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Cones that are cells, and an application to hyperspaces ☆

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Abstract

Let Y be a compact metric space that is not an $(n-1)$-sphere. If the cone over Y is an n-cell, then $Y \times [0,1]$ is an n-cell; if $n \leqslant 4$, then Y is an $(n-1)$-cell. Examples are given to show that the converse of the first part is false (for $n \geqslant 5$) and that the second part does not extend beyond $n = 4$. An application concerning when hyperspaces of simple n-ods are cones over unique compacta is given, which answers a question of Charatonik. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

There are many interesting results and examples concerning cartesian factors of Euclidean spaces and cartesian factors of n-cells (see [17, p. 84] for a brief summary about cartesian factors of n-cells, and see [8] for a more complete discussion). However, the question of when a cone is an n-cell does not seem to have been treated explicitly in the literature. We obtain results about this in Sections 3 and 4. These results are adequately summarized in the abstract above. Thus, we prefer at this time to discuss our initial motivation for obtaining the results.

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Our inquiry into when cones are n-cells was actually motivated by a recent question about hyperspaces of simple n-ods. We discuss the result that led to the question, and then we state the question (which we answer in Section 6).

First, let us note some relevant notation and terminology. A *continuum* is a nonempty, compact, connected metric space. For a continuum X, $C(X)$ denotes the hyperspace of all subcontinua of X with the Hausdorff metric [15]. A *simple n-od* ($n \geqslant 3$) is a continuum that is the union of n arcs emanating from a single point and otherwise disjoint.

Recently, Sergio Macías [12] has proved the following result which corrects an error in [15, p. 333]: Let X be a locally connected continuum; then $C(X)$ is homeomorphic to the cone over a finite-dimensional continuum, Z, if and only if X is an arc, a simple closed curve, or a simple n-od.

We are interested in the part of Macías' result when X is a simple n-od. Let T_n denote a simple n-od. Then the hyperspace $C(T_n)$ is the n-dimensional polyhedron in Fig. 1 (ideas for a proof are in 10.2 and 0.54 of [15]). Macías shows that $C(T_n)$ is a cone by showing that $C(T_n)$ is homeomorphic to the cone over the continuum that we denote by M_n in Fig. 2.

When Macías presented his results in a seminar at the National University of Mexico, Charatonik asked the following question: If Z is a continuum such that $C(T_n)$ is homeomorphic to the cone over Z, then must Z be the continuum in Macías' proof (i.e., M_n)?

We answer Charatonik's question by using results in Sections 3–5. Specifically, we show in Theorem 6.2 that Z must be M_n if and only if $n = 3$ or 4.

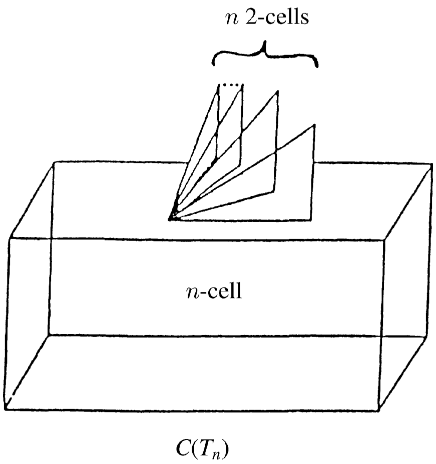

Fig. 1.

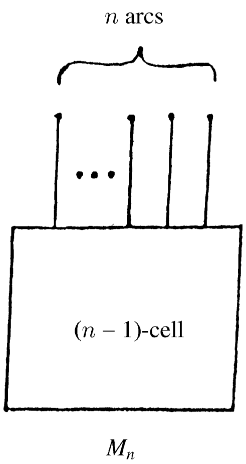

Fig. 2.

2. Notation, terminology, and preliminaries

We will use the *geometric coning operator,* Cone, which we define as follows. Let Y be a compactum (= a nonempty, compact metric space). Let I^∞ denote the Hilbert cube $\prod_{i=1}^{\infty} I_i$, where $I_i = [0,1]$ for each i. Consider Y as embedded in the Hilbert cube $I^\infty \times \{0\}$, and fix a point $v = (p,1) \in I^\infty \times [0,1]$. Then, for any $S \subset Y$, $\text{Cone}(S)$ is the union of all the straight line segments in $I^\infty \times [0,1]$ from points of S to v. We call $\text{Cone}(S)$ the *geometric cone over* S. We call v the *vertex of* $\text{Cone}(S)$ assuming that $S \neq \emptyset$ (note that $\text{Cone}(\emptyset) = \emptyset$). We call S the *base of* $\text{Cone}(S)$.

The letter v always denotes the vertex of a cone. Although it may be inferred from the construction above, we emphasize that *all cones over nonempty subsets of a given compactum Y have the same vertex*.

We state the following two easy-to-prove propositions for convenient reference later.

Proposition 2.1. *For any compactum Y, the geometric coning operator commutes with the closure operator*: $\text{Cone}(\overline{S}) = \overline{\text{Cone}(S)}$ *for all $S \subset Y$.*

Proposition 2.2. *For any compactum Y, the geometric coning operator distributes over nonempty intersections*: $\text{Cone}(\bigcap_{i \in \mathcal{I}} S_i) = \bigcap_{i \in \mathcal{I}} \text{Cone}(S_i)$ *whenever $\bigcap_{i \in \mathcal{I}} S_i \neq \emptyset$ and $S_i \subset Y$ for all $i \in \mathcal{I}$.*

We point out that when Y is a compactum, $\text{Cone}(Y)$ is homeomorphic to the usual topological cone over Y [14, pp. 47–48]. Therefore, since our results are about cones over compacta, our results are about topological cones (even though they are stated using the symbol for the geometric cone). On the other hand, some of our proofs use geometric cones that are not topological cones.

We use the following special symbols: $\approx$ means "is homeomorphic to"; $\times$ is used in denoting cartesian products; $\overline{A}$ or $\text{cl}(A)$ denotes the closure of A; if M is a manifold, ∂M denotes the manifold boundary of M and iM denotes the manifold interior of M; $\mathbb{R}^n$ denotes Euclidean n-space and $\|\cdot\|$ denotes the Euclidean norm;

$$B^n = \{x \in \mathbb{R}^n \colon \|x\| \leqslant 1\} \quad \text{and} \quad S^{n-1} = \{x \in \mathbb{R}^n \colon \|x\| = 1\}.$$

A space that is homeomorphic to B^n is called an *n-cell*; a 1-cell is called an *arc*. A space that is homeomorphic to S^n is called an *n-sphere*; a 1-sphere is called a *simple closed curve*.

Let X be a compactum with metric d. A closed subset, A, of X is said to be a *Z-set in X* provided that for each $\varepsilon > 0$, there is a continuous function $f_\varepsilon : X \to X - A$ such that $d(f_\varepsilon(x), x) < \varepsilon$ for all $x \in X$ [6, p. 2]. We will use the following proposition (which is an easy consequence of the classical result about unstable values in VI2 of [11, p. 77] and parts (3) and (4) of 3.1 of [6, p. 2]):

Proposition 2.3. *A compactum, A, in an n-manifold, M^n, is a Z-set in M^n if and only if $A \subset \partial M^n$.*

Other notation and terminology are standard (and may be found in texts in the references) or will be explained at appropriate places.

