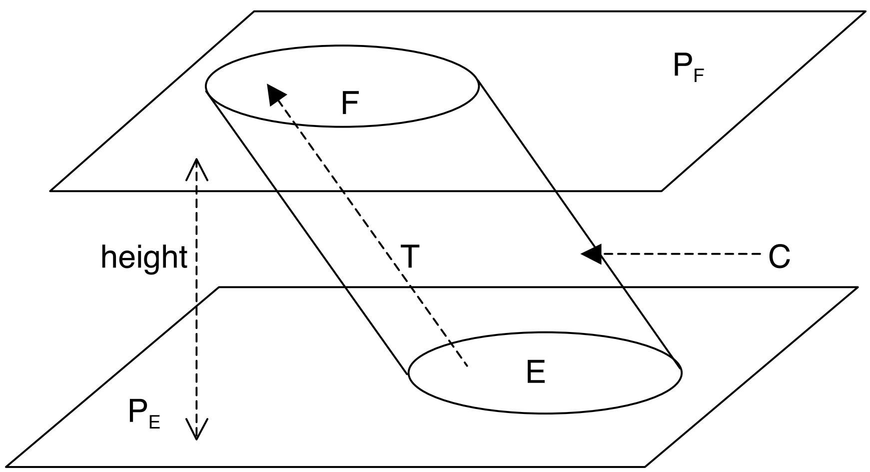
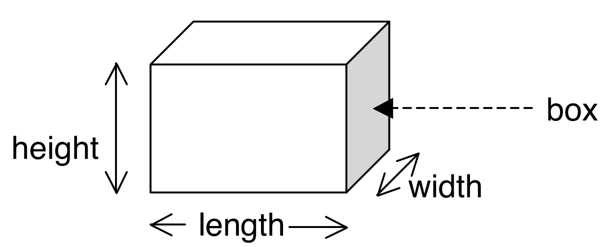
Lesson 22: Volumes of Cylinders, Cones and Balls

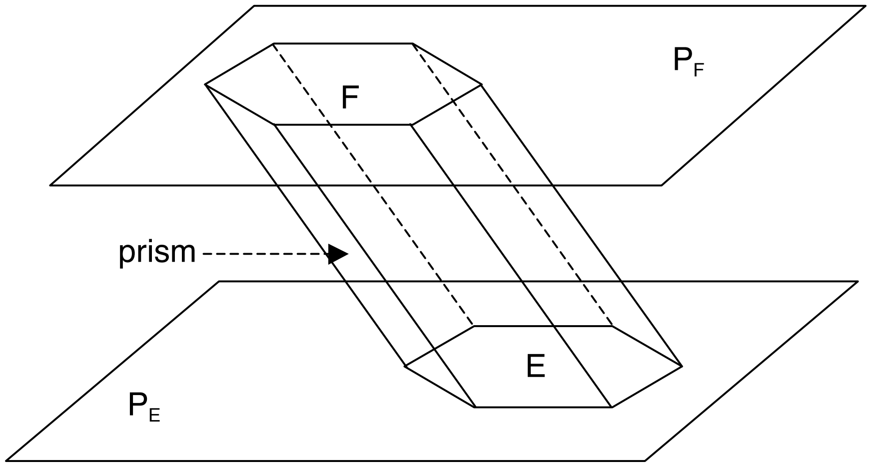
In this lesson we will investigate the volumes of three types of 3-dimensional figures: cylinders, cones and balls. We begin by defining these figures.

We use the term *cylinder* in a very general context. Every *cylinder* C is a figure in 3-dimensional space that has two special subsets called *bases.* The *bases* of C, which we denote by E and F, are subsets of disjoint planes, which we denote by P_E and P_F. (Thus, P_E and P_F are parallel planes.) Furthermore, there is a translation T of 3-dimensional space such that T(E) = F (and $T(P_E) = P_F$). Therefore, E and F are congruent planar sets in 3-dimensional space. The *cylinder* C is then the union of all the line segments $\overline{PT(P)}$ that join a point P in E to its translate T(P) in F. In other words, a point belongs to the cylinder C if and only if it lies on a line segment joining a point P in one base E to its translate T(P) in the other base F. The perpendicular distance between the two planes P_E and P_F is called the *height* of the cylinder C.

