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$\mathscr{Z}$-COMPACTIFICATIONS OF OPEN MANIFOLDS

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Suppose an open n-manifold M^n may be compactified to an ANR $\widehat{M^n}$ so that $\widehat{M^n} - M^n$ is a $\mathscr{Z}$-set in $\widehat{M^n}$. It is shown that (when $n \geqslant 5$) the double of $\widehat{M^n}$ along its "$\mathscr{Z}$-boundary" is an n-manifold. More generally, if M^n and N^n each admit compactifications with homeomorphic $\mathscr{Z}$-boundaries, then their union along this common boundary is an n -manifold. This result is used to show that in many cases $\mathscr{Z}$-compactifiable manifolds are determined by their $\mathscr{Z}$-boundaries. For example, contractible open n-manifolds with homeomorphic $\mathscr{Z}$-boundaries are homeomorphic. As an application, some special cases of a weak Borel conjecture are verified. Specifically, it is shown that closed aspherical n-manifolds ($n \neq 4$) having isomorphic fundamental groups which are either word hyperbolic or $CAT(0)$ have homeomorphic universal covers. © 1999 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

In [29], Siebenmann studied those open n-manifolds ($n \geqslant 6$) which may be compactified in the nicest of ways. In particular, he gave necessary and sufficient conditions for an open manifold to be compactifiable to a manifold with boundary by adding a boundary $(n-1)$-manifold. Similar results for $n = 3$ and 5 may be found in [20] and Section 11.9 of [16]. While these results have found numerous applications, the strict conditions necessary for a manifold to admit this sort of compactification rule out a variety of important (and reasonably nice) open manifolds. For many of these manifolds a more general type of compactification, which we call $\mathscr{Z}$-compactification, seems most appropriate. Roughly speaking, $\mathscr{Z}$-compactification allows one to add a non-manifold boundary, but requires that many of the homotopy properties enjoyed by manifolds with boundary be maintained. Examples of manifolds which do not satisfy Siebenmann's conditions, but which are $\mathscr{Z}$-compactifiable arise in geometric topology, geometric group theory and synthetic differential geometry. They include: Davis' exotic contractible covering spaces [13], universal covers of aspherical manifolds with word hyperbolic or $CAT(0)$ fundamental groups, and all $CAT(0)$ manifolds. In many of these examples, the $\mathscr{Z}$-boundary cannot be a manifold—in fact, it is often non-locally simply connected and fractal in nature. Despite the possible pathology present in a $\mathscr{Z}$-boundary, a great deal of geometric structure is preserved by $\mathscr{Z}$-compactification. In this paper we exhibit and exploit some of that structure.

Our main results are the following. We prove that if two open n-manifolds ($n \geqslant 5$) admit $\mathscr{Z}$-compactifications with homeomorphic $\mathscr{Z}$-boundaries, then their union along these boundaries is a manifold. For example, the double of a $\mathscr{Z}$-compactified n-manifold along its $\mathscr{Z}$-boundary is a manifold. These results and some corollaries are found in Section 4. In Sections 6 and 7 we use this result to prove some uniqueness theorems. In particular, we exhibit conditions under which a $\mathscr{Z}$-compactifiable manifold is determined up to

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homeomorphism by its $\mathcal{Z}$-boundary. As a corollary we prove some special cases of a "weak Borel conjecture". The original (and still unsolved) Borel conjecture states that homotopy equivalent aspherical manifolds are homeomorphic. The weak version is that homotopy equivalent aspherical manifolds have homeomorphic universal covers. We give an affirmative answer to this latter conjecture in dimensions greater than four whenever the fundamental group is word hyperbolic or $CAT(0)$.

With the possible exception of terminology, the notion of a $\mathcal{Z}$-compactification is not new. The concept was developed in the 1970s by Chapman and Siebenmann [10] in their work on Hilbert cube manifolds. As noted there, the definitions (although not all of the theorems) are easily modified to apply to locally compact ANRs. In particular, the definitions apply nicely to finite-dimensional manifolds. Recent work by Bestvina and Mess (see [4, 5]) utilizes a more rigid type of $\mathcal{Z}$-compactification to study boundaries of groups. Their work was later applied by Carlsson and Pedersen (see [9]) to prove Novikov conjectures for the corresponding class of groups.

2. DEFINITIONS

A locally compact separable metric space X is an *absolute neighborhood retract* or *ANR* if it may be embedded as a closed subset of $\mathbb{R}^\infty$ so that there exists a retraction $r: U \to X$, where U is a neighborhood of X in $\mathbb{R}^\infty$. If a retraction $r: \mathbb{R}^\infty \to X$ exists, then X is an *absolute retract* or *AR*. It is well known (see Theorem 5.2.15 of [22]) that an ANR is an AR if and only if it is contractible. When X is finite dimensional, we may replace $\mathbb{R}^\infty$in the above definition with finite-dimensional Euclidean space, $\mathbb{R}^n$ (for n sufficiently large). Hence, a finite-dimensional ANR is sometimes called an *ENR* or *Euclidean neighborhood retract*, and similarly for finite-dimensional ARs. An important characterization (Theorem 5.5.7 of [22]) states that a finite-dimensional locally compact separable metric space is an ANR if and only if it is locally contractible.

A closed subset A of a compact ANR, X, is a *$\mathcal{Z}$-set* if any of the following equivalent conditions is satisfied:

- There is a homotopy $H: X \times I \to X$ with $H_0 = id_X$ and $H_t(X) \cap A = \emptyset$ for all $t > 0$.
- For every $\varepsilon > 0$ there is an ε-homotopy $K: X \times I \to X$ with $K_0 = id_X$ and $K_1(X) \subset X \setminus A$.
- For every $\varepsilon > 0$ there is a map $f: X \to X$ which is ε-close to the identity with $f(X) \subset X \setminus A$.
- For every open set U of X, $U \setminus A \hookrightarrow U$ is a homotopy equivalence.

Let Y be a non-compact ANR. A *$\mathcal{Z}$-compactification* of Y is a compact ANR $\hat{Y}$ containing Y as an open subset and having the property that $\hat{Y} \setminus Y$ is a $\mathcal{Z}$-set in $\hat{Y}$. In this case we call $\hat{Y} \setminus Y$ a *$\mathcal{Z}$-boundary for* Y and denote it $\partial_Z Y$. Note that Y may admit many different $\mathcal{Z}$-boundaries, hence $\partial_Z Y$ is not well defined unless the $\mathcal{Z}$-compactification is specified.

3. EXAMPLES OF $\mathcal{Z}$-COMPACTIFICATIONS

In this section we review some known examples of $\mathcal{Z}$-compactifications. Although we are primarily interested in compactifying finite-dimensional manifolds, we include some discussion of infinite-dimensional and non-manifold examples which are relevant to this paper.

Example 1 (*Manifolds with boundary*). Let M^n be a compact n-manifold with boundary. Then the standard boundary, ∂M^n, is a $\mathcal{Z}$-set in M^n, so M^n is a $\mathcal{Z}$-compactification of

$int(M^n)$ with $\partial_Z(int(M^n)) = \partial M^n$. Hence, $\mathcal{Z}$-compactification may be viewed as a generalization of the manifold compactifications studied by Siebenmann and others.

Example 2 (*Hilbert cube manifolds*). Chapman and Siebenmann have given necessary and sufficient conditions for a Hilbert cube manifold to be $\mathcal{Z}$-compactifiable. Since their characterization plays a significant role this paper, we give a brief description their results. For details, the reader should consult [10].