3. Cones that are n-cells

Assuming that $\mathrm{Cone}(Y)$ is an n-cell, we obtain two results:

(1) either Y is an $(n-1)$-sphere or $Y \times [0,1]$ is an n-cell; and

(2) if $n \leqslant 4$, Y is an $(n-1)$-sphere or an $(n-1)$-cell.

We will see in Section 4 that the converse of the first result is false (for $n \geqslant 5$) and that the natural analogue for $n \geqslant 5$ of the second result is also false. We will use the second result in the proof of the theorem about hyperspaces in Section 6.

We begin with the two Lemmas 3.1 and 3.2. Lemma 3.1 is of some general interest; we will use Lemma 3.1 later as well as in this section. On the other hand, Lemma 3.2 is merely a technical lemma that is designed for only one purpose: to separate the proof of Theorem 3.3 into its two natural components.

Lemma 3.1. *Let Y be a compactum such that* $\mathrm{Cone}(Y)$ *is an n-manifold for some $n \geqslant 2$. Let*

$$\beta(Y) = \left\{ y \in Y \colon \tfrac{1}{2} \cdot y + \tfrac{1}{2} \cdot v \in \partial\, \mathrm{Cone}(Y) \right\}.$$

Then, $\partial\, \mathrm{Cone}(Y) = Y \cup \mathrm{Cone}(\beta(Y))$.

Proof. We first note the following fact, which is easy to prove and which we will use:

(1) For any $y \in Y$ and any $t \in (0,1)$, there is a homeomorphism of $\mathrm{Cone}(Y)$ onto $\mathrm{Cone}(Y)$ taking $(1-t) \cdot y + t \cdot v$ to $\frac{1}{2} \cdot y + \frac{1}{2} \cdot v$.

Now, assume for the moment that $\beta(Y) = \emptyset$. It then follows from (1) that $\partial\, \mathrm{Cone}(Y) \subset Y \cup \{v\}$. Thus, since $n \geqslant 2$ and $\partial\, \mathrm{Cone}(Y)$ is an $(n-1)$-manifold (1.3.4 of [17, p. 3]), we must have that $\partial\, \mathrm{Cone}(Y) \subset Y$. Hence, $v \notin \partial\, \mathrm{Cone}(Y)$. We state what we have proved as follows:

(2) If $v \in \partial\, \mathrm{Cone}(Y)$, then $\beta(Y) \neq \emptyset$.

We see easily from (1) that $\partial\, \mathrm{Cone}(Y) - (Y \cup \{v\}) \subset \mathrm{Cone}(\beta(Y))$. Furthermore, by (2), if $v \in \partial\, \mathrm{Cone}(Y)$ then $v \in \mathrm{Cone}(\beta(Y))$. Therefore, we have that

(3) $\partial\, \mathrm{Cone}(Y) \subset Y \cup \mathrm{Cone}(\beta(Y))$.

Next, we see from (1) that $\mathrm{Cone}(\beta(Y)) - [\beta(Y) \cup \{v\}] \subset \partial\, \mathrm{Cone}(Y)$; thus (since the manifold boundary of any manifold is closed in the manifold), we have that

(4) $\mathrm{Cone}(\beta(Y)) \subset \partial\, \mathrm{Cone}(Y)$.

By (3) and (4), it only remains to prove that $Y \subset \partial\, \mathrm{Cone}(Y)$.

For $\varepsilon > 0$ (ε near 0), consider the maps

$$f_\varepsilon : \mathrm{Cone}(Y) \to \mathrm{Cone}(Y)$$

given by

$$f_\varepsilon(x) = (1-\varepsilon) \cdot x + \varepsilon \cdot v \quad \text{for all } x \in \mathrm{Cone}(Y).$$

These maps show that Y is a Z-set in $\mathrm{Cone}(Y)$. Therefore, by Proposition 2.3, $Y \subset \partial\, \mathrm{Cone}(Y)$. □

There is only one counterexample to the analogue of Lemma 3.1 for $n = 1$: when Y is a one-point set. (The only other compactum Y for which $\mathrm{Cone}(Y)$ is a 1-manifold is a two-point set, in which case Lemma 3.1 for $n = 1$ is true.)

Lemma 3.2. *Let Y be a compactum that is not an $(n-1)$-sphere for some $n \geqslant 1$. Let $M = Y \times [0,1]$, and let q denote the quotient map of M onto $\mathrm{Cone}(Y)$, where $q(Y \times \{1\}) = v$. If $\mathrm{Cone}(Y)$ is an n-cell, then (1)–(3) below hold:*

(1) *M is an n-manifold;*

(2) *$\partial M = (Y \times \{0,1\}) \cup (\beta(Y) \times [0,1])$, where $\beta(Y)$ is as in Lemma 3.1;*

(3) *if U is a neighborhood of v in $\mathrm{Cone}(Y)$, then there is an $(n-1)$-cell, E_U, such that $E_U \subset (\partial M) \cap q^{-1}(U)$ and $Y \times \{1\} \subset i E_U$.*

Proof. The lemma is obviously true for $n = 1$ (since if $n = 1$, $Y = \{\text{pt.}\}$ by our assumption that Y is not a 0-sphere). Therefore, we assume from now on that $n \geqslant 2$ (for the purpose of using Lemma 3.1).

The quotient map q maps $Y \times [0,1)$ homeomorphically onto $\mathrm{Cone}(Y) - \{v\}$. Hence, $Y \times [0,1)$ is an n-manifold. Furthermore, since (for $\beta(Y)$ as in Lemma 3.1)

$$q\big[\big(Y \times \{0\}\big) \cup \big(\beta(Y) \times [0,1)\big)\big] = Y \cup \big[\mathrm{Cone}\,\big(\beta(Y)\big) - \{v\}\big],$$

we have by Lemma 3.1 that

$$\partial\big(Y \times [0,1)\big) = \big(Y \times \{0\}\big) \cup \big(\beta(Y) \times [0,1)\big).$$

Now, using what we have proved about $Y \times [0,1)$ and using the natural homeomorphism of $Y \times [0,1)$ onto $Y \times (0,1]$, we see that $Y \times (0,1]$ is also an n-manifold and that

$$\partial\big(Y \times (0,1]\big) = \big(Y \times \{1\}\big) \cup \big(\beta(Y) \times (0,1]\big).$$

It now follows easily that (1) and (2) of Lemma 3.2 hold.

We prove (3) of Lemma 3.2.

