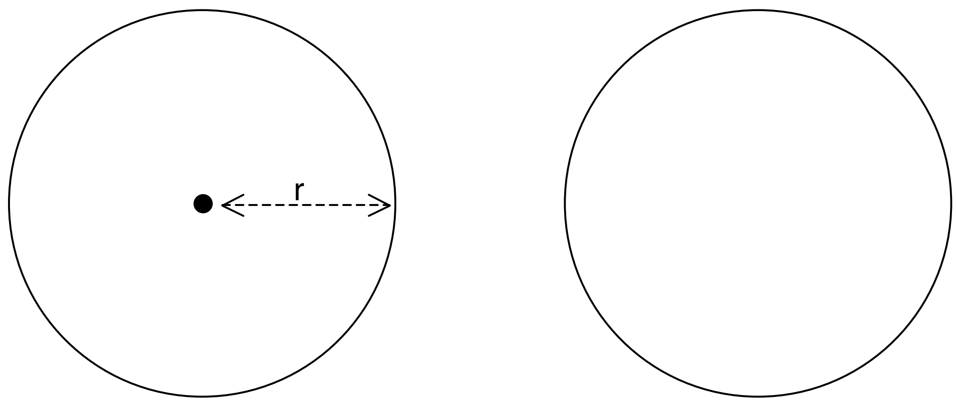
## Addendum to Lesson 22: The Surface Area of a Sphere

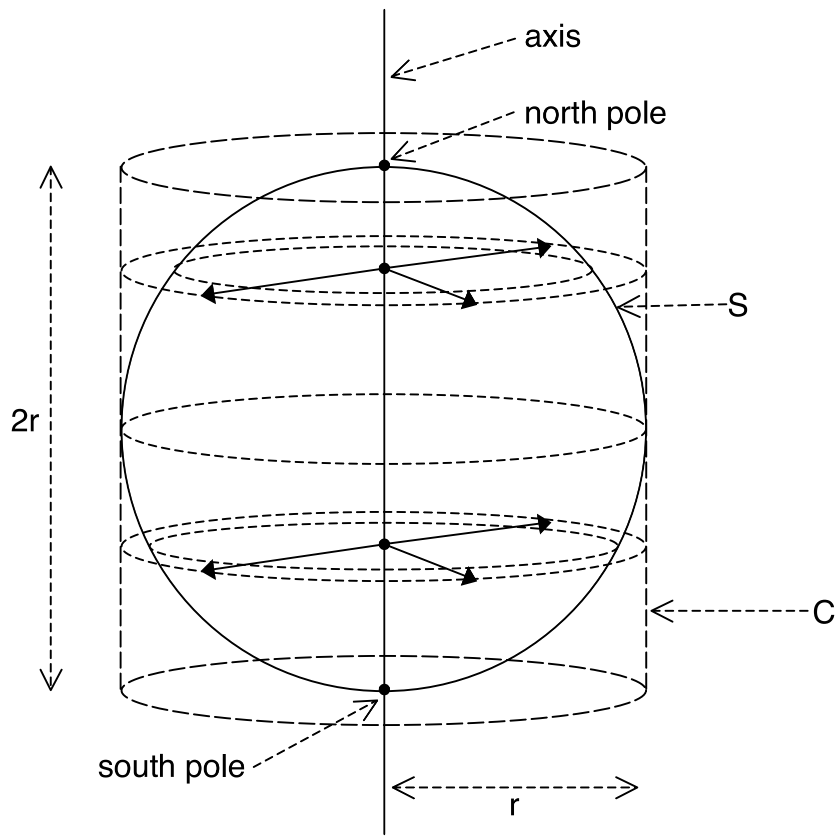
To understand the surface area of a sphere, we will first need to learn something about the surface area of a circular cylinder.

**Activity 9.** Each group should solve the following problem and report its solution to the class. Find a formula for the *lateral surface area* (i.e., the surface area of the side) of a cylinder that has a circular base of radius r and a height of h? To help solve this problem, cut out the three figures below (two circles of radius r and a rectangle of height h) and use tape to construct a cylinder with a circular base.




Our investigation of the areas of figures on spheres is based on an *amazing fact* that was discovered by Archimedes of Syracuse (287 BC to 212 BC). (The Syracuse mentioned here is not a city in the state of New York. It is a city in Sicily, the largest island in the Mediterranean Sea and a region of Italy.) Archimedes is regarded as one of the three greatest mathematicians of all time.