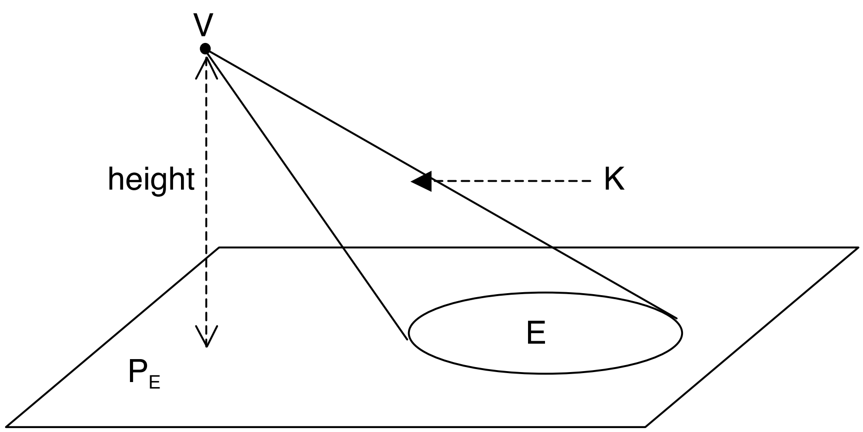


There are some special types of cylinders. If the bases of a cylinder are circular disks, then we call the cylinder a *circular cylinder.* (The cylinder pictured above could be a circular cylinder.) If the bases of a cylinder are polygonal disks, then we call the cylinder a *prism.* (A prism is pictured on the next page.) If the bases of the cylinder are rectangular disks and if the direction of motion of the translation moving one base to the other is perpendicular to the planes containing the bases, then the cylinder is called a *box*. The surface of the box is the union of six rectangles, two of which are the bases. The length, width and height of the box are called its *dimensions.*

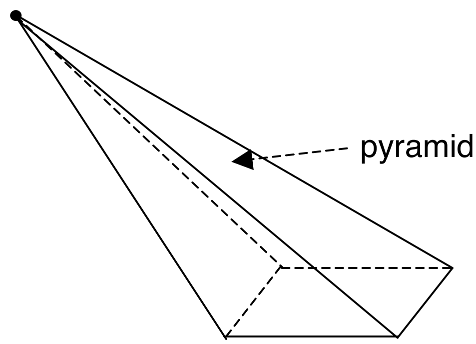


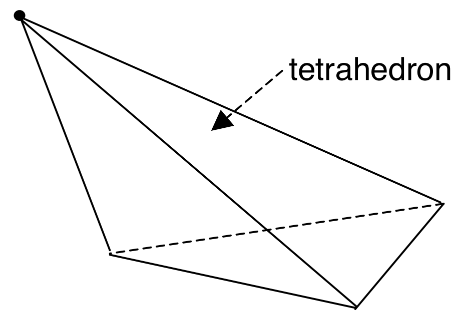


We also use the term *cone* in a very general context. Every *cone* K is a figure in 3-dimensional space that has a special subset called its *base* and a special point called its vertex. The *base* of K, which we denote by E, is a subset of a plane, which we denote by P_E. The *vertex* of K, which we denote by V, does not lie in the plane P_E. The *cone* K is then the union of all the line segments $\overline{VP}$ that join the vertex V a point P in E. In other words, a point belongs to the cone K if and only if it lies on a line segment joining the vertex V to a point P in the base E. The perpendicular distance between the vertex V and the plane P_E is called the *height* of the cone K.

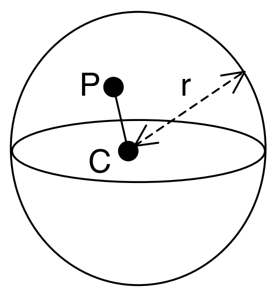


There are some special types of cones. If the base of a cone is a circular disk, then we call the cone a *circular cone.* (The cone pictured above could be a circular cone.) If the base of a cone is a quadrilateral disk, then we call the cone a *pyramid.* If the base of the cone is a triangular disk, then the cone is called a *tetrahedron.* (A pyramid and a tetrahedron are pictured on the next page.)





Recall that a *sphere* with center C and radius r is the 2-dimensional surface in 3-dimensional space consisting of all the points P such that CP = r. The *ball* with center C and radius r is the 3-dimensional figure consisting of all points P in space such that CP ≤ r. Thus the sphere is the surface of the ball.



Cylinders, cones, balls and many other 3-dimensional figures have well defined volumes. We now discuss this concept.