Define the “flipping” homeomorphism φ of $M = Y \times [0,1]$ by $\varphi(y,t) = (y, 1-t)$ for $(y,t) \in M$. Then, $q \circ \varphi$ maps the pair $(\partial M - (Y \times \{0\}), Y \times \{1\})$ homeomorphically onto the pair $(\partial\,\mathrm{Cone}(Y) - \{v\}, Y)$. Let U be a neighborhood of v in $\mathrm{Cone}(Y)$, and let

$$V = q \circ \varphi\big((\partial M) \cap q^{-1}(U)\big).$$

Then, V is a neighborhood of Y in $\partial\,\mathrm{Cone}(Y)$. Hence, to prove (3) it suffices to find an $(n-1)$-cell, D, such that $Y \subset i D$ and $D \subset V - \{v\}$; for then $(q \circ \varphi | \partial M - (Y \times \{0\}))^{-1}$ maps D homeomorphically onto an $(n-1)$-cell E such that $Y \times \{1\} \subset i E$ and $E \subset (\partial M) \cap q^{-1}(U)$.

To find the $(n-1)$-cell D, recall that $\partial\,\mathrm{Cone}(Y) \approx S^{n-1}$. Hence, $\mathbb{R}^{n-1} \approx S^{n-1} - \{\text{point}\} \approx \partial\,\mathrm{Cone}(Y) - \{v\} = Y \cup [\mathrm{Cone}(\beta(Y)) - \{v\}]$. Thus, there is an $(n-1)$-cell, D', such that $Y \subset i D'$ and $D' \subset \partial\,\mathrm{Cone}(Y) - \{v\}$. By compressing D' down the coning arcs of $\mathrm{Cone}(\beta(Y))$ toward $\beta(Y)$, we can isotope D' to an $(n-1)$-cell D such that $Y \subset i D$ and $D \subset V - \{v\}$. This completes the proof of (3). □

Theorem 3.3. *Let Y be a compactum that is not an $(n-1)$-sphere for some $n \geqslant 1$. If $\mathrm{Cone}(Y)$ is an n-cell, then $Y \times [0,1]$ is an n-cell.*

Proof. Let $M = Y \times [0,1]$, and let q be the quotient map of M onto $\mathrm{Cone}(Y)$, where $q(Y \times \{1\}) = v$. We show that q satisfies the Bing Shrinking Criterion, which we state in the context of the present situation as follows (σ denotes the supremum metric for the space of maps from M onto $\mathrm{Cone}(Y)$):

$$\text{for any } \varepsilon > 0, \text{ there is a homeomorphism, } h_\varepsilon, \text{ of } M \text{ onto } M \text{ such that } \sigma(q, q \circ h_\varepsilon) < \varepsilon \text{ and } \mathrm{diam}[h_\varepsilon(Y \times \{1\})] < \varepsilon. \qquad (\#)$$

Once we prove (#), we will know that $M \approx \mathrm{Cone}(Y)$ by Bing's Shrinking Theorem ([6, p. 45] or [13, p. 255]); therefore, we will know that M is an n-cell.

Proof of (#). Fix $\varepsilon > 0$. Let U be a neighborhood of v in $\mathrm{Cone}(Y)$ such that $\mathrm{diam}(U) < \varepsilon$. Then, by (3) of Lemma 3.2, there is an $(n-1)$-cell E_U such that

$$E_U \subset (\partial M) \cap q^{-1}(U) \quad \text{and} \quad Y \times \{1\} \subset i E_U.$$

By (1) of Lemma 3.2, M is an n-manifold; therefore, by Theorem 2 of Brown [4, p. 339], ∂M is collared in M, which means the following: There is a homeomorphism, k, of $(\partial M) \times [0,1)$ onto an open neighborhood of ∂M in M such that $k(y,0) = y$ for all $y \in \partial M$. Thus, since

$$E_U \subset (\partial M) \cap q^{-1}(U),$$

there is a $t \in (0,1)$ such that $k|E_U \times [0,t]$ is an embedding of $E_U \times [0,t]$ in $q^{-1}(U)$ and $k(E_U \times \{0\}) = E_U$. From now on, we consider $E_U \times [0,t]$ as actually being contained in $q^{-1}(U)$ with $E_U \times \{0\} = E_U$.

Now, we proceed to define the homeomorphism h_ε that has the properties in (#). We do this using a homeomorphism, g, on $E_U \times [0,t]$ that we obtain as follows. Recall that $Y \times \{1\} \subset i E_U = i E_U \times \{0\}$. Let $p \in i E_U$. Since $E_U \approx B^n$, there is a homeomorphism h_ε of M such that $h_\varepsilon = \mathrm{id}$ on $\mathrm{cl}(M - (E_U \times [0,t]))$ and such that h_ε squeezes points of $i E_U \times [0,t)$ radially toward $(p,0)$ so that $\mathrm{diam}[h_\varepsilon(Y \times \{1\})] < \varepsilon$. Since $q \circ h_\varepsilon = q$ on $M - q^{-1}(U)$, and since $q \circ h_\varepsilon(q^{-1}(U)) = U = q(q^{-1}(U))$ and $\mathrm{diam}(U) < \varepsilon$, we see that $\sigma(q, q \circ h_\varepsilon) < \varepsilon$. This establishes (#). □

Theorem 3.4. *Let Y be a compactum such that* $\mathrm{Cone}(Y)$ *is an n-cell for some $n \leqslant 4$. Then, Y is an $(n-1)$-sphere or an $(n-1)$-cell.*

Proof. Assume that Y is not an $(n-1)$-sphere. Then, by Theorem 3.3, $Y \times [0,1]$ is an n-cell. Therefore, since $n \leqslant 4$, Y is an $(n-1)$-cell [2, p. 18]. □

4. Examples related to previous theorems

We give examples to show that, for each $n \geqslant 5$, the converse of Theorem 3.3 and the natural extension of Theorem 3.4 are false. The example concerning Theorem 3.4 is particularly important since it is used in proving the hyperspace theorem in Section 6. Our examples are Examples 4.3 and 4.4; we use Propositions 4.1 and 4.2 to verify properties of the examples.

We write $\pi_1(X) = 0$ to mean that the space X is simply connected [10].

Proposition 4.1. *Let M^n be a compact n-manifold for some $n \geqslant 3$. If* $\mathrm{Cone}(M^n)$ *is a manifold, then* $\pi_1(\partial M^n) = 0$.

Proof. Clearly, we may assume for the proof that $\partial M^n \neq \emptyset$. Therefore, by Lemma 3.1, $v \in \partial\,\mathrm{Cone}(M^n)$. Also, since $\mathrm{Cone}(M^n)$ is an $(n+1)$-manifold, $\partial\,\mathrm{Cone}(M^n)$ is an n-manifold without boundary [17, p. 3]. Hence, v has an open neighborhood, U, in $\partial\,\mathrm{Cone}(M^n) - M^n$ such that $U \approx \mathbb{R}^n$. Note that $U - \{v\} \approx \mathbb{R}^n - \{0\}$ and that $\pi_1(\mathbb{R}^n - \{0\}) = 0$ (because $n \geqslant 3$). Thus, $\pi_1(U - \{v\}) = 0$.