A *Hilbert cube manifold* is a separable metric space with the property that each of its points has a closed neighborhood homeomorphic to the Hilbert cube $Q = [0, 1]^\infty$. A non-compact Hilbert cube manifold X is *inward tame at infinity* if for any compact set $A \subset X$, there exists a homotopy $H: (X \setminus A) \times I \to X \setminus A$ so that $H_0 = id$ and $cl_X(H_1(X \setminus A))$ is compact. Equivalently, one may require that X contain arbitrarily large compact subsets A such that $X \setminus A$ is finitely dominated. If this condition holds, one may define an algebraic invariant $\sigma_\infty(X) \in \varprojlim \{\tilde{K}_0\pi_1(X \setminus A) | A \subset X \text{ compact}\}$. Here $\tilde{K}_0\pi_1$ is the projective class group functor and all bonding maps are induced by inclusion. The individual "coordinates" of $\sigma_\infty(X)$ are the Wall finiteness obstructions for the $(X \setminus A)$'s (see [33, 34]). Then $\sigma_\infty(X)$ vanishes iff X contains arbitrarily small neighborhoods of infinity having finite homotopy type (A subset of X is a *neighborhood of infinity* if the closure of its complement is compact.). When $\sigma_\infty(X)$ vanishes, we may define a second algebraic invariant $\tau_\infty(X) \in \varprojlim^1 \{Wh\pi_1(X \setminus A) | A \subset X \text{ compact}\}$. Here $\varprojlim^1$ denotes the first derived limit (see Section 5 for a definition) and $Wh\pi_1$ is the Whitehead group functor. Again, bonding maps are induced by inclusion.

We may now state the theorem.

Theorem 3 (Chapman and Siebenmann [10]). *A Hilbert cube manifold X admits a $\mathcal{Z}$-compactification iff each of the following is satisfied.*

(a) *X is inward tame at infinity.*
(b) $\sigma_\infty(X) \in \varprojlim \{\tilde{K}_0\pi_1(X \setminus A) | A \subset X \textit{ compact}\}$ *is zero.*
(c) $\tau_\infty(X) \in \varprojlim^1 \{Wh\pi_1(X \setminus A) | A \subset X \textit{ compact}\}$ *is zero.*

Remark 1. The notion of inward tameness and the definitions of σ_∞ and τ_∞ can also be applied to arbitrary locally compact ANRs, although it is apparently unknown whether a locally compact ANR satisfying (a)–(c) is $\mathcal{Z}$-compactifiable. On the other hand, a $\mathcal{Z}$-compactifiable ANR, X, must satisfy (a)–(c). One way to see this is to apply Theorem 3 to $X \times Q$ which is a Hilbert cube manifold by a theorem of R. D. Edwards (see Theorem 7.8.1 of [22]).

Example 4 (*Davis' exotic universal covers*). In [13] Davis constructed the first known examples of closed n -manifolds ($n \geqslant 4$) which have contractible universal covering spaces that are not Euclidean space. A construction by Ancel and Siebenmann [3] embeds these universal covers in the n-sphere so that their closures are $\mathcal{Z}$-compatifications and their frontiers the corresponding $\mathcal{Z}$-boundaries.

The $\mathcal{Z}$-boundaries obtained via the above construction are easily seen to be non-manifolds. In fact, the construction may be done so that the $\mathcal{Z}$-boundary is homogeneous and non-locally simply connected at each point. By analyzing the fundamental group at infinity, one can show that the $\mathscr{Z}$-boundaries of these examples can never be a manifolds or even ANRs.

Example 5 (*Rips complexes of word hyperbolic groups*). Let Γ be a word hyperbolic group, and let $P(\Gamma)$ be a contractible Rips complex for G. Then $P(\Gamma)$ may be $\mathscr{Z}$-compactified to $\widehat{P(\Gamma)} = P(\Gamma) \cup \partial\Gamma$, where $\partial\Gamma$ is the Gromov boundary of Γ.

For definitions and details of this example, see [5].

Example 6 (*$CAT(0)$ spaces*). Let Y be a locally compact $CAT(0)$ ANR and let $S_\infty(0)$ denote the visual sphere of Y from some point $0 \in Y$. Then Y admits a $\mathscr{Z}$-compactification $\hat{Y} = Y \cup S_\infty(0)$. Here $S_\infty(0)$ may be viewed as the set of rays emanating from 0. Roughly speaking, the compactification of Y is obtained by adding an end point to each of these rays.

See [14] for further discussion of this topic. For examples of $CAT(0)$ spaces which are manifolds see [1, 2, 14].

An important special case of Example 6 occurs when one begins with a compact non-positively curved ($\equiv$ locally $CAT(0)$) space. Then the metric on X may be lifted to a (globally) $CAT(0)$ metric on the universal cover $\tilde{X}$ where the fundamental group of X acts via isometries. This has lead to the following:

Definition 7. A group G which acts isometrically, properly discontinuously and cocompactly on a $CAT(0)$ space X is called a $CAT(0)$ *group*. In this case the visual sphere of X is called a $CAT(0)$ *boundary* of G.

Remark 2. (a) It has recently been shown (see [11]) that the boundary of a $CAT(0)$ group is not always unique.

(b) Notice that if, in addition, G acts without fixed points (e.g., if G is torsion free), then the projection $X \to X/G$ is a covering map and, since $CAT(0)$ spaces are contractible the quotient is a $K(\pi, 1)$.

The following result allows a potentially significant expansion of Examples 5 and 6. It also prevents the non-uniqueness of $CAT(0)$ boundaries from becoming a significant problem. The result follows from Lemma 1.4 of [4], and could, in fact, be stated for all groups admitting "$\mathscr{Z}$-structures" as defined there.

LEMMA 8. *Suppose K is a finite $K(G, 1)$ where G is either a $CAT(0)$ or word hyperbolic group. If G is word hyperbolic, let ∂G be the (unique) Gromov boundary of G, otherwise, let ∂G be an arbitrary $CAT(0)$ boundary for G. Then, the universal cover $\tilde{K}$ admits a $\mathscr{Z}$-compactification with $\mathscr{Z}$-boundary equal to ∂G.*

Proof. First, recall that since G admits a finite $K(G, 1)$, then G is torsion free (see p. 76 of [18]).

Case 1. If G is word hyperbolic we may apply Example 1.2(i) and Lemma 1.4 of [4] directly.

Case 2. If G is $CAT(0)$, then by definition G acts isometrically, properly discontinuously and cocompactly on a $CAT(0)$ space X. Since G is also torsion free, G also acts without fixed points. Hence, we may apply Example 1.2(ii) and Lemma 1.4 of [4]. ■

4. GLUING MANIFOLDS ALONG $\mathscr{Z}$-SET BOUNDARIES

Let $\widehat{M^n}$ and $\widehat{N^n}$ be $\mathscr{Z}$-compactifications of open n-manifolds (*open manifold* means "noncompact manifold without boundary") and $h: \partial_{\mathscr{Z}}\widehat{M^n} \to \partial_Z\widehat{N^n}$ be a homeomorphism. Denote by $\widehat{M^n} \cup_h \widehat{N^n}$ the quotient space $(\widehat{M^n} \coprod \widehat{N^n})/x \sim h(x)$. We call this a *gluing* of $\widehat{M^n}$ and $\widehat{N^n}$ along their $\mathscr{Z}$-boundaries. If $\widehat{N^n}$ is a second copy of $\widehat{M^n}$ and h is the identity, we call this space the *double* of $\widehat{M^n}$ along its $\mathscr{Z}$-boundary and denote it *Double*$(\widehat{M^n})$. Our main goal in this section is to prove that a gluing of two $\mathscr{Z}$-compactified open n-manifolds along a common $\mathscr{Z}$-boundary is an n-manifold. Although the proof is quite technical, at its core is a fairly standard application of the Edwards–Quinn manifold recognition theorem. The outline of our argument is probably familiar to a number of geometric topologists.