Volume is a function which assigns to a large class of 3-dimensional figures (including cylinders, cones and balls) a non-negative number called the *volume* of the figure. If S is a 3-dimensional figure, then we denote its volume by Vol(S). Thus, Vol(S) ≥ 0. Furthermore, the volume function satisfies the following four basic principles.

The Congruence Invariance Principle. If S and T are congruent 3-dimensional figures (i.e., there is a rigid motion of 3-dimensional space that moves S to T), then Vol(S) = Vol(T). In other words, if S ≅ T, then Vol(S) = Vol(T).

The Additivity Principle. If S is a 3-dimensional figure that is the union of finitely many *non-overlapping* 3-dimensional figures T_1, T_2, … , T_n, then

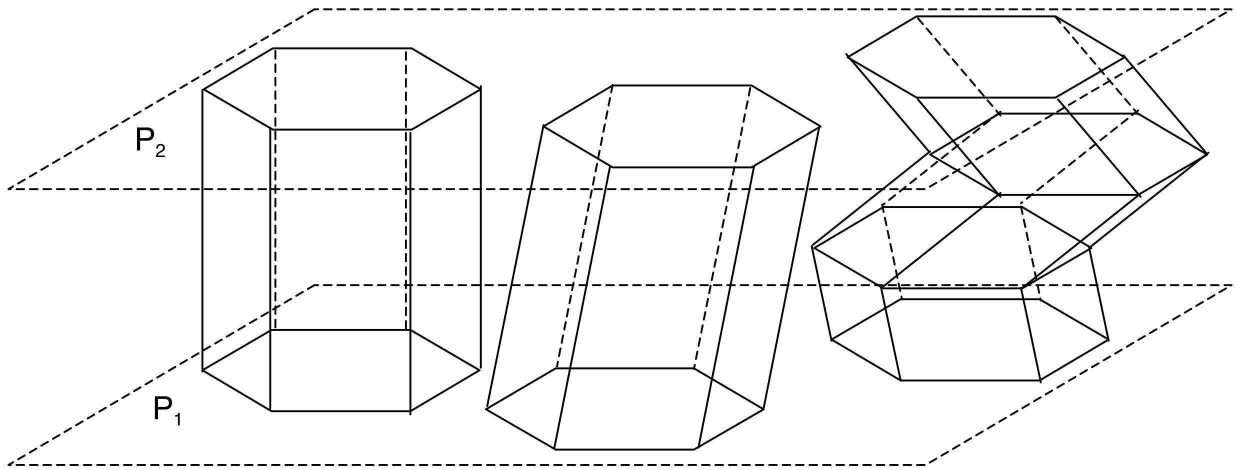
$$\mathrm{Vol}(S) = \mathrm{Vol}(T_1) + \mathrm{Vol}(T_2) + \ldots + \mathrm{Vol}(T_n).$$

(Two 3-dimensional figures are *non-overlapping* if their interiors are disjoint. Their surfaces may or may not touch each other, as long as their interiors are disjoint.)

The Standardization Principle. If S is a box with dimensions a, b and c, then Vol(S) = abc.

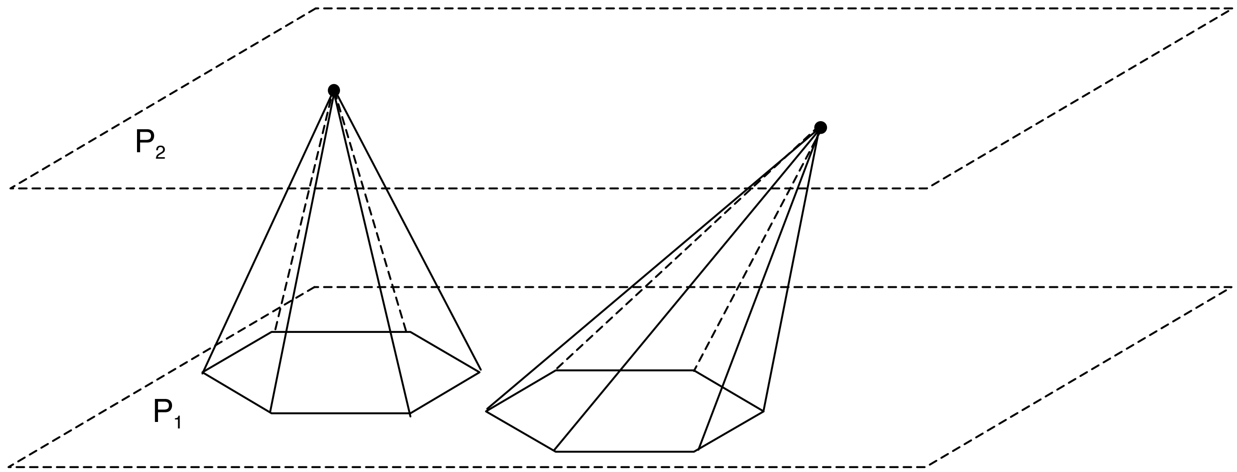
Cavalieri's Principle. Suppose that two 3-dimensional figures S and T lie between two parallel planes P_1 and P_2. If for every plane Q that lies between and is parallel to P_1 and P_2, the two cross-sections $Q \cap S$ and $Q \cap T$ have the same area, then Vol(S) = Vol(T).