Now, let q denote the quotient map of $M^n \times [0,1]$ onto $\mathrm{Cone}(M^n)$, where $q(M^n \times \{1\}) = v$, and let

$$f = q|(\partial M^n) \times (0,1].$$

Clearly, f maps $(\partial M^n) \times (0,1]$ onto $\mathrm{Cone}(\partial M^n) - M^n$; thus, by Lemma 3.1,

$$f\big[(\partial M^n) \times (0,1]\big] = \partial\,\mathrm{Cone}(M^n) - M^n.$$

Therefore, $f^{-1}(U)$ is an open neighborhood of $(\partial M^n) \times \{1\}$ in $(\partial M^n) \times (0,1]$. Thus, since ∂M^n is compact, there exists $t \in (0,1)$ such that $(\partial M^n) \times \{t\} \subset f^{-1}(U)$. Let

$$W = f^{-1}(U) - (\partial M^n) \times \{1\}.$$

Clearly, $f|W$ is a homeomorphism of W onto $U - \{v\}$. Thus, since $\pi_1(U - \{v\}) = 0$, we have that $\pi_1(W) = 0$. Now, note that

$$(\partial M^n) \times \{t\} \subset W \subset (\partial M^n) \times (0,1).$$

Hence, since $(\partial M^n) \times \{t\}$ is obviously a retract of $(\partial M^n) \times (0,1)$, we see that $(\partial M^n) \times \{t\}$ is a retract of W. Thus, since $\pi_1(W) = 0$, we have that $\pi_1((\partial M^n) \times \{t\}) = 0$ [10, p. 150]. Therefore, $\pi_1(\partial M^n) = 0$. □

In the next proposition, $\Sigma(Z)$ denotes the suspension over Z (i.e., $\Sigma(Z)$ is the quotient space obtained from $Z \times [-1,1]$ by shrinking $Z \times \{-1\}$ and $Z \times \{1\}$ to (different) points).

Proposition 4.2. *For any compactum Z,* $\mathrm{Cone}(\mathrm{Cone}(Z)) \approx \mathrm{Cone}(\Sigma(Z))$.

Proof. Set $Q = I^\infty \times [-1,1] \times [0,1]$. For any two subsets A and B of Q, let the *join* of A and B be the union of all the straight line segments joining points of A to points of B, and denote it by $A * B$. We can assume that $Z \subset I^\infty \times \{0\} \times \{0\} \subset Q$. Let $p \in I^\infty$, and set $v = (p,1,0)$, $v' = (p,-1,0)$, and $w = (p,0,1)$. Hence, $v, v', w \in Q$. (See Fig. 3.) Then, $\mathrm{Cone}(Z) = Z * v$ and $\mathrm{Cone}(\mathrm{Cone}(Z)) = (Z * v) * w = Z * (v * w)$ (the join of three sets is associative because the join of sets is the union of the joins of points and the join of three points is clearly associative). Hence, $\mathrm{Cone}(\mathrm{Cone}(Z))$ is the join of Z to the straight line segment $v * w$. Also,

$$\Sigma(Z) = (Z * v) \cup (Z * v') = Z * \{v, v'\}, \quad \text{and}$$
$$\mathrm{Cone}\big(\Sigma(Z)\big) = \big(Z * \{v, v'\}\big) * w = Z * \big(\{v, v'\} * w\big) = Z * \big((v * w) \cup (v' * w)\big).$$

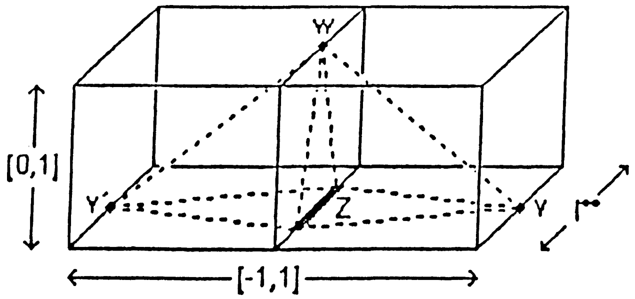

Fig. 3.

In other words, $\mathrm{Cone}(\Sigma(Z))$ is the join of Z to the broken line segment $(v * w) \cup (v' * w)$. Clearly, a homeomorphism from the straight line segment $v * w$ to the broken line segment $(v * w) \cup (v' * w)$ induces a homeomorphism from the join $Z * (v * w)$ to the join $Z * ((v * w) \cup (v' * w))$. We conclude that $\mathrm{Cone}(\mathrm{Cone}(Z)) \approx \mathrm{Cone}(\Sigma(Z))$. □

We are now ready to present our examples.

Our first example concerns the converse of Theorem 3.3. Let us first note that the converse of Theorem 3.3 is true when $n \leqslant 4$: If $Y \times [0, 1]$ is an n-cell and $n \leqslant 4$, then Y is an $(n - 1)$-cell [2, p. 18] and, therefore, $\mathrm{Cone}(Y)$ is an n-cell. However, as the following example shows, the converse of Theorem 3.3 is false for each $n \geqslant 5$ even when Y is assumed to be a manifold.

Example 4.3. For each $k \geqslant 4$, there is a compact, piecewise linear k-manifold, M^k, such that $M^k \times [0, 1]$ is a $(k+1)$-cell and, yet, $\pi_1(\partial M^k) \neq 0$ (see [16] when $k = 4$ and [7] when $k \geqslant 5$). By Proposition 4.1, $\mathrm{Cone}(M^k)$ is not a manifold, much less a $(k+1)$-cell (as would be required for the converse of Theorem 3.3 when $n = k + 1 \geqslant 5$).

The following example shows that Theorem 3.4 does not extend to any $n \geqslant 5$:

Example 4.4. Fix $k \geqslant 3$, and let A be an arc in S^k such that $\pi_1(S^k - A) \neq 0$ (see [9] when $k = 3$ and [3] when $k \geqslant 3$). Let S^k/A denote the quotient space of S^k obtained by shrinking A to a point p. First, we show that $\mathrm{Cone}(S^k/A)$ is not a manifold. Suppose, to the contrary, that $\mathrm{Cone}(S^k/A)$ is a manifold. Then, since $(S^k/A) \times \mathbb{R}^1 \approx S^k \times \mathbb{R}^1$ [1], we see from Lemma 3.1 that

$$\partial\, \mathrm{Cone}(S^k/A) = S^k/A.$$

Thus, since S^k/A is not a manifold [1, p. 1] and manifold boundaries *are* manifolds [17, p. 3], we have a contradiction. Therefore, we have shown that $\mathrm{Cone}(S^k/A)$ is not a manifold. Next, we let $Y = \mathrm{Cone}(S^k/A)$ and we show that $\mathrm{Cone}(Y)$ is a $(k + 2)$-cell. By Proposition 4.2,

$$\mathrm{Cone}(Y) = \mathrm{Cone}\big(\mathrm{Cone}(S^k/A)\big) \approx \mathrm{Cone}\big(\Sigma(S^k/A)\big);$$

therefore, since $\Sigma(S^k/A) \approx S^{k+1}$ [17, p. 84], we have that

$$\mathrm{Cone}(Y) \approx \mathrm{Cone}(S^{k+1}) \approx B^{k+2}.$$

We have shown that Cone(Y) is a $(k+2)$-cell but that Y is not even a manifold. Therefore, for any $n = k+2 \geqslant 5$, the analogue of Theorem 3.4 is false.