Consider a sphere S of radius r. Inscribe this sphere in a cylinder C which has a circular base of radius r and a height of 2r. The discovery of Archimedes implies (among other things) that the surface area of the sphere S is equal to the lateral surface area of the cylinder C in which it is inscribed.




**Activity 10.** The class as a whole should solve the following two problems.

**a)** Use the preceding fact to find a formula for the surface area of a sphere of radius r.

**b)** The radius of the Earth is 4000 miles (to the nearest 100 miles). What is its surface area?

To describe the full thrust of Archimedes' discovery we must first understand the *horizontal radial projection* of the sphere S onto the cylinder C. The horizontal radial projection moves every point of the sphere S except its north and south poles to a point on the lateral surface of the cylinder C. Under horizontal radial projection, each point on the sphere S moves radially away from the north-south axis of the sphere staying in the same horizontal plane until it lands on the lateral surface of the cylinder C. (The six solid arrows in the previous figure emanating from the north-south axis of S indicate the direction of horizontal radial projection.) Thus in the previous figure, each dashed parallel of latitude on the sphere S is moved by horizontal radial projection to the dashed circle in the lateral surface of the cylinder C that is on same horizontal level. Consequently, the equator of the sphere S is moved *to itself* by horizontal radial projection.

**Activity 11.** The class as a whole should solve the following problem. Let J be a meridian of longitude on the sphere S. (Thus J is a semicircle on S joining the north and south poles.) Suppose that the horizontal radial projection moves J onto a set K on the lateral surface of the cylinder C. Describe the set K in as much detail as possible.

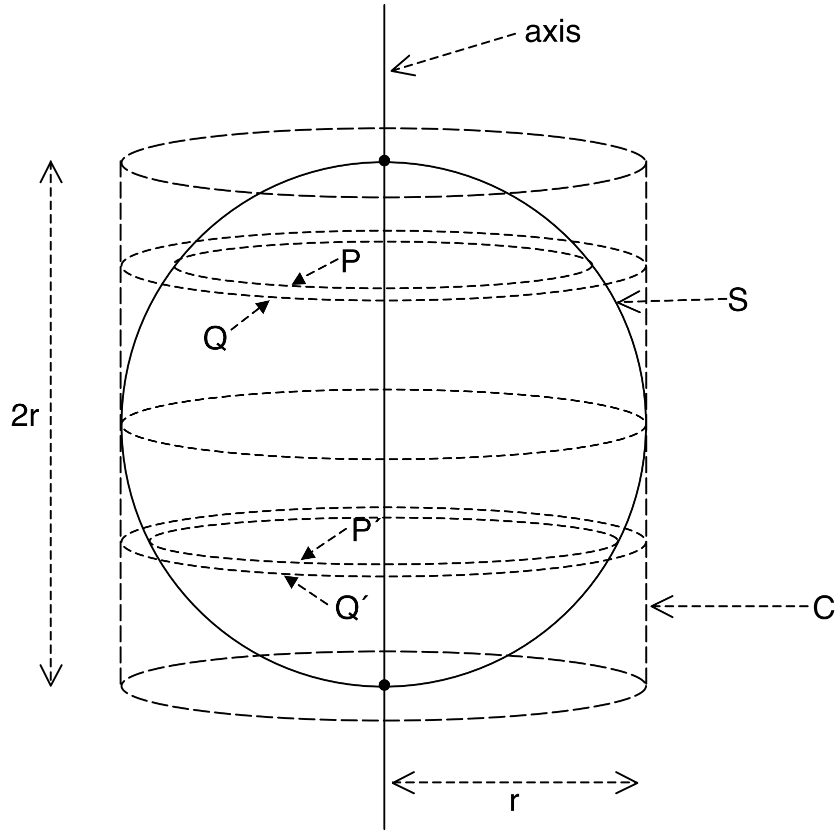
We are now ready to describe Archimedes' discovery.

**Archimedes' discovery:** Horizontal radial projection preserves area.

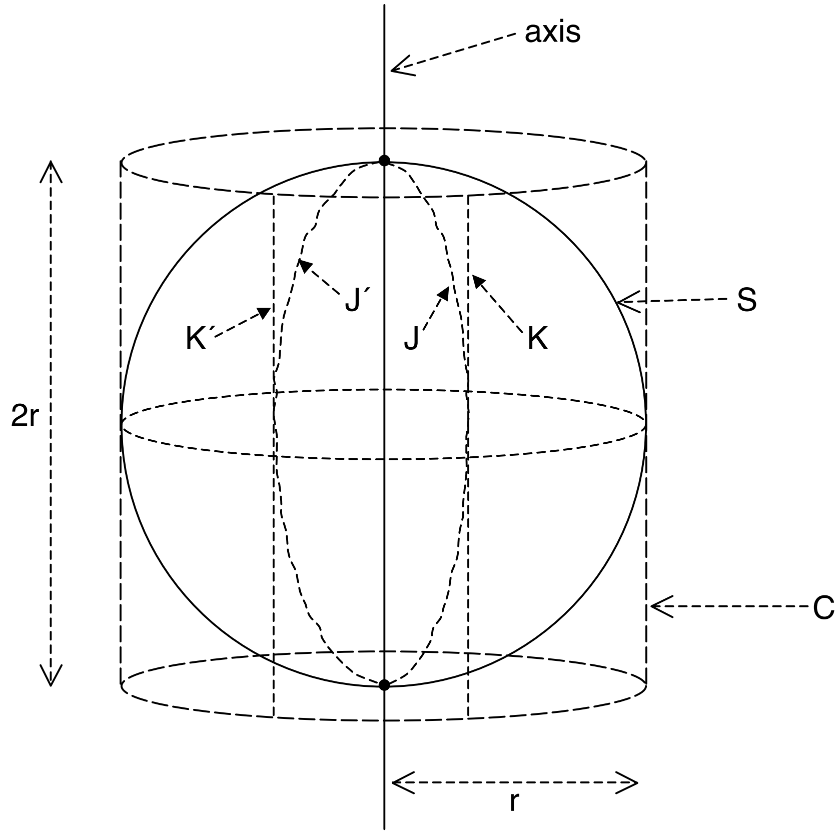
In other words, if A is any set on the sphere S and horizontal radial projection moves the set A to the set B on the lateral surface of C, then B has the same area as A.

Since horizontal radial projection moves the entire sphere S onto the entire lateral surface of C, then Archimedes' discovery tells us that the area of the sphere S equals the area of the lateral surface of C. (We stated this fact previously.)

Suppose that P and P´ are two parallels of latitude on the sphere S. Then horizontal radial projection moves P and P´ to two horizontal circles Q and Q´ on the lateral surface of C. Observe further that horizontal radial projection moves the portion of the sphere S that lies between parallels of latitude P and P´ onto the portion of the lateral surface of C that lies between horizontal circles Q and Q´. Hence, Archimedes' discovery tells us that the portion of S between P and P´ has the same area as the portion of the lateral surface of C between Q and Q´. Also observe that horizontal radial projection moves the portion of the sphere S that lies above the parallel of latitude P (including the north pole) onto the portion of the lateral surface of C that lies above the horizontal circle Q. Hence, Archimedes' discovery implies that the portion of S lying north of P has the same area as the portion of the lateral surface of C lying above Q. A similar observation leads to the conclusion that the portion of S lying south of P has the same area as the portion of the lateral surface of C lying below Q.




Next suppose that J and J´ are (non-antipodal) meridians of longitude on the sphere S. Then horizontal radial projection moves J and J´ to (non-antipodal) vertical line segments K and K´ on the lateral surface of C such that K is tangent to J and K´ is tangent to J´ at the equator. Furthermore, horizontal radial projection moves the (smaller) portion of the sphere S that lies between meridians of longitude J and J´ onto the (smaller) portion of the lateral surface of C that lies between vertical line segments K and K´. Hence, Archimedes' discovery tells us that the portion of S between J and J´ has the same area as the portion of the lateral surface of C between K and K´.




**Activity 12.** The class as a whole should discuss and solve the problems in parts a) through f) of this activity. For these problems assume that the radius of the Earth is 4000 miles (to the nearest 100 miles). **Note:** The remark following this activity contains information that is very useful for solving these problems.

**a)** What is the area of the portion of the Earth's surface that lies north of the equator and south of the 45° N parallel of latitude?

**b)** What proportion of the Earth's surface area lies between the 30° N parallel of latitude and the 30° S parallel of latitude?