THEOREM 9. *Let $\widehat{M^n}$ and $\widehat{N^n}$ be $\mathscr{Z}$-compactifications of open n-manifolds $(n > 4)$ and h: $\partial_Z\widehat{M^n} \to \partial_Z\widehat{N^n}$ be a homeomorphism. Then $\widehat{M^n} \cup_h \widehat{N^n}$ is an n-manifold.*

Before we begin the proof, we state some immediate consequences.

COROLLARY 10. *The double of any $\mathscr{Z}$-compactified open n-manifold $(n > 4)$ along its $\mathscr{Z}$-boundary is an n-manifold.*

COROLLARY 11. *The gluing of any two $\mathscr{Z}$-compactified contractible open n-manifolds $(n > 4)$ along a common $\mathscr{Z}$-boundary is homeomorphic to S^n.*

Proof. First, use techniques like those found below in the proof of Lemma 15 to show that the union is simply connected. Then use standard algebraic topology arguments to show that this gluing has the the homology of an n-sphere. Finally, apply the generalized Poincaré conjecture. ■

COROLLARY 12. *Suppose Σ may be realized as the $\mathscr{Z}$-boundary of a contractible open n-manifold $(n > 4)$. Then there is an involution on S^n with fixed point set Σ.*

Proof. Apply Corollary 11 to the double of the compactified manifold. The involution interchanges the two halves. ■

COROLLARY 13. *Every $\mathscr{Z}$-compactifiable contractible n-manifold $(n > 4)$ may be embedded in S^n as the complement of a contractible compactum.*

Proof. M^n is the complement of a copy of $\widehat{M^n}$ in *Double*$(\widehat{M^n})$. ■

COROLLARY 14. *Suppose a group G acts effectively via isometries on a $CAT(0)$ n-manifold M^n $(n \geqslant 5)$. Then G acts effectively via homeomorphisms on S^n.*

Proof. A $CAT(0)$ space is necessarily contractible. The action extends naturally to the $\mathscr{Z}$-compactification $\widehat{M^n}$ obtained by adding the visual sphere at infinity. This action may be reflected across the boundary to obtain an action on *Double*$(\widehat{M^n})$ which, by Corollary 11, is homeomorphic to S^n. ■

The proof of Theorem 9 utilizes several results from the theory of homology and cohomology manifolds. All of our spaces are finite-dimensional locally compact metric spaces. For a compact pair (X, A), $\bar{H}_*(X, A)$ will denote Steenrod (or equivalently Borel–Moore) homology with integer coefficients (see [6, 23]). Cohomology will be Alexander–Čech cohomology with integer coefficients. A space X is *homologically locally n-connected*, denoted *hlc*^{n}, if for each $x \in X$, and each compact neighborhood U of x, there is a compact neighborhood V of x such that $\tilde{\bar{H}}_k(V) \to \tilde{\bar{H}}_k(U)$ is the trivial map for all $k \leqslant n$. A space which is *hlc*^{n} for all n is called *homologically locally connected* (denoted *hlc*$^\infty$). *Cohomological local* (*n*-) *connectedness* (*clc*^{n} and *clc*$^\infty$) are defined similarly. For each $x \in X$, let $\mathscr{H}_k^x$ denote $\varinjlim \bar{H}_k(X, X\backslash U)$, where the limit is taken over all open neighborhoods of x with maps being induced by inclusion. We call X a *homology n-manifold with boundary* provided for each $x \in X$, $\mathscr{H}_k^x = 0$ for all $k \neq n$ and $\mathscr{H}_n^x = 0$ or $\mathbb{Z}$. In this case we let $\partial X = \{x | \mathscr{H}_n^x = 0\}$. If $\partial X = \emptyset$, we simply call X a *homology n-manifold. Cohomology n-manifolds* (with or without boundary) may be defined similarly (see [6, 7]).

We say that X is *homotopically locally n-connected* (*LC*^{n}) if for each $x \in X$, and each neighborhood U of x, there is a neighborhood $V \subset U$ of x with the property that for all $k \leqslant n$, each map $f: S^k \to V$ extends to $\bar{f}: D^{k+1} \to U$. A space which is *LC*^{n} for all n is called *homotopically locally connected* (*LC*$^\infty$).

For later use, we state and provide references for several key facts from the theory of homology/cohomology manifolds:

1. *A space X is clc*$^\infty$ *iff it is hlc*$^\infty$. See Theorem 6.6 of [6].
2. *Every homology or cohomology manifold with boundary is clc*$^\infty$ (*hence hlc*$^\infty$). For cohomology manifolds this is classical (See, for example, Theorem V.15.7 of [7]). For homology manifolds see Theorem 1 of [24].
3. *Every homology or cohomology n-manifold is locally arc connected.* Local connectedness follows from the *clc*^{0} or *hlc*^{0} properties. From there one may employ results from Sections III.2 and III.3 of [35] to obtain local path connectedness and then the stronger condition of local arc connectedness.
4. *The boundary of a homology n -manifold with boundary is a homology* ($n - 1$)-*manifold* (*without boundary*). See [25].
5. *A space X is a homology n-manifold with boundary iff it is a cohomology n-manifold with boundary.* This observation combines Theorem 7.12 of [6] with Facts 1 and 2. The orientability assumption in [6] is made obsolete by [8].
6. *The union of two homology n-manifolds with boundary along a common boundary is a homology n-manifold.* This combines Theorem 1 of [27] with Fact 5. Again the orientability assumption is obsolete by [8].
7. *A space which is both LC*^{1} *and hlc*$^\infty$ *is LC*$^\infty$. This is Lemma 3 of [15].

The following lemma will also be used in the proof of Theorem 9.

LEMMA 15. *Let X be a locally compact metric space, and suppose $X = A \cup B$ where A and B are each closed and LC*^{1} *and $C = A \cap B$ is LC*^{0}. *Then X is LC*^{1}.

Proof. It is easy to see that X is locally path connected, so we need only show that for each $x \in X$ and neighborhood U of x in X, there is a neighborhood V of x so that loops in V contract in U. If x lies in *int* A or *int* B this follows from the hypotheses, so we assume $x \in C$. Let $U' \subset U$ be a neighborhood of x with compact closure and having the property that loops lying in $U' \cap A$ contract in $U \cap A$ and loops in $U' \cap B$ contract in $U' \cap B$. Then let V be a neighborhood of x lying in U' with the property that points in $V \cap C$ may be connected with a path lying in $U' \cap C$. We will show that each loop in V contracts in U.

By the compactness of $\overline{U'}$ and the LC^1 property for B we may choose for each positive integer k, a $\delta_k > 0$ so that loops in $\overline{U'} \cap B$ having diameter less than δ_k bound singular disks in U having diameter less than $1/k$. Similarly, for each δ_k just chosen, choose $\varepsilon_k > 0$ so that points in $V \cap C$ which are ε_k-close may be connected in $U' \cap C$ by paths having diameter $< \delta_k$.

Now let $\alpha : S^1 \to V$ be a loop. If $\alpha(S^1)$ is contained in $V \cap A$ or $V \cap B$ then α contracts in $U \cap A$ or $U \cap B$, so we are finished. Otherwise, we build a homotopy in U from α to α' with $\alpha'(S^1) \subset U' \cap A$. Then by our choice of U', α' contracts in $U \cap A$. Hence, α contracts in U. To build the homotopy, note that $\alpha^{-1}(V \backslash A)$ is a (possibly infinite) collection $\mathscr{A} = \{A_1, A_2, \dots\}$ of pairwise disjoint open subarcs of S^1. If α_i denotes $\alpha(\bar{A}_i)$, then each α_i is a path in $V \cap B$ with end points in $V \cap C$.