5. Two useful results

We prove the results in Propositions 5.2 and 5.3 for use in the next section.

Lemma 5.1. *Let Y be a compactum such that* Cone(Y) *embeds in $\mathbb{R}^n$. If Y contains an $(n-1)$-sphere Z, then $Y = Z$.*

Proof. Assume throughout the proof that Cone$(Y) \subset \mathbb{R}^n$. Let

$$U = \mathrm{Cone}(Z) - Z.$$

Note that the vertex v of Cone(Y) is a point of U (since $Z \subset Y$ and v is also the vertex of Cone(Z)).

We show that U is open in $\mathbb{R}^n$ as follows. Since $Z \approx S^{n-1}$, Cone$(Z) \approx B^n$ and $\partial\,\mathrm{Cone}(Z) = Z$. Hence, $U \approx B^n - S^{n-1}$. Therefore, by Invariance of Domain [17, p. 3], U is open in $\mathbb{R}^n$.

Now, since $v \in U \subset$ Cone(Z) and since U is open in $\mathbb{R}^n$, clearly v is not arcwise accessible from $\mathbb{R}^n -$ Cone(Z). On the other hand, there is certainly an arc in [Cone(Y) − Cone(Z)] $\cup \{v\}$ from any point of Cone(Y) − Cone(Z) to v. Therefore, there is only one conclusion to draw: $Y = Z$. □

Proposition 5.2. *Let X be a continuum such that $C(X)$ embeds in $\mathbb{R}^n$, where $n \geqslant 3$. If $C(X) \approx$* Cone(Y) *for some compactum Y, then S^{n-1} does not embed in Y.*

Proof. By our assumptions, Cone(Y) embeds in $\mathbb{R}^n$. Suppose that S^{n-1} embeds in Y. Then, by Lemma 5.1, $Y \approx S^{n-1}$. Thus, since $C(X) \approx$ Cone(Y) by assumption, we now have that $C(X) \approx B^n$; however, $C(X) \not\approx B^n$ since $n \geqslant 3$ (1.208.4 of [15, p. 199]). □

We make three comments about the proposition that we just proved.

(1) The conclusion to Proposition 5.2 can not be strengthened to say that S^{n-2} does not embed in Y. This follows by letting X be a simple n-od and letting Y be Macías' continuum M_n in Fig. 2 (recall from the Introduction that $C(X) \approx$ Cone(M_n) [12]).

(2) If we assume as in Proposition 5.2 that $C(X)$ embeds in $\mathbb{R}^n$ ($n \geqslant 3$), but we only assume that Cone(Y) embeds in $C(X)$, then it can happen that S^{n-1} embeds in Y (e.g., let X be a simple n-od and let $Y = S^{n-1}$). However, under the assumptions just mentioned, S^n does not embed in Y: For if S^n embeds in Y and Cone(Y) embeds in $C(X)$, then B^{n+1} embeds in $C(X)$ and, hence, $C(X)$ does not embed in $\mathbb{R}^n$.

(3) Regarding the assumption in Proposition 5.2 that $n \geqslant 3$, we note that the only three continua X for which $C(X)$ embeds in $\mathbb{R}^2$ are an arc, a simple closed curve, and a single point [15, p. 238]. Furthermore, Proposition 5.2 for $n = 2$ is false since $C(X) \approx \text{Cone}(S^1)$ when X is an arc or a simple closed curve [15, pp. 30–31].

Our next proposition is a general result about cones. It involves the notion of a dimensional component, which we define as follows.

Let Z be a space. For $z \in Z$, let $\dim_z(Z)$ denote the dimension of Z at z. For any integer $n \geqslant 0$, or for $n = \infty$, let

$$d_n(Z) = \{z \in Z\colon \dim_z(Z) = n\}.$$

Then, by a *dimensional component of* Z we mean a maximal connected subset of $d_n(Z)$ for some n such that $d_n(Z) \neq \emptyset$.

We let π denote the natural projection of $\text{Cone}(Y) - \{v\}$ onto Y given by

$$\pi\big((1-t)\cdot y + t\cdot v\big) = y \quad \text{for all } (1-t)\cdot y + t\cdot v \in \text{Cone}(Y) - \{v\}.$$

Proposition 5.3. *Let Y be a compactum, and let D be a dimensional component of* $\text{Cone}(Y)$. *Let $M = \pi(D - \{v\})$. Then, $\overline{D} = \text{Cone}(\overline{M})$. In more descriptive terms, the closure of a dimensional component of* $\text{Cone}(Y)$ *is the cone over a subcompactum of Y.*

Proof. First, we prove the following fact:

For each $y \in Y$, $\text{Cone}(\{y\}) - \{v\}$ is contained in a single dimensional component of $\text{Cone}(Y)$. (1)

Proof of (1). Fix $y \in Y$. Let $p_t = (1-t)\cdot y + t\cdot v$ for $0 \leqslant t < 1$. If $0 < s < t < 1$, then

$$\dim_{p_s}\big(\text{Cone}(Y)\big) = \dim_{p_t}\big(\text{Cone}(Y)\big)$$

since there is a homeomorphism of $\text{Cone}(Y)$ onto $\text{Cone}(Y)$ that takes p_s to p_t. If $0 < t < 1$, then $\dim_{p_0}(\text{Cone}(Y)) = \dim_{p_t}(\text{Cone}(Y))$ by the following reasoning: Arbitrarily small neighborhoods of p_t can be "truncated" to obtain homeomorphic copies of arbitrarily small neighborhoods of p_0 so that boundaries truncate to boundaries; arbitrarily small neighborhoods of p_0 can be "doubled" to obtain homeomorphic copies of arbitrarily small neighborhoods of p_t so that boundaries double to boundaries; truncation and doubling do not raise boundary dimension by the subspace theorem [11, p. 26] and the sum theorem [11, p. 30]. Therefore, we have shown that $\text{Cone}(Y)$ has the same dimension at every point of $\text{Cone}(\{y\}) - \{v\}$. Thus, since $\text{Cone}(\{y\}) - \{v\}$ is connected, we have proved (1).

Now, we use (1) to prove that $\text{Cone}(\overline{M}) \subset \overline{D}$. By (1), we see that

$$\text{Cone}\big(\{y\}\big) - \{v\} \subset D \quad \text{for each } y \in M.$$

Hence, $\text{Cone}(M) - \{v\} \subset D$ and $v \in \overline{D}$. Thus, $\text{Cone}(M) \subset \overline{D}$. Therefore, it follows using Lemma 2.1 that $\text{Cone}(\overline{M}) \subset \overline{D}$.