Cavalieri's Principle implies that the three figures shown here (two cylinders and a "staggered cylinder") have the same volume. (All cross-sections are hexagons of the same area.)



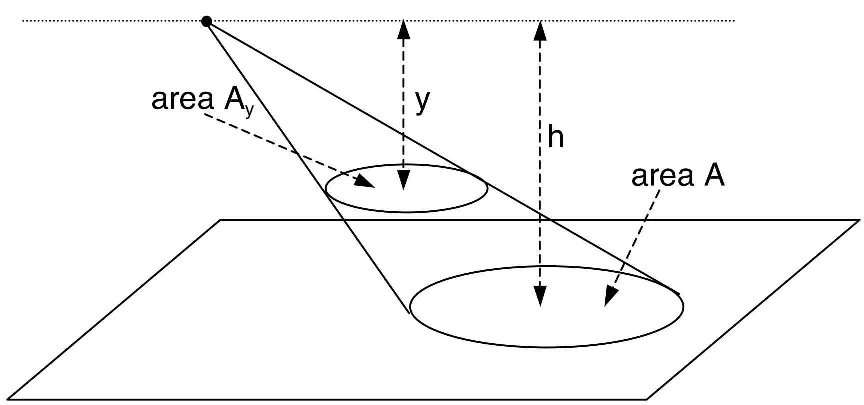
Suppose that C and D are cylinders and P_1 and P_2 are parallel planes such that P_1 contains one base of C and one base of D while P_2 contains the other base of C and the other base of D. If the bases of C have the same area as the bases of D, then Cavalieri's Principle implies that Vol(C) = Vol(D), even if C has a circular base and D has a square base. This is because: if the bases of C and D have the same area, then so do all the cross-sections of C and D determined by planes that are parallel to P_1 and P_2.

Cavalieri's Principle also implies that the two cones shown here have the same volume. These two cones have the same height and their bases have the same area. It then follows that at each height, the cross-sections of the two cones have the same area. Hence, Cavalieri's Principle applies to these two cones.



We expand on the information stated in the previous paragraph. Suppose a cone has height h and base area A. Consider a cross-section of the cone which lies on a plane, let y denote the perpendicular distance from the vertex of the cone to the plane, and let A_y denote the area of the cross-section. Then A_y is determined by the following following equation.