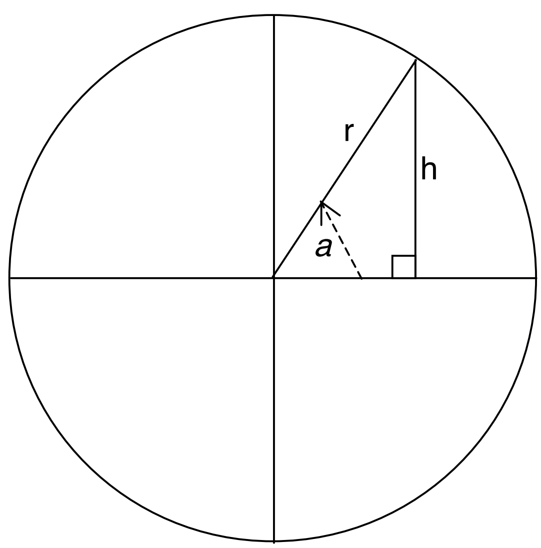
**c)** The Earth's southern polar cap is the portion of the Earth's surface that lies south of the 85° S parallel of latitude. What is the area of the Earth's southern polar cap?

**d)** What is the area of the (smaller) portion of the Earth's surface that lies between the 35° W meridian of longitude and the 115° W meridian of longitude?

**e)** What is the area of the portion of the Earth's surface that lies between the 95° E meridian of longitude and the 150° W meridian of longitude and contains the international date line?

**f)** What is the area of the (smaller) portion of the Earth's surface that is bounded by the 70ºS parallel of latitude, the 15º S parallel of latitude, the 80º E meridian of longitude and the 155º E meridian of longitude?

**Remark.** Suppose that the legs of a right triangle are horizontal and vertical, the length of the vertical leg is *h*, the length of the hypotenuse is *r*, and the measure of the angle opposite the vertical leg is *a* degrees. In the figure below, this triangle is drawn so that its hypotenuse is a radius of a circle of radius *r*, and the vertex of the triangle that has angle measure *a* is the center of the circle. The ratio *h/r* depends only on the angle measure *a.* The table below gives the ratios *h/r* associated with angle measures that are multiples of 5º. If you are given the radius *r* of a circle and the angle measure *a*, then you can use the associated value of *h/r* found in the table to calculate the height *h*. (If you know *r* and *a,* find the value of *h/r* in the table corresponding to this *a,* and calculate: $h = (h/r) \times r$.)




| *a* | *h/r* |
|---|---|
| 0º | 0.000000000 |
| 5º | 0.087155743 |
| 10º | 0.173648178 |
| 15º | 0.258819045 |
| 20º | 0.342020143 |
| 25º | 0.422618262 |
| 30º | 0.500000000 |
| 35º | 0.573576436 |
| 40º | 0.642787610 |
| 45º | 0.707106781 |
| 50º | 0.766044443 |
| 55º | 0.819152044 |
| 60º | 0.866025404 |
| 65º | 0.906307787 |
| 70º | 0.939692621 |
| 75º | 0.965925826 |
| 80º | 0.984807753 |
| 85º | 0.996194698 |
| 90º | 1.000000000 |

(Those students who have learned some trigonometry should recognize that the ratio *h/r* is simply the *sine* of the angle measure *a.*)

**Homework Problem 7.** For these problems assume that the diameter of the planet Mars is 6800 kilometers. Again the remark following Activity 12 contains information that is very helpful for solving these problems.

**a)** What is the area of the portion of Mars' surface that lies south of the equator and north of the 55º S parallel of latitude?

**b)** What proportion of Mars' surface area lies between the 60º N parallel of latitude and the 60º S parallel of latitude?

**c)** Mars' northern polar cap is the portion of Mars' surface that lies north of the 80º N parallel of latitude. What is the area of Mars' northern polar cap?

**d)** What is the area of the (smaller) portion of Mars' surface that lies between the 65º E meridian of longitude and the 140º E meridian of longitude?

**e)** What is the area of the (smaller) portion of Mars' surface that lies between the 170º W meridian of longitude and the 50º E meridian of longitude?

**f)** What is the area of the (smaller) portion of Mars' surface that is bounded by the 15ºS parallel of latitude, the 25º N parallel of latitude, the 25º E meridian of longitude and the 50º W meridian of longitude?