It remains to prove that $\overline{D} \subset \mathrm{Cone}(\overline{M})$. To prove this, we first need to prove a fact about the dimensional component, D_v, of Cone(Y) containing v; namely, we prove that

$$D_v \neq \{v\}. \tag{2}$$

Proof of (2). We take two cases.

Case 1. $\dim(Y) = n < \infty$. Then, $\dim(\mathrm{Cone}(Y)) = n + 1$ (see, e.g., 8.0 of [15, p. 301]). Hence, Cone(Y) contains an $(n+1)$-dimensional Cantor manifold K (Theorem VI8 of [11, p. 94]). We prove that

$$\dim_v\bigl(\mathrm{Cone}(Y)\bigr) = n + 1. \tag{$*$}$$

If $v \in K$, then $\dim_v(K) = n + 1$ by (A) of [11, p. 93]; thus since $\dim(\mathrm{Cone}(Y)) = n + 1$, $(*)$ holds. Therefore, to prove $(*)$, we assume that $v \notin K$. Now, suppose that $(*)$ is false, i.e., $\dim_v(\mathrm{Cone}(Y)) \leqslant n$. Then, since $v \notin K$, there is an open neighborhood, V, of v such that $K \cap \overline{V} = \emptyset$ and $\dim(\overline{V} - V) \leqslant n - 1$. Hence, we can push K up towards v with an isotopy, $\{h_t\}_{0\leqslant t\leqslant 1}$, such that $h_0(K) = K$ and $h_1(K) \cap V \neq \emptyset$. Let $U = \mathrm{Cone}(Y) - \overline{V}$. We argue that there exists t_o such that $h_{t_o}(K) \cap U \neq \emptyset$ and $h_{t_o}(K) \cap V \neq \emptyset$: Let

$$s = \sup\bigl\{t \in [0,1]\colon\ h_t(K) \subset \mathrm{Cone}(Y) - V\bigr\}.$$

Since $h_t(K) \approx K$ for each $t \in [0,1]$ and since $\dim(\overline{V} - V) \leqslant n - 1$, we see that $h_t(K) \not\subset \overline{V} - V$ for any $t \in [0,1]$. Hence, since $h_s(K) \subset \mathrm{Cone}(Y) - V$, we see that $h_s(K) \cap U \neq \emptyset$. Thus, since U is open in Cone(Y), there exists $t_o > s$ such that $h_{t_o}(K) \cap U \neq \emptyset$; furthermore, since $t_o > s$ and $h_{t_o}(K) \not\subset \overline{V} - V$, $h_{t_o}(K) \cap V \neq \emptyset$. It follows easily that $h_{t_o}(K) - (\overline{V} - V)$ is not connected. However, this is a contradiction since $h_{t_o}(K)$ is an $(n + 1)$-dimensional Cantor manifold and $\dim(\overline{V} - V) \leqslant n - 1$. Therefore, we have proved $(*)$.

Now, we complete the proof of (2) for Case 1. Let $z \in K - \{v\}$ (z exists since $\dim(K) = n + 1$). Let $y = \pi(z)$. Since $\dim(\mathrm{Cone}(Y)) = n + 1$ and $\dim_z(K) = n + 1$, clearly $\dim_z(\mathrm{Cone}(Y)) = n + 1$. Hence, by (1), Cone$(Y)$ is $(n + 1)$-dimensional at each point of $\mathrm{Cone}(\{y\}) - \{v\}$. Therefore, we see from $(*)$ that $\mathrm{Cone}(\{y\}) \subset D_v$. This proves (2) under the assumption in Case 1.

Case 2. $\dim(Y) = \infty$. Then, by combining a result of Tumarkin [18] with Theorem VI8 of [11, p. 94], we have that one (or both) of the following holds:

(a) Y contains an infinite-dimensional Cantor manifold, Y_∞;

(b) Y contains an n_i-dimensional Cantor manifold, Y_{n_i}, for each i, where $n_1 < n_2 < \cdots < n_i < \cdots$.

Assume first that (a) holds. Then the proof of $(*)$ in Case 1 can be easily adapted to prove that $\dim_v(\mathrm{Cone}(Y)) = \infty$. Hence, $\mathrm{Cone}(Y_\infty) \subset D_v$; therefore, (2) holds assuming (a). Next, assume that (b) holds. Let $p_i \in Y_{n_i}$ for each i, and assume without loss of generality that $\{p_i\}_{i=1}^\infty$ converges to, say, p. Now, fix $q \in \mathrm{Cone}(\{p\}) - \{p\}$. Suppose that q has an open neighborhood, V, in Cone(Y) such that $\overline{V} \cap Y = \emptyset$ and $\dim(\overline{V} - V) = k < \infty$. Fix $n_j \geqslant k + 2$ such that $p_{n_j} \in \pi(V)$. Let $\{h_t\}_{0\leqslant t\leqslant 1}$ be an isotopy of Y_{n_j} in Cone(Y) such that $h_0(Y_{n_j}) = Y_{n_j}$ and $h_1(Y_{n_j}) \cap V \neq \emptyset$. Then, as in the proof of $(*)$ in Case 1, there exists t_o such that $h_{t_o}(Y_{n_j}) - (\overline{V} - V)$ is not connected, a contradiction (since $k \leqslant n_j - 2$). Hence, we have shown that $\dim_q(\mathrm{Cone}(Y)) = \infty$ for each $q \in \mathrm{Cone}(\{p\}) - \{p\}$. Thus,

$\mathrm{Cone}(\{p\}) - \{p\} \subset D_v$; therefore, (2) holds assuming (b). This completes the proof of (2) under the assumption in Case 2.

Therefore, we have proved (2).

Now, we prove that $\overline{D} \subset \mathrm{Cone}(\overline{M})$. It is easy to see that $D \subset \mathrm{Cone}(M)$ (use (2) if $D = D_v$). Therefore, $\overline{D} \subset \mathrm{Cone}(\overline{M})$ by Proposition 2.1. □

6. Application to hyperspaces

Using previous results, we answer Charatonik's question that we discussed in the Introduction.

We use the following notation throughout the section. For any $n \geqslant 3$, T_n denotes a simple n-od and M_n denotes Macías' continuum (Fig. 2, Section 1). If K_n is a compactum such that $\mathrm{Cone}(K_n) \approx C(T_n)$, then Fig. 1 in Section 1 depicts $\mathrm{Cone}(K_n)$; with this in mind, Q_n denotes the maximal n-cell in $\mathrm{Cone}(K_n)$, and $F_1, \dots, F_n$ denote the maximal 2-cells ("fins") that comprise the closure of $\mathrm{Cone}(K_n) - Q_n$.

In the following lemma, we give some technical information for convenient reference in the proof of Theorem 6.2. Most of the proof of the lemma is based on Fig. 1 in Section 1.