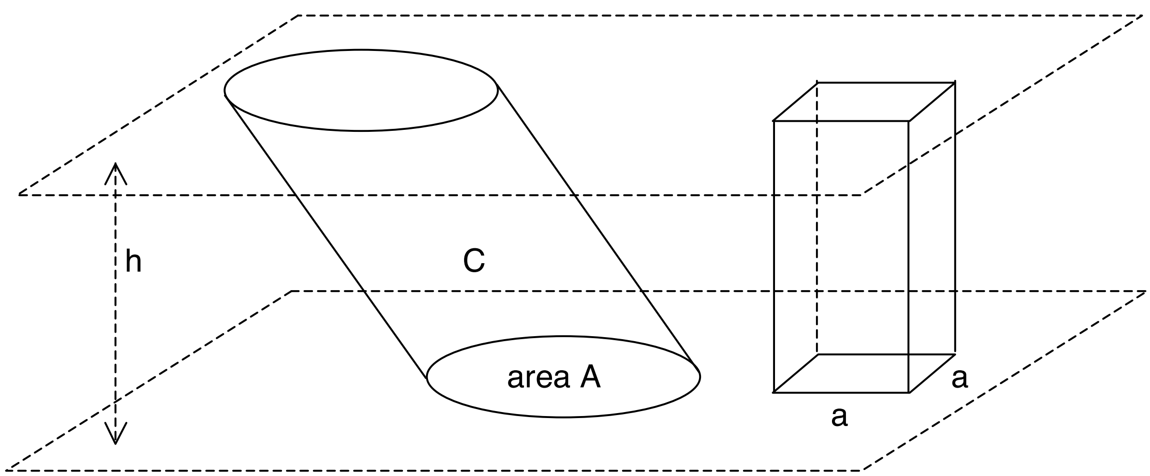
$$A_y = \left(y/h\right)^2 A.$$



Hence, if two cones both have height h and base area A, then their cross-sections at a (perpendicular) distance y from their vertices both have areas determined by this equation. Thus, these cross-sectional areas are equal. Therefore, in the situation of two cones with equal heights and equal base areas, Cavalieri's principle tells us that the two cones have equal volumes.

Activity 1. Each group should test the truth of Cavalieri's Principle by considering geometric models of two *cylinders* with the same base area and height, one of which is "straight" while the other is "slanted". Check whether the same amount of sand is needed to fill each model. Also test Cavalieri's Principle using geometric models of two *cones* with the same base area and height, one straight and one slanted.

Activity 2. The class as a whole should answer these questions. Let C be a cylinder with bases of area A and height h.
a) What are the areas of the cross-sections of C?
b) Let $a = \sqrt{A}$. Is there a box with length a, width a and height h?. If so, what are the areas of the cross-sections of this box?
c) What does Cavalieri's Principle tell you about the relation between the volume of the cylinder C and the box?
d) Can you use your answers to the previous parts of this activity to write a formula for the volume of the cylinder in terms of A and h?



Activity 3. The class as a whole should answer this question. Suppose C is a cylinder and K is a cone with the same base areas and heights. Fill K with sand and pour this sand into C. How many times must you repeat this process before C is filled with sand?

Activity 4. Each group should test the class's answer to Activity 3 with geometric models of a cylinder C and a cone K with the same base areas and heights. Fill K with sand and pour this sand into C as many times as it takes to fill C. Report this number to the class.

Activity 5. The class should write a formula for the volume of a cone in terms of the area A of its base and its height h.

Activity 6. The class as a whole should answer this question. Suppose H is a half-sphere of radius r and K is a circular cone with base radius r and height 2r. Fill H with sand and pour this sand into K. How many times must you repeat this process before K is filled with sand?

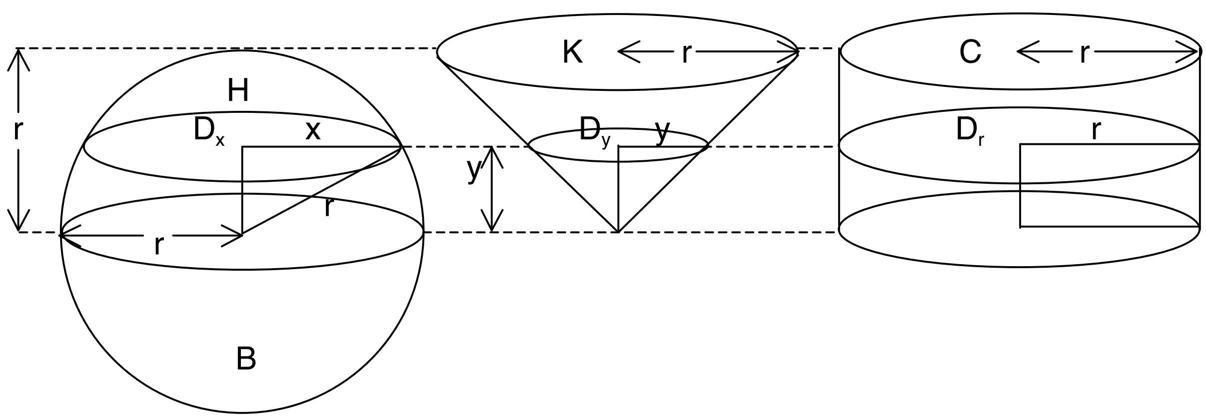
Activity 7. Each group should test the class's answer to Activity 6 with geometric models of a half-sphere H of radius r and a circular cone K with base radius r and height 2r. Fill H with sand and pour this sand into K as many times as it takes to fill C. Report this number to the class.