By choice of V, we may connect the end points of α_1 by a path α'_1 in $U' \cap C$. Then $\alpha_1 \cup \alpha'_1$ is a loop in $U' \cap B$. Since this loop must contract in $U \cap B$, there is a homotopy (rel end points) of α_1 to α'_1 in $U \cap B$. If $\mathscr{A}$ is finite, repeat this for each α_i to homotope α into $U' \cap A$ as desired.

If $\mathscr{A}$ is infinite we must take more care. Necessarily, $diam(A_i) \to 0$ as $i \to \infty$. Then, by uniform continuity, $diam(\alpha_i) \to 0$ as $i \to \infty$. Again we homotope each α_i in $U \cap B$ (in order) to an arc α'_i in $U' \cap (X)$, but now we use controls made possible by the existence of the δ_k's and ε_k's chosen earlier to ensure that these homotopies converge to a homotopy which moves α to the desired α'. ■

Proof of Theorem 9. We show that $X = \widehat{M^n} \cup_h \widehat{N^n}$ is an n-manifold by verifying all criteria of the Edwards–Quinn manifold recognition theorem. (Theorem VII.40.4 of [12]). In particular, we will show that X is a finite-dimensional ANR homology n-manifold which satisfies the disjoint disks property (DDP) and has trivial Quinn resolvability obstruction.

Finite dimensionality of X follows from the closed sum theorem (Theorem 4.3.7 of [22]) once we show that $\widehat{M^n}$ and $\widehat{N^n}$ are finite dimensional. By Theorem 4.5.13 of [22], $\widehat{M^n}$ is n-dimensional iff there exist η-maps from $\widehat{M^n}$ to an n-dimensional polyhedron for arbitrarily small $\eta > 0$. To obtain such a map, first push $\widehat{M^n}$ into M^n with an $(\eta/4)$-move (using the $\mathscr{Z}$-set hypothesis), then compose with an $(\eta/2)$-map from M^n to an n-dimensional polyhedron. The same argument shows that $\widehat{N^n}$ and thus X are n-dimensional.

Next, we show that X is a homology n-manifold. In light of Fact 6, it suffices to show that $\widehat{M^n}$ and $\widehat{N^n}$ are homology n-manifolds with boundaries $\partial_{\mathscr{Z}}\widehat{M^n}$ and $\partial_{\mathscr{Z}}\widehat{N^n}$, respectively. Let $x \in \widehat{M^n}$. If $x \in M^n$, then

$$\mathscr{H}_k^x \cong \begin{cases} 0 & \text{if } k \neq n \\ \mathbb{Z} & \text{if } k = n \end{cases}$$

since M^n is an n-manifold. If $x \in \partial_{\mathscr{Z}}\widehat{M^n}$, we must show that $\mathscr{H}_k^x = 0$ for all k. As noted on p. 510 of [25], it suffices to show that $\varinjlim H_*(\widehat{M^n}, \widehat{M^n} \backslash U) = 0$, where H_* denotes ordinary singular homology. To this end, fix a neighborhood U' of x in $\widehat{M^n}$. Since $\partial_{\mathscr{Z}}\widehat{M^n}$ is a $\mathscr{Z}$-set, there is a homotopy $K_t : \widehat{M^n} \to \widehat{M^n}$ with $K_0 = id_{\widehat{M^n}}$ and $K_1(\widehat{M^n}) \subset M^n$. Let $\lambda : \widehat{M^n} \to [0, 1]$ be a map sending x to 1 and $\widehat{M^n} \backslash U'$ to 0, and define $J_t : \widehat{M^n} \to \widehat{M^n}$ by $J_t(x) = K_{t \cdot \lambda(x)}(x)$. Then J_t pulls $\widehat{M^n}$ away from x while keeping $\widehat{M^n} \backslash U'$ fixed. Let U'' be a neighborhood of x disjoint from $J_1(\widehat{M^n})$. Then $H_*(\widehat{M^n}, \widehat{M^n} \backslash U') \to H_*(\widehat{M^n}, \widehat{M^n} \backslash U'')$ is the trivial map. It follows that $\varinjlim H_*(\widehat{M^n}, \widehat{M^n} \backslash U) = 0$.

Next, we observe that X is an ANR. Since X is finite dimensional, it suffices to show that X is LC^∞ (Theorem 7.1 of [19]). Since X is a homology n-manifold we need only show that X is LC^1 (see Facts 2 and 7). Since $\widehat{M^n}$ and $\widehat{N^n}$ are ANRs (hence, locally contractible) and $\partial_{\mathscr{Z}}\widehat{M^n}$ is locally path connected by Fact 3, the LC^1 property follows from Lemma 15.

The Quinn resolvability obstruction for a connected ENR homology manifold Y is an integral invariant $I(Y)$ which vanishes if and only if there exists a cell-like map from a manifold onto Y (see [26]). A key property of I is that for any open subset $U \subset Y$, $I(U) = I(Y)$. Since our space X contains open subsets which are manifolds, it follows that $I(X)$ is trivial.

The final step in our proof is to show that X satisfies the *DDP*; i.e. we must show that given $\varepsilon > 0$ and maps $f, g : D^2 \to X$, there exist maps $f', g' : D^2 \to X$ so that $d(f, f') < \varepsilon$, $d(g, g') < \varepsilon$, and $f'(D^2) \cap g'(D^2) = \emptyset$.

Before beginning, we remind the reader that for any closed subset A of D^2, any small perturbation of $f|_A$ can be extended to a small perturbation of f. This follows from a controlled version of the Borsuk homotopy extension principle and the fact that X is an ANR.

Next, observe that for any simplex σ, every map $\phi : \partial\sigma \to \widehat{M}$ *with small image extends to a map* $\Phi : \sigma \to \widehat{M}$ with small image such that $\Phi(int(\sigma)) \subset M$. This follows from local contractibility of $\widehat{M}$ along with the fact that $Z = \widehat{M} \backslash M$ is a $\mathscr{Z}$-set in $\widehat{M}$. Of course, $\widehat{N}$ enjoys a similar property.

Assuming that ε, f and g have been chosen, we build the desired approximations in three steps.

Step 1. Perturbing f to it make it "transverse" to Z.

In this step we arbitrarily closely approximate f by a map (of the same name) so that $f^{-1}(Z)$ is a one-dimensional submanifold of D^2 such that $\partial f^{-1}(Z) = f^{-1}(Z) \cap \partial D^2$. Furthermore, we arrange that $f(D^2) \cap Z$ be a one-dimensional subset of Z. This construction is similar to the "classical" construction that would work if X were a manifold and Z were a bicollared codimension-one submanifold. It exploits the fact that $\widehat{M}$ and $\widehat{N}$ are ANRs and Z is a $\mathscr{Z}$-set in each.

