $\{w_n: n \geq 1\} \cup \{x\}$ is compact and f is proper, then $f^{-1}(\{w_n: n \geq 1\} \cup \{x\})$ is compact. It follows that by passing to a subsequence, we can assume that $\{s_n\}$ converges to a point t of J. Then necessarily $f(t) = x$. Since $e \circ f: J \to Y$ is continuous, then $\{e \circ f(s_n)\}$ converges to $e \circ f(t)$. Thus, $\{e(w_n)\}$ converges to $e(x)$. We have reached a contradiction.

Next we argue that e is proper. For suppose not. Then there is a compactum C in Y such that $e^{-1}(C)$ is not compact. Consequently, $e^{-1}(C)$ must contain a sequence $\{x_n\}$ that has no converging subsequence. By passing to a subsequence, we can assume there is an element $f: J \to X$ of $\mathscr{F}$ which passes through $\{x_n\}$. Hence, for each $n \geqslant 1$ we can select a point t_n in J such that $f(t_n) = x_n$. Therefore, no subsequence of $\{t_n\}$ can converge. Since $e \circ f$ is proper, then $(e \circ f)^{-1}(C)$ is a compactum. As $\{t_n\}$ lies in $(e \circ f)^{-1}(C)$, we see that some subsequence of $\{t_n\}$ must converge. We have reached a contradiction.

Finally we argue that e^{-1} is continuous. Let C be a closed subset of X. We shall prove that $e(C)$ is a closed subset of Y. Let $y \in Y - e(C)$. There is an open neighborhood N of y whose closure, $\operatorname{cl} N$, is compact. The propriety of e implies that $e^{-1}(\operatorname{cl} N)$ is compact. Thus, $C \cap e^{-1}(\operatorname{cl} N)$ is compact. Therefore, $e(C \cap e^{-1}(\operatorname{cl} N)) = e(C) \cap \operatorname{cl} N$ is a compact and, hence, closed subset of Y. It follows that $N - (e(C) \cap \operatorname{cl} N) = N - e(C)$ is an open neighborhood of y which misses $e(C)$. $\square$

We shall now demonstrate that each bounded sequence in the plane has a subsequence through which a C^∞ arc passes and a subsequence through which a regular C^1 arc passes. We deduce both of these results from the following lemma.

Lemma 10. *Given strictly decreasing sequences $\{t_n\}$ and $\{x_n\}$ converging to 0 such that $t_{n+1} \leqslant \frac{1}{2} t_n$, there is a non-decreasing onto map $\alpha: \mathbb{R} \to \mathbb{R}$ with the following properties.*

(1) *$\alpha(t_n) = x_n$ for each $n \geqslant 1$.*

(2) *$\alpha \mid \mathbb{R} - \{0\}$ is C^∞.*

(3) *$\{t \in \mathbb{R}: \alpha'(t) = 0\} \subset \{0\} \cup (\bigcup_{n=1}^\infty [t_{n+1}, \frac{1}{2} t_n]) \cup \{t_1\}$.*

(4) *α is a C^k map (at 0) for every integer $k \geqslant 1$ for which $\{x_n/(t_n)^k\}$ converges to 0.*

Proof. Using the map ζ defined in the proof of Theorem 3, we specify a C^∞ map $\eta: \mathbb{R} \to [0, 1]$ by

$$\eta(t) = \int_{-\infty}^{t} \zeta(t)\zeta(1-t)\,\mathrm{d}t \Big/ \int_{-\infty}^{\infty} \zeta(t)\zeta(1-t)\,\mathrm{d}t.$$

Then $\eta(-\infty, 0] = 0$, $\eta[1, \infty) = 1$ and $\eta'(t) > 0$ for $0 < t < 1$.

For each $n \geqslant 1$, define the C^∞ map $\alpha_n: \mathbb{R} \to [x_{n+1}, x_n]$ by

$$\alpha_n(t) = x_{n+1} + (x_n - x_{n+1}) \cdot \eta((2t/t_n) - 1).$$

Then $\alpha_n(-\infty, \frac{1}{2} t_n] = x_{n+1}$, $\alpha_n[t_n, \infty) = x_n$, α'_n is positive on $(\frac{1}{2} t_n, t_n)$, and for $k \geqslant 1$

$$\alpha_n^{(k)}(t) = \left(\frac{x_n - x_{n+1}}{(t_n)^k}\right) 2^k \eta^{(k)}((2t/t_n) - 1).$$

Define the C^∞ maps $\alpha_0:\mathbb{R}\to[x_1,\infty)$ and $\alpha_-:\mathbb{R}\to(-\infty,0]$ by $\alpha_0(t)=x_1+\zeta(t-t_1)$ and $\alpha_-(t)=-\zeta(-t)$. Then $\alpha_0(-\infty,t_1]=x_1$, $\alpha_0'(t)>0$ for $t>x_1$ and $\alpha_0[t_1,\infty)=[x_1,\infty)$; and $\alpha_-[0,\infty)=0$, $\alpha_-'(t)>0$ for $t<0$ and $\alpha_-(-\infty,0]=(-\infty,0]$.

The desired map $\alpha:\mathbb{R}\to\mathbb{R}$ is defined by

$$\alpha(t)=\begin{cases}\alpha_0(t) & \text{for } t\geq t_1,\\ \alpha_n(t) & \text{for } \frac{1}{2}t_n\leq t\leq t_n,\\ x_{n+1} & \text{for } t_{n+1}\leq t\leq\frac{1}{2}t_n,\\ \alpha_-(t) & \text{for } t\leq 0.\end{cases}$$

The only property of α which requires proof is (4). We proceed by induction on k. Assume (4) is valid for the positive integer k, and suppose $\{x_n/(t_n)^{k+1}\}$ converges to 0. Then $\{x_n/(t_n)^k\}$ also converges to 0, and the inductive hypothesis implies $\alpha^{(k)}(0)$ exists. Hence $\alpha^{(k)}(0)=\alpha_-^{(k)}(0)=0$. Let $M_j=\max\{\eta^{(j)}(t): 0\leq t\leq 1\}$ for each $j\geq 0$. To prove that $\alpha^{(k+1)}(0)$ exists and equals zero it suffices to establish that $\lim_{t\downarrow 0}\alpha^{(k)}(t)/t=0$. The latter follows from the observation that if $\frac{1}{2}t_n\leq t\leq t_n$, then

$$\left|\frac{\alpha^{(k)}(t)}{t}\right|\leq\left(\frac{x_n-x_{n+1}}{(t_n)^k t}\right)2^k M_k\leq\left(\frac{x_n}{(t_n)^{k+1}}\right)2^{k+1}M_k.$$

To prove the continuity of $\alpha^{(k+1)}(t)$ at $t=0$, it suffices to establish that $\lim_{t\downarrow 0}\alpha^{(k+1)}(t)=0$. The latter follows from the observation that if $\frac{1}{2}t_n\leq t\leq t_n$, then

$$|\alpha^{(k+1)}(t)|\leq\left(\frac{x_n-x_{n+1}}{(t_n)^{k+1}}\right)2^{k+1}M_{k+1}\leq\left(\frac{x_n}{(t_n)^{k+1}}\right)2^{k+1}M_{k+1}. \quad \square$$

Now we show how to use Lemma 10 to pass a C^∞ arc through a subsequence of a given bounded sequence in the plane.

Theorem 11. *If $\{z_n\}$ is a bounded sequence in $\mathbb{R}^2$ (I^2), then there is a C^∞ arc in $\mathbb{R}^2$ (I^2) which passes through a subsequence of $\{z_n\}$.*

Proof. Since $\{z_n\}=\{(x_n,y_n)\}$ is bounded, then by passing to subsequences, we can assume that $\{z_n\}$ converges to $z_0=(x_0,y_0)$ and that $\{x_n\}$ and $\{y_n\}$ are strictly monotone. (If a subsequence of either $\{x_n\}$ or $\{y_n\}$ is constant, then a subsequence of $\{z_n\}$ lies on a straight line, and we are done. So we can assume this doesn't happen.) Furthermore, we can assume that $|x_n-x_0|<1/2^{(n^2)}$ and $|y_n-y_0|<1/2^{(n^2)}$. These inequalities allow us to use Lemma 10 to obtain non-decreasing onto C^∞ maps $\alpha:\mathbb{R}\to\mathbb{R}$ and $\beta:\mathbb{R}\to\mathbb{R}$ such that $\alpha(1/2^{n-1})=|x_n-x_0|$, $\beta(1/2^{n-1})=|y_n-y_0|$ and such that α' and β' are zero only at points of the countable set $\{0,1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\cdots\}$. It follows that α and β are actually homeomorphisms of $\mathbb{R}$. Let $\sigma=(x_1-x_0)/|x_1-x_0|$ and $\tau=(y_1-y_0)/|y_1-y_0|$. Define the isometry T of $\mathbb{R}^2$ by $T(x,y)=(\sigma x+x_0,\tau y+y_0)$. Define the map $f:\mathbb{R}\to\mathbb{R}^2$ by $f(t)=T(\alpha(t),\beta(t))$. Then $f(1/2^{n-1})=z_n$ for each $n\geq 1$. So $f([0,1])$ is a C^∞ arc which passes through $\{z_n\}$.

Now suppose $\{z_n\}$ lies in I^2. $T^{-1}(I^2)$ is a square in the plane which contains $(\alpha(0), \beta(0))$ and $(\alpha(1), \beta(1))$, and whose sides are parallel to the x and y axes. Since α and β are increasing functions, it follows that $(\alpha(t), \beta(t)) \in T^{-1}(I^2)$ for each $t \in [0, 1]$. Thus $f([0, 1]) \subset I^2$. □

We use Lemma 10 a second time to pass a regular C^1 arc through a subsequence of a given bounded sequence in the plane.

Theorem 12. *If $\{z_n\}$ is a bounded sequence in $\mathbb{R}^2$ (I^2), then there is a regular C^1 arc in $\mathbb{R}^2$ (I^2) which passes through a subsequence of $\{z_n\}$.*

Proof. Since $\{z_n\}$ is bounded and the unit circle is compact, we can assume $\{z_n\}$ converges to a point z_0 and $\{(z_n - z_0)/|z_n - z_0|\}$ converges to a point u with $|u| = 1$. For each $n \geqslant 1$, let $w_n = (x_n, y_n) = (z_n - z_0)/u$. (Here we are performing division in the field of complex numbers.) Then $\{w_n\}$ converges to $(0, 0)$ and $\{w_n/|w_n|\}$ converges to $(1, 0)$. So $\{x_n/|w_n|\}$ converges to 1 and $\{y_n/|w_n|\}$ converges to 0. Therefore, $\{x_n\}$, $\{y_n\}$ and $\{y_n/x_n\}$ all converge to 0. Furthermore, by passing to subsequences, we can assume $0 < x_{n+1} \leqslant \frac{1}{2}x_n$ and $\{y_n\}$ converges strictly monotonically to 0. (If all but finitely many y_n's are 0, then a subsequence of $\{z_n\}$ lies on a straight line, and we are done. So we can assume this doesn't happen.)

Lemma 10 provides a non-decreasing C^1 map $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha(x_n) = |y_n|$ for each $n \geqslant 1$. Hence $\alpha(0) = \alpha'(0) = 0$. Let $\sigma = y_1/|y_1|$. Define the isometry T of $\mathbb{R}^2$ by $T(x + \mathrm{i}y) = (u \cdot (x + \mathrm{i}\sigma y)) + z_0$. (Here we are performing multiplication in the field of complex numbers, and 'i' denotes a square root of -1.) Define the map $f : \mathbb{R} \to \mathbb{R}^2$ by $f(t) = T(t, \alpha(t))$. Then $f(x_n) = z_n$ for each $n \geqslant 1$. So $f|[0, 1]$ passes through a subsequence of $\{z_n\}$. f is C^1 because $f'(t) = u \cdot (1 + \mathrm{i}\sigma\alpha'(t))$. f is regular because $|f'(t)| = |1 + \mathrm{i}\sigma\alpha'(t)| \geqslant 1$. f is injective because both T and the map $t \to (t, \alpha(t))$ are injective.