Activity 8. The class should write a formula for the volume of a ball of radius r in terms of r.

We have just discovered a formula for the volume of a ball of radius r by *experiment* based on already knowing formulas for the volumes of a cylinder and a cone. We can also derive this formula *theoretically* by using Cavalieri's Principle. (We will still need to know the formulas for the volumes of a cylinder and a cone.) We end this lesson by presenting this simple and elegant derivation.

Let B be a ball of radius r. Let H be the upper half-ball of B. Thus H is a half-ball of radius r. Let K be a cone of height r whose base is a circular disk of radius r. Let C be a cylinder of height r whose base is a circular disk of radius r.

Let H, K and C lie between two planes. The upper plane contains the north pole of H, the base of K and the upper base of C. The lower plane contains the "equatorial



base" of H (a circular disk of radius r), the vertex of K and the lower base of C.

Let $0 \le y \le r$, and consider the cross-sections of H, K and C at height y. The cross-section of H at height y is a circular disk D_x of radius x where $x^2 + y^2 = r^2$. The cross-section of K at height y is a circular disk D_y of radius y. The cross-section of C at height y is a circular disk D_r of radius r. Now:

$$\text{Area}(D_x) = \pi x^2, \quad \text{Area}(D_y) = \pi y^2, \quad \text{and} \quad \text{Area}(D_r) = \pi r^2.$$

Remembering that $x^2 + y^2 = r^2$, we have:

$$\text{Area}(D_x) + \text{Area}(D_y) = \pi x^2 + \pi y^2 = \pi(x^2 + y^2) = \pi r^2 = \text{Area}(D_r).$$

Thus, the areas of the cross-sections of H and K at height y add up to the area of the cross-section of C at height y. In other words, the area of the cross-section of $H \cup K$ at height y equals the area of the cross-section of C at height y. Therefore, Cavalieri's Principle tells us that

$$\text{Vol}(H \cup K) = \text{Vol}(C).$$

The Additivity Principle tells us that

$$\text{Vol}(H \cup K) = \text{Vol}(H) + \text{Vol}(K).$$

Thus,

$$\text{Vol}(H) + \text{Vol}(K) = \text{Vol}(C).$$

Since H is one-half of the ball B of radius r, then

$$\text{Vol}(H) = ({}^1/_2)\text{Vol}(B).$$

Hence,

$$({}^1/_2)\text{Vol}(B) + \text{Vol}(K) = \text{Vol}(C).$$

Since K is a cone with height r and with base area πr^2, then

$$\text{Vol}(K) = ({}^1/_3)(\pi r^2)r = ({}^1/_3)\pi r^3.$$

Since C is cylinder with height r and with base area πr^2, then

$$\text{Vol}(C) = (\pi r^2)r = \pi r^3.$$

Therefore,

$$({}^1/_2)\text{Vol}(B) + ({}^1/_3)\pi r^3 = \pi r^3.$$

Hence,

$$({}^1/_2)\text{Vol}(B) = \pi r^3 - ({}^1/_3)\pi r^3 = ({}^2/_3)\pi r^3.$$

Thus,

$$\text{Vol}(B) = 2({}^2/_3)\pi r^3 = ({}^4/_3)\pi r^3.$$

This completes our *proof* of the formula for the volume of a ball B of radius r

$$\text{Vol}(B) = ({}^4/_3)\pi r^3$$

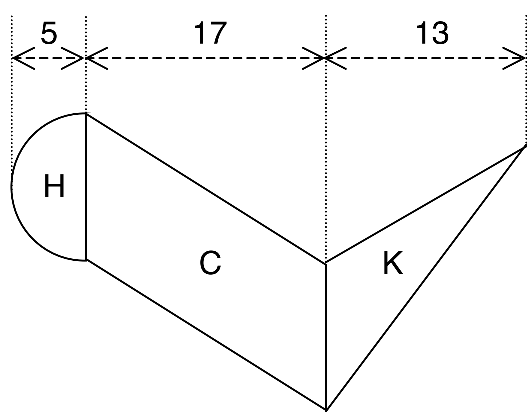
based on Cavalieri's Principle and previous knowledge of the formulas for the volumes of a cylinder and a cone.