Lemma 6.1. *If K_n is a compactum such that $C(T_n) \approx \mathrm{Cone}(K_n)$ for some $n \geqslant 3$, then* (6.1.1)–(6.1.5) *are true*:

(6.1.1) *K_n is a continuum*;

(6.1.2) $\mathrm{Cone}(K_n) = Q_n \cup (\bigcup_{i=1}^{n} F_i)$;

(6.1.3) *the dimensional components of* $\mathrm{Cone}(K_n)$ *are Q_n and $F_i - Q_n$ for each $i \leqslant n$, $\overline{Q}_n = Q_n$ and $\overline{F_i - Q_n} = F_i$ for each $i \leqslant n$*;

(6.1.4) *$Q_n \cap F_i = \partial Q_n \cap \partial F_i$ is an arc for each $i \leqslant n$*;

(6.1.5) *the vertex v of* $\mathrm{Cone}(K_n)$ *is the unique point in $\bigcap_{i=1}^{n} \partial F_i = F_j \cap F_k$ for any $j \neq k$.*

Proof. To prove (6.1.1), note the following easy-to-prove fact: The vertex of the cone over any nonconnected space separates the cone. Also, note from Fig. 1 that no point of $\mathrm{Cone}(K_n)$ separates $\mathrm{Cone}(K_n)$. Thus, K_n must be connected. This proves (6.1.1).

The statements in (6.1.2)–(6.1.4) follow easily by inspecting Fig. 1.

Finally, we prove (6.1.5). Note that the vertex of any cone has arbitrarily small, *open* neighborhoods in the cone whose closures are homeomorphic to the cone. Also, note from Fig. 1 that the point in $\bigcap_{i=1}^{n} \partial F_i$ is the only point of $\mathrm{Cone}(K_n)$ with such open neighborhoods. Therefore, the point in $\bigcap_{i=1}^{n} \partial F_i$ must be the vertex of $\mathrm{Cone}(K_n)$. That the equality in (6.1.5) holds for any $j \neq k$ is, of course, evident from Fig. 1. □

We answer Charatonik's question with the following theorem.

Theorem 6.2. *If $n = 3$ or 4 and $C(T_n) \approx \mathrm{Cone}(K_n)$ for some compactum K_n, then $K_n \approx M_n$. However, for each $n \geqslant 5$ there is a continuum, L_n, such that $C(T_n) \approx \mathrm{Cone}(L_n)$ but $L_n \not\approx M_n$.*

Proof. To prove the first part of the theorem, assume that $n = 3$ or 4 and that $C(T_n) \approx \mathrm{Cone}(K_n)$ for some compactum K_n. By (6.1.3) and Proposition 5.3, there are compacta E and $A_1, \ldots, A_n$ in K_n such that (for Q_n and F_i as above Lemma 6.1)

(1) $Q_n = \mathrm{Cone}(E)$, and

(2) $F_i = \mathrm{Cone}(A_i)$ for each $i \leqslant n$.

Note that by (1), (2), and (6.1.2), we have that

(3) $K_n = E \cup (\bigcup_{i=1}^{n} A_i)$.

Therefore, once we prove (4)–(7) below, we will know that $K_n \approx M_n$ (see Fig. 2 in Section 1).

Since Q_n is an n-cell and $n = 3$ or 4, we see from (1) and Theorem 3.4 that E is either an $(n-1)$-cell or an $(n-1)$-sphere; since the vertex v of $Q_n = \mathrm{Cone}(E)$ lies in ∂Q_n (by (6.1.4) and (6.1.5)), clearly E can not be an $(n-1)$-sphere. Therefore,

(4) E is an $(n-1)$-cell.

Next, recall that each F_i is a 2-cell; hence, by (2) and Theorem 3.4, each A_i is either a 1-sphere or an arc. However, no A_i can be a 1-sphere since $v \in \partial\,\mathrm{Cone}(A_i)$ for each i by (2) and (6.1.5). Thus, we have that

(5) A_i is an arc for each $i \leqslant n$.

By (2) and (6.1.5), $\mathrm{Cone}(A_j) \cap \mathrm{Cone}(A_k) = F_j \cap F_k = \{v\}$ for $j \neq k$; hence, we have that

(6) $A_j \cap A_k = \emptyset$ whenever $j \neq k$.

By Proposition 2.2 and by (1) and (2), $\mathrm{Cone}(E \cap A_i) = \mathrm{Cone}(E) \cap \mathrm{Cone}(A_i) = Q_n \cap F_i$; hence, by (6.1.4), $\mathrm{Cone}(E \cap A_i)$ is an arc. Thus, $E \cap A_i$ must be a one-point set or a two-point set. Suppose that $E \cap A_i$ is a two-point set. Then, clearly,

$$v \notin \partial\,\mathrm{Cone}(E \cap A_i) = \partial(Q_n \cap F_i);$$

however, we know from (6.1.5) and Fig. 1 that $v \in \partial(Q_n \cap F_i)$. Having thus established a contradiction, we now know that $E \cap A_i$ consists of only one point, say x_i. Therefore,

$$\mathrm{Cone}\big(\{x_i\}\big) = Q_n \cap F_i.$$

Hence, by (6.1.4), $\mathrm{Cone}(\{x_i\}) = \partial Q_n \cap \partial F_i$. Thus, by (1),

$$\mathrm{Cone}\big(\{x_i\}\big) \subset \partial\,\mathrm{Cone}(E).$$

Therefore, by Lemma 3.1 and Invariance of Domain [17, p. 3], it follows that $x_i \in \partial E$ (recall (4)). Similarly (using (2) and (5)), we see that $x_i \in \partial A_i$ for each i. Therefore, we have proved that

(7) $E \cap A_i = \{x_i\}$ and $x_i \in \partial E \cap \partial A_i$ for each $i \leqslant n$.

On considering (3)–(7) and keeping Fig. 2 in mind, we see that $K_n \approx M_n$. This proves the first part of Theorem 6.2.

To prove the second part of Theorem 6.2, fix $n \geqslant 5$. Write Macías's continuum M_n as follows: $M_n = D \cup (\bigcup_{j=1}^{n} E_j)$, where D is an $(n-1)$-cell and, for $1 \leqslant j \leqslant n$, E_j is an arc such that $D \cap E_j = \partial D \cap \partial E_j = \{p_j\}$, a single point. Let $k = n - 2$, and let $Y = \mathrm{Cone}(S^k/A)$ be as in Example 4.4. Then, by Example 4.4, Y is not an $(n-1)$-cell but $\mathrm{Cone}(Y)$ is an n-cell. Choose distinct points $q_1, q_2, \ldots, q_n$ in $(S^k/A) - \{A\}$, where A is regarded both as an arc in S^k and as a point in S^k/A. Form the space L_n by attaching n

pairwise disjoint arcs $I_1, I_2, \ldots, I_n$ to Y so that $I_j \cap Y = \partial I_j \cap \partial Y = \{q_j\}$ for $1 \leqslant j \leqslant n$. Thus, q_j is an end point of I_j for $1 \leqslant j \leqslant n$. We will show that L_n is not homeomorphic to Macías' continuum M_n but that $C(T_n) \approx \mathrm{Cone}(L_n)$.