To begin, we take a triangulation T of D^2 which is so fine that the f-images of the simplices of T are very small. We then use the property that Z is a $\mathscr{Z}$-set to perturb f slightly so that $f(T^0) \cap Z = \emptyset$, where T^0 denotes the 0-skeleton of T. Define the function $\phi^0 : T^0 \to \{0, 1\}$ by $\phi^0(v) = 0$ if $f(v) \in M$ and $\phi^0(v) = 1$ if $f(v) \in N$, and extend ϕ^0 to a simplicial map $\phi : T \to [0, 1]$. Observe that since ϕ is transverse to $\frac{1}{2}$, then $\phi^{-1}(\frac{1}{2})$ is a one-dimensional submanifold of D^2 such that $\partial\phi^{-1}(\frac{1}{2}) = \phi^{-1}(\frac{1}{2}) \cap \partial D^2$. Now if σ is a 1-simplex of T such that $f(\partial\sigma) \subset M$, then the observation preceding Step 1 allows us to perturb f slightly keeping $f|_{\partial\sigma}$ fixed so that $f(\sigma) \subset M$. This procedure can be carried out for every 1-simplex and every 2-simplex σ of T. Thus, we can assume that for any 1-simplex or 2-simplex σ of T, $f(\sigma) \subset M$ whenever $\phi(\sigma) = 0$ and $f(\sigma) \subset N$ whenever $\phi(\sigma) = 1$. Now, suppose that σ is a 1-simplex of T such that $\frac{1}{2} \in \phi(\sigma)$. Then ϕ maps σ homeomorphically onto $[0, 1]$. Hence, $\sigma \cap \phi^{-1}(1/2)$ is a single point p in $int(\sigma)$, and f maps the endpoints of σ to opposite sides of Z. Thus $f(int(\sigma)) \cap Z \neq \emptyset$. Therefore, we can perturb f keeping $f|_{\partial\sigma}$ fixed so that $f(p) \in Z$. Then the observation preceding Step 1 allows us to perturb f so that $f(\sigma \cap \phi^{-1}([0, \frac{1}{2}))) \subset M$ and $f(\sigma \cap \phi^{-1}((\frac{1}{2}, 1])) \subset N$. By carrying out this procedure for every 1-simplex in T, we establish these inclusion relations for every 1-simplex σ of T. Next, suppose σ is a 2-simplex of T such that $\frac{1}{2} \in \phi(\sigma)$. Then $\sigma \cap \phi^{-1}(\frac{1}{2})$ is an arc α such that $\partial\alpha = \alpha \cap \partial\sigma$. Since Z is locally arc connected (Fact 3) and since $f(\partial\alpha) \subset Z$, we can perturb f keeping $f|_{\partial\alpha}$ fixed so that $f(\alpha)$ is an embedded arc in Z joining the two points of $f(\partial\alpha)$. Set $A = \sigma \cap \phi^{-1}([0, \frac{1}{2}])$ and $B = \sigma \cap \phi^{-1}([\frac{1}{2}, 1])$. Then A and B are two-dimensional disks such that $A \cap B = (\partial A) \cap (\partial B) = \alpha$, $f(\partial A) \subset \widehat{M}$, $f(\partial B) \subset \widehat{N}$, and $f^{-1}(Z) \cap (\partial A \cup \partial B) = \alpha$. Then the observation preceding Step 1 allows us to perturb f keeping $f|_{\partial A}$ and $f|_{\partial B}$ fixed so that $f(A) \subset \widehat{M}$, $f(B) \subset \widehat{N}$, $f(A \backslash \alpha) \subset M$, and $f(B \backslash \alpha) \subset N$. By carrying out this procedure for every 2-simplex of T, we obtain $f^{-1}(Z) = \phi^{-1}(\frac{1}{2})$. Thus $f^{-1}(Z)$ is a one-dimensional submanifold of D^2 such that $\partial f^{-1}(Z) = f^{-1}(Z) \cap \partial D^2$, and $f(D^2) \cap Z = f(f^{-1}(Z)) = f(\phi^{-1}(\frac{1}{2}))$ is a finite union of arcs. Thus, $f(D^2) \cap Z$ is a one-dimensional subset of Z. Set $L = f(D^2) \cap Z$.

Step 2. Perturbing g so it is transverse to Z and $f(D^2) \cap g(D^2) \cap Z = \emptyset$.

Before constructing the desired approximation to g, we must observe that if U is a small non-empty connected open subset of Z, then $U \backslash L$ is also non-empty and path connected. The key to this observation will be a version of Alexander duality for homology manifolds. Begin with the homology exact sequence

$$\bar{H}_1^c(U, U \backslash L) \to \bar{H}_0^c(U \backslash L) \to \bar{H}_0^c(U) \to \bar{H}_0^c(U, U \backslash L)$$

where $\bar{H}_*^c$ denotes Borel–Moore homology with compact supports and coefficients in $\mathbb{Z}$. By Theorem 9.3 of [7] $\bar{H}_1^c(U, U \backslash L) \cong \bar{H}_c^{n-1}(L \cap U, \mathscr{O})$, and $\bar{H}_0^c(U, U \backslash L) \cong \bar{H}_c^n(L \cap U, \mathscr{O})$, where $\bar{H}_c^*$ denotes sheaf cohomology and $\mathscr{O}$ denotes the orientation sheaf of U. By [8], we may choose U so small that $\mathscr{O}$ is constant. Then these cohomology groups are also with $\mathbb{Z}$-coefficients, and further reference to $\mathscr{O}$ may be omitted. To see that Theorem 9.3 of [7] can be applied as desired, first observe that since $U \backslash L$ is open in U, then it is also "locally closed" in U, and then note that since $L \cap U$ is a closed subset of U and compact supports are "paracompactifying", then $L \cap U$ is "c-taut" in U. See pp. 8, 15, and 52 of [7] for discussions of these terms.

Next, we use the fact that $n - 1 > 1 = \dim L$ to conclude that $\bar{H}_c^{n-1}(L \cap U)$ and $\bar{H}_c^{n-1}(L \cap U)$ are both trivial (see p. 144 of [7], or p. 236 of [17]). Hence, by our homology exact sequence, $\bar{H}_0^c(U \backslash L) \cong \bar{H}_0^c(U)$. Since U is connected, Theorem 5.11 of [7] implies that $\bar{H}_0^c(U) \cong \mathbb{Z}$. Thus, $\bar{H}_0^c(U \backslash L) \cong \mathbb{Z}$, and Theorem 5.11 of [7] implies that $U \backslash L$ is non-empty and connected. Since $U \backslash L$ is also locally path connected (Fact 3) we may conclude that $U \backslash L$ is path connected.

Now, we will apply the argument of Step 1 to g to make g "transverse" to Z. Thus, we will arbitrarily closely approximate g by a map (of the same name) so that $g^{-1}(Z)$ is a one-dimensional submanifold of D^2 such that $\partial g^{-1}(Z) = g^{-1}(Z) \cap \partial D^2$. In addition, we will arrange that $g(D^2) \cap L = \emptyset$. To achieve this additional condition, we must take extra care at two points. Assume that $\psi : T \to [0, 1]$ is a simplicial map which plays the same role with respect to the map $g : D^2 \to X$ that the simplicial map ϕ: $T \to [0, 1]$ played in Step 1 with respect to the map $f : D^2 \to X$. The first point requiring extra care is when σ is a 1-simplex of T such that $\sigma \cap \psi^{-1}(\frac{1}{2}) \neq \emptyset$. Then $\sigma \cap \psi^{-1}(\frac{1}{2})$ is a single point p, and we must choose $g(p)$ to lie in $Z \backslash L$. If an initial choice of $g(p)$ lies in L, we can move $g(p)$ off L by a small move, because $U \backslash L \neq \emptyset$ for any neighborhood U of $g(p)$ in Z. The second point requiring extra care is when σ is a 2-simplex of T such that $\frac{1}{2} \in \psi(\sigma)$. Then $\sigma \cap \psi^{-1}(\frac{1}{2})$ is an arc α such that $\partial\alpha = \alpha \cap \partial\sigma$, and $g(\partial\alpha) \subset Z \backslash L$. Since Z is locally arc connected, we can perturb g so that g embeds α in Z. If our initial choice of $g(\alpha)$ intersects L, we can move $g(\alpha)$ off L by a small move, because $U \backslash L$ is path connected for any small path connected neighborhood U of $g(\alpha)$ in Z. We complete Step 2 as we completed Step 1, with the result that both $f : D^2 \to X$ and $g : D^2 \to X$ are transverse to Z and $f(D^2) \cap g(D^2) \cap Z = \emptyset$.