Now suppose $\{z_n\}$ lies in I^2. $T^{-1}(I^2)$ is a square which contains the straight-line segment between $(0, 0) = T^{-1}(z_0)$ and $(x_n, |y_n|) = T^{-1}(z_n)$ for each $n \geqslant 1$. Since $\{|y_n|/x_n\}$ converges to 0, it follows that $T^{-1}(I^2)$ contains the segment $[0, \varepsilon] \times \{0\}$ for some $\varepsilon > 0$. Choose $n \geqslant 1$ so large that $x_n \leqslant \varepsilon$. Then $T^{-1}(I^2)$ contains the triangle Δ with vertices $(0, 0)$, $(x_n, 0)$ and $(x_n, |y_n|)$. There is a $\delta > 0$ such that $\delta \leqslant x_n$ and $0 \leqslant \alpha(t)/t \leqslant |y_n|/x_n$ for $0 < t \leqslant \delta$. Hence $(t, \alpha(t)) \in \Delta$ for $0 \leqslant t \leqslant \delta$. It follows that $f(t) \in I^2$ for $0 \leqslant t \leqslant \delta$. We define $g : \mathbb{R} \to \mathbb{R}^2$ by $g(t) = f(\delta t)$. Then $g|[0, 1]$ is a regular C^1 arc in I^2 which passes through a subsequence of $\{z_n\}$. □

Combining Proposition 9 with Theorems 11 and 12, we have:

Corollary 13. *The collection of C^∞ arcs in I^2 fixes the topology.*

Corollary 14. *The collection of regular C^1 arcs in I^2 fixes the topology.*

5. Collections of paths and rays in the plane

We now show how to vary the topology of $\mathbb{R}^2$ while preserving the continuity and propriety of an appropriate collection of proper maps into $\mathbb{R}^2$.

Proposition 15. *Let $\alpha, \beta : [0, \infty) \to \mathbb{R}$ be maps such that $\alpha(x) < \beta(x)$ for each $x \geq 0$. For each $t \geq 0$, let*

$$W_t = \{(x, y) \in \mathbb{R}^2 : x > t \text{ and } \alpha(x) < y < \beta(x)\}.$$

Let Q denote the space obtained from $(-\infty, \infty]$ by identifying the points 0 and ∞. Then there is a bijective map $\Phi : \mathbb{R}^2 \to Q \times \mathbb{R}$ such that $\Phi | \mathbb{R}^2 - W_t$ is proper for each $t \geq 0$.

Proof. $\Phi = G \circ H$ where $G : \mathbb{R}^2 \to Q \times \mathbb{R}$ is a bijective map such that $G | (-\infty, t] \times \mathbb{R}$ is proper for each $t \geq 0$, and H is a homeomorphism of $\mathbb{R}^2$ such that $H(\mathbb{R}^2 - W_t) \subset (-\infty, t] \times \mathbb{R}$ for each $t \geq 0$. The construction of G is obvious. $H = h_2 \circ h_1$ where h_1 and h_2 are homeomorphisms of $\mathbb{R}^2$. h_1 only moves points vertically, h_1 maps the graph of α onto $A = \{(x, -x + \alpha(0)) : x \geq 0\}$, and h_1 maps the graph of β onto $B = \{(x, x + \beta(0)) : x \geq 0\}$. h_2 only moves points horizontally and to the left, $h_2(A) = \{0\} \times (-\infty, \alpha(0)]$, and $h_2(B) = \{0\} \times [\beta(0), \alpha)$. □

Observe that $\mathbb{R}^2$ is not homeomorphic to $Q \times \mathbb{R}$. The Jordan Curve Theorem tells us that every simple closed curve in $\mathbb{R}^2$ separates $\mathbb{R}^2$. However, $Q \times \mathbb{R}$ clearly contains many non-separating simple closed curves.

Corollary 16. *Let α, β, W and W_t, $t \geq 0$, be as in Proposition 15. Suppose $\mathcal{F}$ is a collection of proper maps into $\mathbb{R}^2$ with the property that for every element $f : J \to \mathbb{R}^2$ of $\mathcal{F}$, there is a $t \geq 0$ such that $f(J) \cap W_t = \emptyset$. Then $\mathcal{F}$ does not fix the topology of $\mathbb{R}^2$.*

Proof. Clearly if $f : J \to \mathbb{R}^2$ is any element of $\mathcal{F}$, then $\Phi \circ f : J \to Q \times \mathbb{R}^2$ is a proper map. So if $\mathbb{R}^2$ is equipped with a nonstandard topology by using Φ to pull back the topology on $Q \times \mathbb{R}$, then this nonstandard topology makes each element of $\mathcal{F}$ continuous and proper. □

We now show that for an appropriate choice of α and β, the collection of polynomial rays together with the collection of all paths in $\mathbb{R}^2$ satisfies the hypothesis of Corollary 16.

Theorem 17. *For each $t \geq 0$, let*

$$W_t = \left\{(x, y) \in \mathbb{R}^2 : x \geq t \text{ and } \frac{1}{x+1} \leq t \leq \frac{2}{x+1}\right\}.$$

If $f : J \to \mathbb{R}^2$ is either a path or a polynomial ray, then there is a $t \geq 0$ such that $f(J) \cap W_t = \emptyset$.

Proof. We need only consider the case in which f is a polynomial ray. Assume there is a sequence $\{t_j\}$ in $[0,\infty)$ such that $f(t_j)\in W_j$ for each $j\geq 1$. $f(t)=\sum_{k=0}^{n}(a_k+ib_k)t^k$ where a_k and b_k are real numbers for $0\leq k\leq n$. ('i' denotes a square root of -1.) Then $f(t)=g(t)+ih(t)$ where $g(t)=\sum_{k=0}^{n}a_kt^k$ and $h(t)=\sum_{k=0}^{n}b_kt^k$. Since $(g(t_j),h(t_j))\in W_j$ for each $j\geq 1$, then $\{g(t_j)\}$ converges to ∞ and $\{h(t_j)\}$ is a sequence of positive numbers converging to 0. As $\{g(t_j)\}$ converges to ∞, so does $\{t_j\}$. Since h is a non-constant polynomial, it follows that $\{(h(t_j))\}$ converges to ∞. We have reached a contradiction. □

From Corollary 16 and Theorem 17, we conclude:

Corollary 18. *The collection of polynomial rays and all paths in* $\mathbb{R}^2$ *does not fix the topology.*

Corollary 19. *No polynomial ray in* $\mathbb{R}^2$ *passes through a subsequence of* $\{(n,1/(n+1)): n\geq 1\}$.

Next we show how to pass a regular injective entire ray through a subsequence of any unbounded sequence in the plane. Our proof is based on the following lemma.

Lemma 20. *If* $\{x_n\}$ *and* $\{y_n\}$ *are sequence of real numbers such that* $x_{n+1}\geq 2x_n>0$ *for each* $n\geq 1$ *and* $\sum_{n=1}^{\infty}|y_n|/x_n<\infty$, *then there is an entire map* $f:\mathbb{R}\to\mathbb{R}$ *such that* $f(x_n)=y_n$ *for each* $n\geq 1$.

Proof. For each $n\geq 1$, since $\sum_{k\neq n}|z/x_k|\leq\sum|z|/2^{k-1}x_1\leq 2|z|/x_1$ for $z\in\mathbb{R}^2$, then Theorem 15.6 of [3] implies that an entire map $f_n:\mathbb{R}^2\to\mathbb{R}^2$ is defined by $f_n(z)=\prod_{k\neq n}(1-z/x_k)$, and that $f_n(z)=0$ if and only if $z=x_k$ for some $k\neq n$. Let $a_n=y_n/f_n(x_n)$ for each $n\geq 1$. We shall argue that an entire map $f:\mathbb{R}^2\to\mathbb{R}^2$ is defined by $f(z)=\sum_{n=1}^{\infty}a_nf_n(z)$. This f has the desired properties: $f(\mathbb{R})\subset\mathbb{R}$ because $f_n(\mathbb{R})\subset\mathbb{R}$ for each $n\geq 1$, and $f(x_n)=y_n$ for each $n\geq 1$.

We now show that f is well-defined and entire. Theorem 15.5 of [3] tells us that $\delta=\prod_{k=1}^{\infty}(1-1/2^k)>0$. For $n>1$,

$$|f_n(x_n)|=\prod_{k=1}^{n-1}\left(\frac{x_n}{x_k}-1\right)\cdot\prod_{k=n+1}^{\infty}\left(1-\frac{x_n}{x_k}\right)\geq\left(\frac{x_n}{2x_1}\right)\delta$$

because $(x_n/x_1)-1\geq x_n/2x_1$ and $x_n/x_k\leq 1/2^{n-k}$ for $k>n$. Hence

$$|a_n|\leq\frac{|y_n|}{x_n}\cdot\frac{2x_1}{\delta}\quad\text{for } n\geq 1.$$

Lemma 15.3 of [3] implies that $|f_n(z)|\leq e^{(2|z|/|x_1|)}$ for each $z\in\mathbb{R}^2$. Hence

$$\sum_{n=1}^{\infty}|a_nf_n(z)|\leq\left(\sum_{n=1}^{\infty}\frac{|y_n|}{x_n}\right)\frac{2x_1}{\delta}\,e^{(2|z|/x_1)}.$$

Since $\sum_{n=1}^{\infty} |y_n|/x_n < \infty$, then Theorem 10.27 of [3] implies that f is well-defined and entire.

Theorem 21. *If* $\{z_n\}$ *is an unbounded sequence in* $\mathbb{R}^2$, *then there is a regular injective entire ray which passes through a subsequence of* $\{z_n\}$.

Proof. We can assume $\{|z_n|\}$ converges to ∞ and $\{z_n/|z_n|\}$ converges to u where $|u| = 1$. Let $(x_n, y_n) = z_n/u$ for $n \geqslant 1$. (Here we are performing division in the field of complex numbers.) Then $(x_n/|z_n|, y_n/|z_n|)$ converges to $(1, 0)$. Hence $\{x_n\}$ converges to ∞ and $\{y_n/x_n\}$ converges to 0. By passing to subsequences we can assume $x_{n+1} \geqslant 2x_n > 0$ for each $n \geqslant 1$ and $\sum_{n=1}^{\infty} |y_n|/x_n < \infty$. We now apply Lemma 2 to obtain an entire map $f: \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(x_n) = y_n$ for each $n \geqslant 1$ and $f(\mathbb{R}) \subset \mathbb{R}$.

We define the entire map $g: \mathbb{R}^2 \to \mathbb{R}^2$ by $g(z) = u \cdot (z + \mathrm{i}f(z))$. (Here we are performing multiplication in the field of complex numbers, and 'i' denotes a square root of -1.) Clearly $g(x_n) = z_n$ for each $n \geqslant 1$. Since $g'(z) = u \cdot (1 + \mathrm{i}f'(z))$, then $|g'(t)| = |1 + \mathrm{i}f'(t)| \geqslant 1$ for $t \in \mathbb{R}$; so $g|[0, \infty)$ is regular. $g|[0, \infty)$ is injective because both the map $t \to (t, f(t))$ and multiplication by u are injective. $\square$

Combining Theorems 11, 12 and 21 with Proposition 9, we have:

Corollary 22. *The collection of* C^∞ *arcs and regular injective entire rays in* $\mathbb{R}^2$ *fixes the topology.*

Corollary 23. *The collection of regular* C^1 *arcs and regular injective entire rays in* $\mathbb{R}^2$ *fixes the topology.*

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