Clearly, $L_n \not\approx M_n$ because the dimensional component of M_n of dimension $n-1$ is the $(n-1)$-cell D, but the dimensional component of L_n of dimension $n-1$ is Y which is not an $(n-1)$-cell.

We show that $C(T_n) \approx \mathrm{Cone}(L_n)$ by showing that $\mathrm{Cone}(M_n) \approx \mathrm{Cone}(L_n)$.

Set

$$J = \mathrm{Cone}\big(\{p_1, p_2, \ldots, p_n\}\big) \subset \mathrm{Cone}(D),$$

and set

$$K = \mathrm{Cone}\big(\{q_1, q_2, \ldots, q_n\}\big) \subset \mathrm{Cone}(Y).$$

Then, J and K are n-ods embedded in the $(n-1)$- spheres $\partial\,\mathrm{Cone}(D)$ and $\partial\,\mathrm{Cone}(Y)$, respectively. We will show that there is a homeomorphism, h, of $\partial\,\mathrm{Cone}(D)$ onto $\partial\,\mathrm{Cone}(Y)$ such that $h(J) = K$. Because a homeomorphism between the boundaries of two n-cells extends to a homeomorphism between the n-cells (by coning), h will extend to a homeomorphism from $\mathrm{Cone}(D)$ onto $\mathrm{Cone}(Y)$; also, because a homeomorphism between arcs in the boundaries of two 2-cells extends to a homeomorphism between the 2-cells [17, p. 47], $h|J$ will extend to a homeomorphism from $\mathrm{Cone}(\bigcup_{j=1}^n E_j)$ onto $\mathrm{Cone}(\bigcup_{j=1}^n I_j)$. Hence, h will extend to a homeomorphism from $\mathrm{Cone}(M_n)$ onto $\mathrm{Cone}(L_n)$. Therefore, to prove that $\mathrm{Cone}(M_n) \approx \mathrm{Cone}(L_n)$, it suffices to produce a homeomorphism h of $\partial\,\mathrm{Cone}(D)$ onto $\partial\,\mathrm{Cone}(Y)$ such that $h(J) = K$.

Recall that if X is a subset of a space W, then $W - X$ is *locally simply connected at* a point $x \in X$ if for every neighborhood, U, of x in W, there is a neighborhood, V, of x in W such that every loop in $V - X$ is null homotopic in $U - X$. Also, note the following fact, which follows from [5]:

(8) If $e_0, e_1 : Z \to N$ are two homotopic embeddings of a 1-dimensional compact polyhedron, Z, into a topological manifold, N, of dimension $\geqslant 4$ such that $N - e_i(Z)$ is locally simply connected at each point of $e_i(Z)$ for $i = 0$ and 1, then there is a homeomorphism, g, of N onto N such that $g \circ e_0 = e_1$.

We will now argue that $\partial\,\mathrm{Cone}(Y) - K$ is locally simply connected at each point of K. We will accomplish this by showing that each point of K has arbitrarily small neighborhoods, U, in $\partial\,\mathrm{Cone}(Y)$ such that $U - K$ is simply connected. Note that K has three types of points: its vertex v, the points $q_1, q_2, \ldots, q_n$ of its base, and the points of $K - \{v, q_1, q_2, \ldots, q_n\}$. Each q_i has arbitrarily small neighborhoods in $(S^k/A) - \{A\}$ that are homeomorphic to $\mathbb{R}^{n-2}$. It follows that each q_i has arbitrarily small neighborhoods, U, in $\partial\,\mathrm{Cone}(Y)$ such that $U - K$ is homeomorphic to $(\mathbb{R}^{n-2} \times \mathbb{R}^1) - (\{0\} \times [0, \infty))$. It also follows that each point of $K - \{v, q_1, q_2, \ldots, q_n\}$ has arbitrarily small neighborhoods, U, in $\partial\,\mathrm{Cone}(Y)$ such that $U - K$ is homeomorphic to $(\mathbb{R}^{n-2} - \{0\}) \times \mathbb{R}^1$. Both $(\mathbb{R}^{n-2} \times \mathbb{R}^1) - (\{0\} \times [0, \infty))$ and $(\mathbb{R}^{n-2} - \{0\}) \times \mathbb{R}^1$ are simply connected because $n \geqslant 5$. The vertex v has arbitrarily small neighborhoods, U, in $\partial\,\mathrm{Cone}(Y)$ such that $U - K$ is homeomorphic to $((S^k/A) - \{q_1, q_2, \ldots, q_n\}) \times \mathbb{R}^1$. Now, $((S^k/A) - \{q_1, q_2, \ldots, q_n\}) \times \mathbb{R}^1$ is homeomorphic to $(S^k - \{n \text{ points}\}) \times \mathbb{R}^1$ by [1], and the latter space is simply connected because $S^k - \{n \text{ points}\}$ is simply connected (since $k = n - 2 \geqslant 3$). This establishes that $\partial\,\mathrm{Cone}(Y) - K$ is locally simply connected at each point of K.

An argument similar to the one just given shows that $\partial\,\mathrm{Cone}(D) - J$ is locally simply connected at each point of J.

Now, let h_0 be any homeomorphism of the $(n-1)$-sphere $\partial\,\mathrm{Cone}(D)$ onto the $(n-1)$-sphere $\partial\,\mathrm{Cone}(Y)$ (recall Example 4.4). Then, $h_0(J)$ is a simple n-od embedded in $\partial\,\mathrm{Cone}(Y)$ such that $\partial\,\mathrm{Cone}(Y) - h_0(J)$ is locally simply connected at each point of $h_0(J)$. Also, the identity maps $\mathrm{id}_{h_0(J)}$ and id_K are homotopic in $\partial\,\mathrm{Cone}(Y)$ because both of the simple n-ods, $h_0(J)$ and K, are contractible. Hence, by (8) above, there is a homeomorphism, h_1, of $\partial\,\mathrm{Cone}(Y)$ onto $\partial\,\mathrm{Cone}(Y)$ such that $h_1(h_0(J)) = K$. Let $h = h_1 \circ h_0 : \partial\,\mathrm{Cone}(D) \to \partial\,\mathrm{Cone}(Y)$. Then, clearly, h is a homeomorphism of $\partial\,\mathrm{Cone}(D)$ onto $\partial\,\mathrm{Cone}(Y)$ such that $h(J) = K$. As we explained earlier, h extends to a homeomorphism from $\mathrm{Cone}(M_n)$ onto $\mathrm{Cone}(L_n)$. Therefore, $C(T_n) \approx \mathrm{Cone}(L_n)$. □

Remark. In relation to results in Section 3, the second author has obtained results about when the cone over Y is the Hilbert cube. His paper, *Cones and suspensions that are Hilbert cubes*, is in the Bol. Soc. Mat. Mexicana (3) 4 (1998) 285–289.

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