Step 3. Final adjustments of f and g.

Since $X \backslash Z = M \cup N$ is a manifold of dimension $\geqslant 5$, then f and g can be perturbed without moving $f|_{f^{-1}(Z)}$ and $g|_{g^{-1}(Z)}$ so that $f|_{f^{-1}(M \cup N)} : f^{-1}(M \cup N) \to M \cup N$ and $g|_{g^{-1}(M \cup N)}$: $g^{-1}(M \cup N) \to M \cup N$ are in general position. This makes $f(D^2) \cap g(D^2) \cap (M \cup N) = \emptyset$. We conclude that $f(D^2) \cap g(D^2) = \emptyset$. ■

5. PROPER h-COBORDISMS AND A CONNECTION TO $\mathscr{Z}$-COMPACTIFICATIONS

Our next goal is to exhibit cases in which $\mathscr{Z}$-compactifiable manifolds with homeomorphic $\mathscr{Z}$-boundaries are necessarily homeomorphic. A key ingredient in our proofs will be the proper s-cobordism theorem [30], together with a "naturality" result from [10] which provides a link between $\mathscr{Z}$-compactification and proper h-cobordism theory. In this section we give a brief description of these results. For details the reader should refer to the original papers.

Recall that a map is *proper* if the preimage of each compactum is compact. A homotopy equivalence is proper provided the corresponding homotopies may be chosen to be proper maps. A cobordism (W^{n+1}, M^n, N^n) is a proper h-cobordism if $M^n \hookrightarrow W^{n+1}$ and $N^n \hookrightarrow W^{n+1}$ are each proper homotopy equivalences.

THEOREM 16 (Proper s-cobordism theorem). *A proper h-cobordism* (W^{n+1}, M^n, N^n) *where $n > 4$ is homeomorphic to $M^n \times [0, 1]$ iff $M^n \hookrightarrow W^{n+1}$ is an infinite simple homotopy equivalence.*

Remark 3. (a) This result remains true when $n = 4$ provided certain fundamental group conditions are satisfied. For more information, see Chap. 7 of [16].

(b) As usual, when $n \geqslant 5$, "homeomorphic" may be replaced with "PL homeomorphic" or "diffeomorphic" provided one begins in the corresponding category.

The difficulty in applying this result is in identifying which maps are infinite simple homotopy equivalences. Infinite simple homotopy types are much more complicated than finite simple homotopy types. For example, even when the spaces involved are simply connected, a proper homotopy equivalence may fail to be simple. Siebenmann has defined a pair of algebraic invariants which (when computable) allow one to determine whether a proper homotopy equivalence is simple. Fortunately for us, there is a connection between these invariants and those appearing in Theorem 3.

Let $f: X \to Y$ be a proper homotopy equivalence between locally finite CW-complexes. By replacing Y with the mapping cylinder of f we may assume that X is a subcomplex of Y and f is inclusion. Write $Y = \bigcup_{i=1}^{\infty} C_i$ where C_i is compact and $C_i \subset int(C_{i+1})$ and let $D_i = C_i \cap X$ for each i. The first obstruction to f being a simple homotopy equivalence is denoted $\sigma_\infty(f)$. It lies in $\varprojlim \{\tilde{K}_0 \pi_1(Y \backslash C_i)\}$, where the coordinates of $\sigma_\infty(f)$ are the relative finiteness obstructions for the pairs $(Y \backslash C_i, X \backslash D_i)$. Roughly speaking, if $\sigma_\infty(f)$ vanishes we may dissect the inclusion $X \hookrightarrow Y$ into an infinite collection of relatively finite inclusions. This allows us to define the second invariant $\tau'(f)$ which lies in the group

$$\left(G \times \prod_{i=1}^{\infty} G_i\right) \Big/ \{(-px_1, x_1 - px_2, x_2 - px_3, x_3 - px_4, \cdots) | x_i \in G_i\}.$$

Here $G = Wh(\pi_1(Y))$ and $G_i = Wh(\pi_1(Y \backslash C_i))$ where the C_i have been appropriately chosen. By abuse of notation, p denotes each of the inclusion-induced maps. A less fine, but important "sub-invariant", $\tau_\infty(f)$, is obtained by neglecting the G-coordinate. More precisely $\tau_\infty(f)$ lies in the derived limit

$$\varprojlim{}^1 \{G_i\} \equiv \left(\prod_{i=1}^{\infty} G_i\right) \Big/ \{(x_1 - px_2, x_2 - px_3, x_3 - px_4, \cdots) | x_i \in G_i\}.$$

Siebenmann shows that f is an infinite simple homotopy equivalence iff both $\sigma_\infty(f)$ and $\tau'(f)$ vanish. Hence, by Theorem 16, a proper h-cobordism (W^{n+1}, M^n, N^n) is a product iff $\sigma_\infty(i)$ and $\tau'(i)$ vanish, where $i: M^n \to W^{n+1}$ is inclusion. If $\sigma_\infty(i)$ and $\tau_\infty(i)$ (but not necessarily $\tau'(i)$) vanish, one may conclude that W^{n+1} is a product on some neighborhood of infinity.

The following result which combines Theorems 5.2 and 6.2 of [10] provides a crucial link between $\mathscr{Z}$-compactifications and infinite simple homotopy types.

THEOREM 17 (Naturality for σ_∞ and τ_∞). *Let $f: X \to Y$ be a proper homotopy equivalence between locally compact ANRs. Then*

(a) *If X and Y are inward tame at infinity, then $\sigma_\infty(Y) = \sigma_\infty(f) + f_*(\sigma_\infty(X))$.*

(b) *If X and Y are inward tame at infinity and $\sigma_\infty(Y) = 0 = \sigma_\infty(X)$, then $\tau_\infty(Y) = \tau_\infty(f) + f_*(\tau_\infty(X))$.*

Remark 4. In [10], these results are stated for X and Y Hilbert cube manifolds. To see that they hold more generally one may again apply the theorem of Edwards mentioned in Remark 1.

6. CONTRACTIBLE n-MANIFOLDS AND A WEAK BOREL CONJECTURE

In this section we present the simplest of our uniqueness theorems. A more general result will be given in the next section. We focus on the special case because it is more elegant and because it leads quickly to our primary application.

THEOREM 18. *Suppose M^n and N^n are contractible open n-manifolds ($n > 4$) which admit $\mathscr{Z}$-compactifications having homeomorphic $\mathscr{Z}$-boundaries. Then M^n and N^n are homeomorphic.*

Remark 5. The above theorem might be compared to a well-known result by Kervaire (see [21]) which states that a compact contractible n-manifold ($n \geqslant 5$) is determined, up to homeomorphism, by its boundary. A four-dimensional version of that result may be found in [16].

Our primary application of Theorem 18 is related to the following famous open problem.

CONJECTURE 19 (The Borel conjecture). *If P and Q are closed aspherical manifolds with isomorphic fundamental groups, then they are homeomorphic.*

Since a solution has been so illusive, we suggest the following.

CONJECTURE 20 (A weak Borel conjecture). *If P and Q are closed aspherical manifolds with isomorphic fundamental groups, then their universal covers are homeomorphic.*

The following gives an affirmative answer to the latter conjecture for some important classes of fundamental groups.

COROLLARY 21. *Let P^n and Q^n be closed aspherical n-manifolds ($n > 4$) with isomorphic fundamental groups. If this group is word hyperbolic or $CAT(0)$ then P^n and Q^n have homeomorphic universal covers.*

Proof. Since P^n and Q^n are aspherical, their universal covers are contractible. Hence, the result follows from Lemma 8 and Theorem 18. ■

Remark 6. (a) As with Lemma 8, this result can be stated more generally for all groups supporting a $\mathscr{Z}$-structure as defined in [4].

(b) All known closed aspherical manifolds having exotic; i.e., non-Euclidean, universal covers have been produced in [13] or [14] and have $CAT(0)$ fundamental groups (some of which are also word hyperbolic). The classical Borel conjecture remains open for these manifolds.

(c) By Theorem 4.1 of [5] and Remark 2.9 of [4], Corollary 21 is also true for $n = 3$ since the universal covers will necessarily be $\mathbb{E}^3$. In some sense, Corollary 21 is the high-dimensional analog of these three-dimensional results.

Proof of Theorem 18. Let $\widehat{M^n}$ and $\widehat{N^n}$ be the $\mathscr{Z}$-compactifications and $h: \partial_{\mathscr{Z}} M \to \partial_{\mathscr{Z}} N$ a homeomorphism. Let B^{n+1} be the $(n+1)$-ball of radius 1 in $\mathbb{R}^{n+1}$ with Euclidean metric d and let S^n be its boundary. By Corollary 11, we may identify $\widehat{M^n} \cup_h \widehat{N^n}$ with S^n. Let Z denote the common copy of $\partial_{\mathscr{Z}} M$ and $\partial_{\mathscr{Z}} N$ lying in S^n and let $W^{n+1} = B^{n+1} \backslash Z$. Then the boundary of W^{n+1} is the disjoint union of M^n and N^n.

Claim 1. *(W^{n+1}, M^n, N^n) is a proper h-cobordism.*

It suffices to construct a strong deformation retraction $R: B^{n+1} \times I \to B^{n+1}$ of B^{n+1} onto $\widehat{M^n}$ with the additional property that $R^{-1}(Z) = Z \times I$. Then the restriction of R to $W^{n+1} \times I$ will be a proper deformation retraction of W^{n+1} onto M^n. By symmetry there will be a proper deformation retraction of W^{n+1} onto N^n.

First, note that by definition of $\mathscr{Z}$-compactification $\widehat{M^n}$ is an ANR and $M^n \hookrightarrow \widehat{M^n}$ is a homotopy equivalence. Thus, $\widehat{M^n}$ is a contractible ANR, hence, an AR. Since B^{n+1} is contractible, we may apply Theorems VII.1.1 and VII.2.1 of [19] to obtain a strong deformation retraction $S: B^{n+1} \times I \to B^{n+1}$ of B^{n+1} onto $\widehat{M^n}$. By one of our $\mathscr{Z}$-set characterizations, there is a homotopy $H: \widehat{M^n} \times I \to \widehat{M^n}$ with $H_0 = id_{\widehat{M^n}}$ and $H_t(\widehat{M^n} \cap Z) = \emptyset$ for all $t > 0$. Define $K: B^{n+1} \times I \to B^{n+1}$ by $K(x,t) = H(S(x,1), t \cdot d(x, \widehat{M^n}))$. Notice that $(K_1)^{-1}(Z) = Z$. Then $T: B^{n+1} \times I \to B^{n+1}$, defined by

$$T(x,t) = \begin{cases} S(x, 2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2} \\ K(x, 2t-1) & \text{if } \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

is a strong deformation retraction of B^{n+1} onto $\widehat{M^n}$ with the property that $(T_1)^{-1}(Z) = Z$. Hence, under T, the only points of B^{n+1} having tracks ending in Z are points of Z.

Now, using scalar multiplication in $\mathbb{R}^{n+1}$, define $R: B^{n+1} \times I \to B^{n+1}$ by

$$R(x,t) = [1 - (d(x, \widehat{M^n}) \cdot t \cdot (1-t))] \cdot T(x,t).$$

Since $0 \leqslant t \cdot (1-t) \leqslant \frac{1}{4}$ and $d(x, \widehat{M^n}) \leqslant 2$ for all x, the scalars range from $\frac{1}{2}$ to 1 with value equal to 1 whenever $x \in \widehat{M^n}$. Hence, for $x \in \widehat{M^n}$, $R(\{x\} \times I) = T(\{x\} \times I) = \{x\}$. For $x \notin \widehat{M^n}$, $R(x,0) = T(x,0) = x$ and $R(x,1) = T(x,1) \notin Z$; moreover, if $0 < t < 1$, then $1 - (d(x, \widehat{M^n}) \cdot t \cdot (1-t))$ is strictly less than 1, so $R(\{x\} \times (0,1)) \subset int(B^{n+1})$. Thus, for $x \notin \widehat{M^n}$, $R(\{x\} \times I) \cap Z = \emptyset$.

By the above we see that R fixes Z; and points outside of Z have tracks missing Z. Hence, $R^{-1}(Z) = Z \times I$, as desired.

Claim 2. *$i: M^n \hookrightarrow W^{n+1}$ is an infinite simple homotopy equivalence.*

Since M^n is $\mathscr{Z}$-compactifiable it satisfies conditions (a)–(c) of Theorem 3 (see Remark 1). Next, note that $B^{n+1} = W^{n+1} \cup Z$ is a $\mathscr{Z}$-compactification of W^{n+1}. Hence, W^{n+1} also satisfies (a)–(c) of Theorem 3. Now, apply Theorem 17 to conclude that $\sigma_\infty(i)$ and $\tau_\infty(i)$ are both zero. Since M^n is simply connected $Wh(\pi_1(M^m))$ is trivial, therefore, the triviality of $\tau_\infty(i)$ implies the triviality of $\tau'(i)$. Hence, by the discussion following Theorem 16, i is an infinite simple homotopy equivalence.

The theorem now follows from the proper s-cobordism theorem. ■

Remark 7. Steve Ferry has suggested an alternative proof for this result which relies on continuously controlled surgery theory.

7. A MORE GENERAL UNIQUENESS THEOREM

A tamely embedded compact codimension 0 submanifold (with boundary) C of an open n-manifold M^n is called a *compact core* provided $C \hookrightarrow M^n$ is a homotopy equivalence. If $\partial C \hookrightarrow M^n \backslash int(C)$ is also a homotopy equivalence, we call C a *geometric compact core*. In general, we call a manifold with boundary, F^n, a *homotopy collar* provided ∂F^n is compact and the inclusion $\partial F^n \hookrightarrow F^n$ is a homotopy equivalence. Hence, an open manifold contains a geometric core if and only if it contains a neighborhood of infinity which is a homotopy collar.

Obviously, any open manifold which contains a compact core is homotopy equivalent to a finite complex. A partial converse is provided by Stallings' Embedding Theorem [31], which guarantees that any open PL n-manifold M^n which is homotopy equivalent to a finite k-complex K with $n-k \geqslant 3$ contains a k-dimensional subcomplex K' for which $K' \hookrightarrow M^n$ is a homotopy equivalence. A compact core may be obtained by taking a regular neighborhood of K'. (Recent work by Venema [32] shows that when $n-k=2$ a compact core need not exist.)

The following shows that compact cores obtained in the above manner are, in fact, geometric.

LEMMA 22. *Suppose K' is a k-dimensional subcomplex of an open PL n-manifold M^n, $K' \hookrightarrow M^n$ is a homotopy equivalence, and $n-k \geqslant 3$. Then any regular neighborhood C* of *K' in M^n is a geometric compact core.*

Proof. We must show that $\partial C \hookrightarrow M^n \backslash int(C)$ is a homotopy equivalence. It suffices to show that $\pi_i(M^n \backslash int(C), \partial C) = 0$ for all i.

For $i = 1$ the diagram

$$\begin{tikzcd} \pi_1(C) \arrow[r, "\cong"] & \pi_1(M^n) \\ \pi_1(\partial C) \arrow[u, "\cong"] \arrow[r] & \pi_1(M^n \backslash int(C)) \arrow[u, "\cong"] \end{tikzcd}$$

where the vertical isomorphisms are obtained by standard general position arguments, shows that $\pi_1(\partial C) \to \pi_1(M^n \backslash int(C))$ is an isomorphism, hence, $\pi_1(M^n \backslash int(C), \partial C) = 0$.

For $i > 1$, lift the inclusion $C \hookrightarrow M^n$ to an inclusion $\tilde{C} \hookrightarrow \widetilde{M^n}$ of universal covers which is also a homotopy equivalence. Hence, by excision, $H_i(\widetilde{M^n} \backslash int(\tilde{C}), \partial \tilde{C}) = H_i(\widetilde{M^n}, \tilde{C}) = 0$. Using the isomorphisms from the above diagram it is easy to see that $\widetilde{M^n} \backslash int(\tilde{C})$ and $\partial \tilde{C}$ are both connected and simply connected, so by the Hurewicz theorem, $\pi_i(\widetilde{M^n} \backslash int(\tilde{C}), \partial \tilde{C}) = 0$. It follows that $\pi_i(M^n \backslash int(C), \partial C) = 0$ ■

We now show that simply connected $\mathscr{Z}$-compactifiable homotopy collars are determined by their boundaries.

Theorem 23. *Suppose F^n and G^n are simply connected n-dimensional homotopy collars ($n > 4$) which admit $\mathscr{Z}$-compactifications having homeomorphic $\mathscr{Z}$-boundaries. Then F^n and G^n are homeomorphic.*

Corollary 24. *Suppose M^n and N^n are open n-manifolds ($n > 4$) admitting $\mathscr{Z}$-compactifications with homeomorphic $\mathscr{Z}$-boundaries. If M^n and N^n contain geometric cores C_M and C_N with simply connected boundaries, then $M^n \backslash int(C_M)$ and $N^n \backslash int(C_N)$ are homeomorphic.*

Remark 8. Theorem 18 can be obtained from this corollary. First, note that an embedded n-ball is a geometric compact core for any contractible n-manifold (apply Lemma 22). Hence, two contractible open n-manifolds ($n > 4$) with homeomorphic $\mathscr{Z}$-boundaries are homeomorphic on the complements of open n-balls. The Alexander trick may be used to extend this homeomorphism.

Proof of Theorem 23. Let $\widehat{F^n}$ and $\widehat{G^n}$ be the $\mathscr{Z}$-compactifications, $h: \partial_{\mathscr{Z}} F^n \to \partial_{\mathscr{Z}} G^n$ be a homeomorphism, and Z denote the common copy of $\partial_{\mathscr{Z}} M$ and $\partial_{\mathscr{Z}} N$ lying in $X = \widehat{F^n} \cup_h \widehat{G^n}$. Since ∂F^n and ∂G^n are compact and disjoint from Z, we may conclude from Theorem 9 that X is an n-manifold with boundary equal to $\partial F^n \cup \partial G^n$.

Claim. *The cobordism $(X, \partial F^n, \partial G^n)$ is homeomorphic to $(\partial F^n \times [0, 1], \partial F^n \times \{0\}, \partial \times \{1\})$.*

We will show that $(X, \partial F^n, \partial G^n)$ is a simply connected h-cobordism. The claim then follows from the classical (compact) h -cobordism theorem.

Since F^n and G^n are simply connected and the inclusions $\partial F^n \hookrightarrow F^n$, $F^n \hookrightarrow \widehat{F^n}$, $\partial G^n \hookrightarrow G^n$ and $G^n \hookrightarrow \widehat{G^n}$ are all homotopy equivalences, each of these spaces is simply connected. An argument similar to that used for Lemma 15 then shows that X is simply connected. Hence, by the Hurewicz and Whitehead theorems and duality, the claim follows once we show that $H_i(X, \partial F^n) = 0$ for all i.

Fix $i \geqslant 0$ and let α be a singular relative i-cycle for $(X, \partial F^n)$. We already know that $H_i(\widehat{F^n}, \partial F^n) = 0$, so it suffices to show that α is homologous to a cycle with support contained in $\widehat{F^n}$. Since $\widehat{F^n}$ is an ANR subset of X, there is an open neighborhood T of $\widehat{F^n}$ in X and a homotopy H of X which homotopes T onto $\widehat{F^n}$ (see Proposition 3.4 of [19]). Since $H^*(G^n, \partial G^n) \equiv 0$, duality for non-compact manifolds implies that $H_i^{lf}(G^n) = 0$ (H_*^{lf} denotes homology based on locally finite chains). By subdividing $\alpha' = \alpha \cap G^n$ finer and finer near Z, we may view α' as a locally finite i-cycle in G^n. Choose a locally finite $(i+1)$-chain β in U_N with $\partial \beta = \alpha'$. Since $G^n \backslash T$ is compact, a finite "subchain" of β provides a homology between α and some $\gamma \subset T$. We may now use H to push γ into $\widehat{F^n}$ completing the proof of the claim.

Next, consider the compact $(n+1)$ -manifold with boundary, $X \times [0, 1]$. By the above claim, we may identify $\widehat{F^n}$ and $\widehat{G^n}$ as subsets of $X \times \{1\}$. Since $\widehat{F^n} \hookrightarrow X \times \{1\} \hookrightarrow X \times [0, 1]$ are homotopy equivalences and each space is an ANR, $X \times [0, 1]$ strong deformation retracts onto $\widehat{F^n}$ (Theorems VII.8.1 and VII.2.1 of [19]). Similarly, $X \times [0, 1]$ strong deformation retracts onto $\widehat{G^n}$.

The remainder of the proof now mimics that of Theorem 18 with $X \times [0, 1]$ taking the place of B^{n+1}. In particular, one shows that $(X \times [0, 1] \backslash Z, F^n, G^n)$ is a relative proper h-cobordism (where F^n, G^n and Z are regarded as subsets of $X \times \{1\}$) and $(\partial F \times [0,1]) \cup (X \times \{0\}) \cup (\partial G \times [0,1])$ is already equipped with a product structure. In place of the radial structure on B^{n+1}, which was used in the proof of Theorem 18, one uses a collar structure on $\partial(X \times [0,1])$. To complete the proof, an easily obtainable "relative" version of the proper s-cobordism must be used since F^n and G^n have boundary. See p. 87 of [28] for a discussion of the compact version of this theorem. ■

Remark 9. A similar proof may be used to obtain variations on this theorem. For example, the assumption that F^n and G^n be simply connected may be weakened to an assumption that $Wh(\pi_1(F^n)) = 0$ if, in addition, we place conditions on the homeomorphism h to ensure that $(X, \partial F^n, \partial G^n)$ is an h-cobordism. If (in this case) we only wish to conclude that F^n and G^n contain homeomorphic neighborhoods of infinity, the comments immediately preceding Theorem 17 allow us to omit the assumption that $Wh(\pi_1(F^n)) = 0$ altogether.

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