Homework Problem 1. One of the great pyramids in Egypt has a square base whose sides are 900 feet long and its height is 400 feet. The pyramid is solid except for a huge excavated burial chamber in its interior which is a box that is 300 feet long, 200 feet wide and 50 feet high. (All measurements are to the nearest foot.) What is the volume of the solid part of the pyramid?

Homework Problem 2. A twisted column 10 feet high is constructed from a large number of thin square cement plates that are 3 feet long on each side. As the column rises, the square plates twist in the counterclockwise direction. The squares complete two complete 360° twists as they rise from the bottom of the column to the top. What is the volume of the column?

Homework Problem 3. A large teepee has a circular base of radius 12 feet and its height is 18 feet. Suppose that a circular ceiling has been installed in the teepee 10 feet above the floor. What is the volume of the portion of the teepee that is below the ceiling?

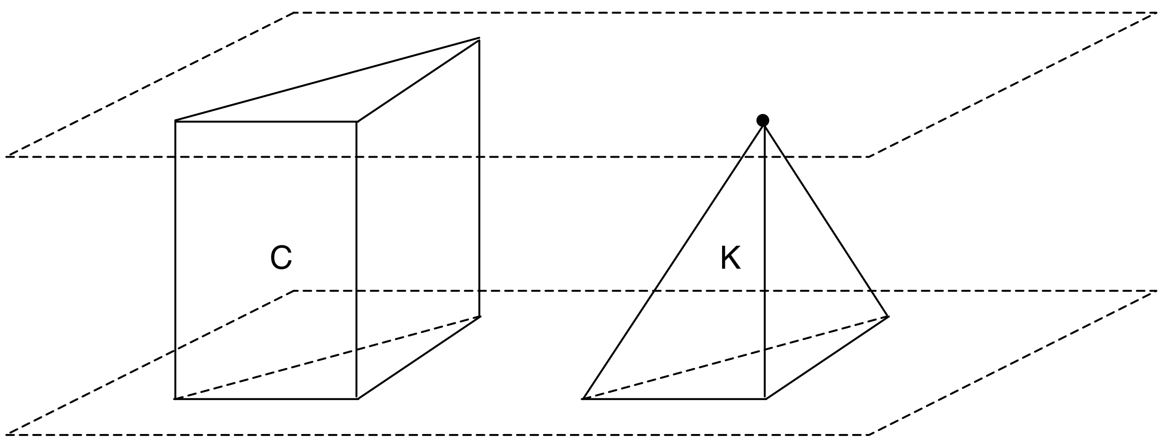
Homework Problem 4. Consider a 3-dimensional figure that consists of a slanted cylinder C with a circular base, a slanted cone K with a circular base that is mounted on one base of C, and a half ball H mounted on the other base of C. Suppose that the radius of the base of the cylinder C, the radius of the base of the cone K and the radius of the half ball H are all the same: 5 inches. Also suppose that the cone K and the half ball H are mounted on the two bases of the cylinder C so that the two bases of C are exactly covered by K and H. In other words, K and H are mounted on the bases of C so that no portion of either base of C is exposed. Furthermore, the height of the cylinder C is 17 inches and the height of the cone K is 13 inches. Here is a 2-dimensional schematic picture of this figure:



What is the volume of this 3-dimensional figure?

Homework Problem 5. Suppose that aliens from galaxy far far away come to Earth with a giant straight knife and they slice a wedge out of the Earth in the same way that you would slice a wedge from an orange. They make one straight cut along the 24º W meridian of longitude, and they make a second straight cut along the 57º E meridian of longitude. (The cuts are planar – i.e. flat – and end at the Earth's axis, just as the cuts that remove a wedge from an orange run straight from meridians of longitude on the orange's surface to its axis.) The aliens then remove this wedge and haul it back to their galaxy. The radius of the Earth is 4000 miles (to the nearest 100 miles). What is the volume of the wedge?

Homework Problem 6. Let C be a cylinder and K a cone of equal heights and both with congrent triangular bases, as shown here. Prove that Vol(C) = 3Vol(K) by decomposing C into three cones of equal volume, one of which is a copy of K.



Make models of the three cones out of cardstock. Demonstrate that they can be assembled to make a copy of C. Also demonstrate that in each pair of cones, bases can be chosen so that the bases are congruent and the heights (on these bases) are equal.

Hint: