## Addendum to Lesson 21: Estimating the Circumference of the Earth in Milwaukee on any Afternoon of the Year

Review. First we briefly review Eratosthenes' measurement of the circumference of the Earth. In the third century BC Eratosthenes, the chief librarian of the great Library at Alexandria, Egypt, made a remarkably accurate estimate the circumference of the Earth. He accomplished this by estimating the circumference of the great circle that contains the meridian of longitude that passes through Alexandria. He knew that this great circle crossed the Tropic of Cancer 5000 stades ( $=500$ miles ) south of Alexandria in a town called Syene. Thus, he knew that at solar noon on the day of the summer solstice the Sun was directly overhead in Syene. At solar noon on the day of the summer solstice, Eratosthenes measured the angle that the Sun's rays made with the vertical in Alexandria and got a value of $7.2^{\circ}$. The geometry of this situation (see the figure on the next page) yields the equation

$$
\frac{c}{500}=\frac{360^{\circ}}{7.2^{\circ}}
$$

where c represents the circumference of the Earth in miles. Solving this equation for c yields

$$
c=\left(\frac{360}{7.2}\right) 500=50 \times 500=25,000 \text { miles }
$$

Thus, Eratosthenes' estimate of the circumference of the Earth was 25,000 miles, a value which is within one half of $1 \%$ of modern. (In fact, we now know that there were errors of about 7\% in each of the two values that Eratosthenes used as inputs for his estimate - the distance from Alexandria to the Tropic of Cancer, and the angle between the Sun's rays and the vertical at solar noon on the day of the summer solstice in Alexandria. Fortuitously, these two errors cancelled each other to give an estimate of spectacular accuracy.)


Solar Noon in Alexandria and Syene on the Day of the Summer Solstice

## Estimating the Circumference of the Earth in Milwaukee on any Afternoon of the Year.

Our estimation process, like Eratosthenes', will measure the circumference of the Earth by estimating the circumference of a particular great circle. Like the great circle used by Eratosthenes, our great circle is determined by the intersection of a plane with the surface of the Earth. The plane which determines the circle used by Eratosthenes has specific properties which make it useful in measuring the Earth's circumference in Alexandria. We will review these properties, and then modify them to determine a plane that can be used to measure the Earth's circumference in Milwaukee.

The plane that generates the great circle that was measured by Eratosthenes has three salient properties.

- The plane passes through center of the Earth. This property is necessary to insure that the plane intersects the surface of the Earth in a great circle.
- The plane passes through the center of the Sun. This property insures that a light ray traveling from the Sun to a point on the great circle remains in the plane rather than intersecting the plane at a single point. Such rays can then be represented by lines which remain in the plane as in the figure above.
- The plane passes through Alexandria. This property insures that the plane contains the line from the center of the Earth to Alexandria. This line represents the vertical direction in Alexandria. Also the plane contains the line from the Sun to Alexandria. Hence, the angle between the Sun's rays and the vertical in Alexandria lies in the plane. Having this angle lie in the plane is essential for carrying out the geometric argument that led to Eratosthenes' estimate of the circumference of the Earth.

To summarize: the three points - the center of the Earth, the center of the Sun and Alexandria - must lie on the plane that generates the great circle measured by Eratosthenes. Furthermore, this plane is the only plane that passes through these three points. In other words, the plane is uniquely determined by the three points.

In our endeavor to measure the Earth's circumference from Milwaukee, we will imitate Eratosthenes' approach. We will measure the circumference of a great circle that passes through Milwaukee. This great circle will arise from the intersection of a plane with the surface of the Earth. By analogy with Eratosthenes' method, this plane will pass through three points.

- The plane passes through center of the Earth. This property insures that the plane intersects the surface of the Earth in a great circle.
- The plane passes through the center of the Sun. This property insures that light rays traveling from the Sun to a point on the great circle can be represented by lines which remain in the plane.
- The plane passes through Milwaukee. This property insures that the angle between the Sun's rays and the vertical in Milwaukee lies in the plane. Having this angle lie in the plane will be essential for carrying out the geometric argument that will lead to our estimate of the circumference of the Earth.

Hence, the unique plane determined by the three points - the center of the Earth, the center of the Sun and Milwaukee - intersects the Earth in a great circle. This great circle is the one whose circumference we will estimate to measure the circumference of the Earth. Let $P$ denote the plane determined by the three points - the center of the Earth, the center of the Sun, and Milwaukee, and let $C$ denote the great circle in which $P$ intersects the Earth's surface.

On the next page is a picture of the plane $P$ determined by the center of the Earth, the center of the Sun and Milwaukee and the great circle C in which this plane intersects the Earth's surface.


The Great Circle C Generated by the Plane P Which Passes Through the Center of the Earth, The Center of the Sun, and Milwaukee

Once Eratosthenes had focused his attention on the great circle generated by the plane passing through the center of the Earth, the center of the Sun and Alexandria, he then used some special knowledge about a point on this great circle that had a particularly nice geometric property. The point to which I'm referring is the one that is closest to the Sun. The nice geometric property that this point possesses is that the Sun's rays approach it vertically - from directly overhead. The special knowledge that Eratosthenes had about this point was that he knew the location of the point (at the moment of solar noon on the day of the summer solstice) and he knew the distance of this point from Alexandria. (Actually, Eratosthenes believed slightly erroneously that this point was at the town of Syene, and he had estimated the distance from Alexandria to Syene.) Once he knew the distance from Alexandria to this special point, and he had measured the angle between the Sun's rays and the vertical in Alexandria (at solar noon on the day of the summer solstice), he used a simple bit of geometric reasoning to arrive at an estimate of the Earth's circumference.

To complete our estimate of the Earth's circumference, we must imitate Eratosthenes and choose a point with a nice geometric property on the great circle C generated by the plane $P$ that passes through the center of the Earth, the center of the

Sun and Milwaukee. In the figure on the previous page, I have indicated three candidates for this point. I have marked three points on $C$ that have nice geometric properties: the point closest to the Sun and the two points on the terminator - sunrise and sunset. The point closest to the Sun has the property, stated above, that the Sun's rays approach it vertically. The points of sunrise and sunset have the property that the Sun's rays approach these points horizontally - perpendicular to the vertical. If, at the moment we measure the angle the Sun's rays make with the vertical in Milwaukee, we can also acquire knowledge of the distance from Milwaukee to one of these points, then we, like Eratosthenes, can arrive at an estimate of the Earth's circumference through a simple bit of geometric reasoning.

The challenge for us is to acquire this distance information. The problem is complicated by the fact that our great circle C moves along the Earth's surface as the Earth rotates about its axis. We can visualize this by imagining the line $L$ determined by the centers of the Earth and the Sun. The plane $P$ that generates our great circle $C$ contains this line $L$ and passes through Milwaukee. So as the Earth rotates about its axis and Milwaukee traverses its parallel of latitude, the plane $P$ pivots back and forth on the line L . The extreme positions of the pivoting plane P occur when Milwaukee passes through the terminator - at sunrise and sunset. The middle position of the pivoting motion of the plane $P$ occurs when Milwaukee is at the point in its parallel of latitude that is either closest to or farthest from the Sun - at solar noon or solar midnight. (Solar noon and solar midnight in Milwaukee are the only two moments of the day when the plane $P$ passes through the north and south poles and the great circle $C$ contains the meridian of longitude through Milwaukee.) Try to visualize the motion of the plane P as the Earth rotates. Using a globe may help.

It would simplify our task if we could take advantage of the alignment that occurs at solar noon between the great circle C and the meridian of longitude through Milwaukee. (Eratosthenes took advantage of exactly this alignment.) To avail ourselves of this coincidence, we would have to perform our measurement of the angle between the Sun's rays and the vertical at the moment the alignment occurs - solar noon. Unfortunately, our class does not meet at this time. So we must instead confront the more general situation that arises if we perform the measurement of the angle between the Sun's rays and the vertical at a time with no special geometric significance when the great circle C doesn't coincide with any well known great circle on the Earth's surface. Our inability to perform our measurement at solar noon complicates our procedure, but does not make it impossible, as we will see.

We must decide which of the three special points on the great circle $C$ - sunrise, sunset or the point closest to the Sun - to use in our procedure. (Eratosthenes used the point closest to the Sun.) Because our class meets in the late afternoon when the sunset point is closer to Milwaukee than the other two points, we will use the sunset point in our procedure. (As we will see below, the nature of our procedure makes it
easier to work with a point that is closer to Milwaukee. If the class met early in the morning we might choose to work with the sunrise point; while if the class met near noon, we might choose to work with the point closest to the Sun - where solar noon is occurring.)

Now let's reconsider the picture of the plane $P$ that passes through the center of the Earth, the center of the Sun and Milwaukee and that intersects the Earth's surface in the great circle C. Of the three geometrically interesting points on C, I have labeled only the sunset point. Let $v$ denote the measure of the angle between the Sun's rays and


The Plane P and the Great Circle C
the vertical in Milwaukee. Then $v$ is also equal to the measure of the angle between two lines meeting at the center of the Earth: one joining the center of the Earth to the center of the Sun, and the other joining the center of the Earth to Milwaukee. Also let $z$ denote the measure of the angle between two lines meeting at the center of the Earth: one joining the center of the Earth to Milwaukee, and the other joining the center of the Earth to the sunset point on the great circle C. From the figure above, we see that

$$
z=90^{\circ}-v
$$

Let $d$ denote the distance along the great circle C from Milwaukee to the sunset point on C. Also let $c$ denote the circumference of the Earth $=$ the circumference of the great circle $C$. From the figure above, we obtain the equation

$$
\frac{c}{d}=\frac{360}{z}
$$

Hence,

$$
c=\left(\frac{360}{z}\right) d
$$

Hence, to obtain an estimate of the circumference of the Earth c, one needs to measure $z$ and $d$. It it easy to calculate $z$ : first measure the angle between the Sun's rays and the vertical in Milwaukee and call this angle measure $v$, and then set $z=90^{\circ}-v$. The difficult part of this estimate is the value of $d$.

We now discuss the challenges posed by measuring the distance $d$ along the great circle C from Milwaukee to the sunset point at the same moment we determine the measure $v$ of the angle between the Sun's rays and the vertical in Milwaukee. We face four obstacles.

- We must identify the great circle $C$ at the moment we perform our measurement of $v$.
- We must represent C on a map.
- We must find the point on the map that represents the sunset point on $C$ at the moment we performed the measurement of $v$.
- We must calculate the distance from Milwaukee to the sunset point.

How can we identify the location of the great circle $C$ on the Earth's surface at the moment we measure the angle between the Sun's rays and the vertical in Milwaukee? The answer is that the shadow of a vertical line at Milwaukee lies along the great circle C. This is because the plane $P$ contains: the vertical line at Milwaukee, the rays from the Sun that pass through points of this vertical line, and the great circle $C$ which goes through the foot of this vertical line. Normally, the rays of the Sun pass through the vertical line and illuminate part of the great circle C. However, if we erect a stick along the vertical line, then the stick blocks the rays of the Sun and the shadow of the stick traces out part of the great circle C. Thus, the shadow of a vertical stick in Milwaukee points along the great circle C . This shadow allows us to identify the great circle C by taking its compass heading.

To use the shadow of a vertical line to measure the compass heading of the great circle C, first draw a straight line on a piece of paper and write "N" (for north) at one end of the line. Put the piece of paper in a clipboard, place the clipboard on the ground, and use a magnetic compass to align the clipboard so that the line points to magnetic north with the " N " at the north end of the line. Tie a weight to one end of a piece of string and hold the other end to create a vertical line that casts a shadow across the clipboard, and trace this shadow on the piece of paper. Write " S *" (for "away from the Sun" or "toward the sunset point") at the end of this line that is farthest from the Sun. Since the Sun lies in the southwest part of the sky in the afternoons in Milwaukee,
then this line will point from southwest to northeast with the " $S^{*}$ " at the northeast end. Now carefully measure the angle between line which points toward magnetic north ("N") and the line which traces the shadow of the vertical and points toward the sunset point ("S*"). Unfortunately magnetic north doesn't coincide with true north, because the magnetic north pole and the true north pole are at different locations. ${ }^{1}$ From the perspective of Milwaukee in 2010, the magnetic north pole lies about $3^{\circ} 33^{\prime}$ west of the true north pole. So our initial angle measure must be corrected by moving the north pole $3^{\circ} 33^{\prime}$ to the east. Since the angle we are measuring lies to the east of north, this has the effect of decreasing the angle measure by $3^{\circ} 33^{\prime}$. Let a denote this angle measure. Thus $a=$ the initial angle measure minus $3^{\circ} 33^{\prime}$.

We remind the reader that the great circle C moves on the Earth's surface as the Earth rotates about its axis. We are interested in the position of $C$ at the moment we measure the angle between Sun's rays and the vertical in Milwaukee and thereby obtain the values $v$ and $z$. Hence, at the exact same moment we measure $v$ and $z$, we must also measure the angle between magnetic north and the shadow of the vertical in Milwaukee to obtain the value a. Also we must record the date and exact time that we perform these simultaneous measurements. Our procedure will require the date and time information as well as the values of $z$ and $a$.

We now know that the great circle C runs through Milwaukee making an angle of a degrees to the east of true north. We would like to represent $C$ on a map by a line through Milwaukee. Unfortunately, if one draws a straight line through Milwaukee on a standard wall map (e.g., a Mercator projection) with a heading of a degrees to the east of true north, that straight line will not represent a great circle. In general, straight lines on wall maps do not represent great circles. ${ }^{2}$ To overcome this difficulty, we have

[^0]http://www.ngdc.noaa.gov/geomag/GeomagneticPoles.shtml and
http://www.ngdc.noaa.gov/geomagmodels/Declination.jsp.
In January, 2011, the shift of the Earth's magnetic poles caused the closure of a runway at the Tampa International Airport so that numbers painted on the runway giving its compass heading could be updated. (See:
http://www.dailymail.co.uk/sciencetech/article-1344899/Shift-magnetic-north-pole-affects--Tampa-
airport.html?ito=feeds-newsxml.)
${ }^{2}$ On a typical wall map (Mercator projection), the only straight lines that represent great circles are the horizontal line that represents the equator and the vertical lines that represent meridians of longitude. Horizontal lines other than the line representing the equator represent parallels of latitude that are not great circles. Lines that are neither horizontal nor vertical represent curves on the Earth's surface that spiral up to the north pole and down to the south pole, and are clearly not great circles.
arranged for the creation of a special map called an Equidistant Azimuthal Projection Centered on City of Milwaukee. ${ }^{3}$ This map has several special properties that make it ideal for our purposes.

- Any straight line drawn through the point on the map representing Milwaukee represents a great circle on the Earth's surface.
- If two lines are drawn on the map through the point representing Milwaukee, then the measure of the angle they make is the same as the measure of the angle made on the Earth's surface by the two great circles which these lines represent.
- There is a fixed scale factor that relates distance measured on the map between the point representing Milwaukee and any other point A on the map to the distance on the surface of the Earth between Milwaukee and the point on the Earth's surface represented by A.
- Parallels of latitude and meridians of longitude are drawn on this map, so that the latitude and longitude coordinates of any point on the map can be determined to within less than one degree.

You will find an image of this map on the final page of this addendum. (Make an enlarged copy of this image if possible.) On the enlarged copy of the map, draw a line through the point representing Milwaukee that makes a heading of a degrees to the east of north. Call this line $\mathrm{C}^{*}$. The line $\mathrm{C}^{*}$ represents the great circle C .

We must now find the point on $\mathrm{C}^{\star}$ that corresponds to the sunset point on C . To accomplish this we will consult the website
http://www.usno.navy.mil/USNO/astronomical-applications/data-services/rs-one-dayworld.

This website allows us to enter the date that we performed the measurements of the quantities $z$ and $a$, and Milwaukee's time zone. (Milwaukee's time zone is 6 hours west of Greenwich if daylight savings time was not in effect when the measurements were taken; but if daylight savings was in effect, then Milwaukee's time zone is 5 hours west of Greenwich.) If we then enter the latitude and longitude coordinates of any point, the website will tell us the time that sunset occurred at the given point in terms of Milwaukee time. Our procedure is to read the latitude and longitude coordinates of points on the line $\mathrm{C}^{*}$ from the map, and to enter them into the website to learn their sunset times. We continue the procedure until we find the point on $\mathrm{C}^{*}$ at which sunset occurred at the same moment that that we performed the measurements of the quantities $z$ and $a$. This point is the sunset point. We have now obtained the latitude and longitude coordinates of the sunset point.

[^1]http://www4.uwm.edu/cgis/.

Finally, we must calculate the distance $d$ from Milwaukee to the sunset point along the great circle C. ( $d$ is also the distance from Milwaukee to the sunset point along the surface of the Earth.) To calculate $d$, go to the website

## http://www.chemical-ecology.net/java/lat-long.htm.

Enter the latitude and longitude coordinates of Milwaukee ${ }^{4}$, and enter the latitude and longitude coordinates that we just discovered for the sunset point. The website will then tell us the distance $d$ between these two points.

Once we have determined that value of $d$, we use it together with the value for $z$ found earlier to calculate our estimate of the circumference $c$ of the Earth from the equation

$$
c=\left(\frac{360}{z}\right) d
$$

Activity 3. Carry out the procedure just described to estimate the circumference of the Earth. Below we have rewritten this procedure. We have broken it into seven numbered steps. The steps provide more detail about how to use the websites and some other issues. Also they may be easier to follow than the explanation given above when you are actually carrying out the estimation process.

## Procedure for Estimating the Circumference of the Earth in Milwaukee

1. Hold a clipboard with a piece of paper on it vertically and aim it toward the Sun. (The plane of the clipboard should then contain a vertical line and the line from your eye to the Sun.) Hold a thin rod (e.g., a pencil) perpendicular to the clipboard touching the clipboard near the edge closest to the Sun. Trace the line created by the rod's shadow on the paper, and write " $S$ " (for "Sun") at the end of this line that is nearest the Sun. Also hang a string with a weight on it down the middle of the clipboard to create a vertical line and trace this line, and write "U" (for "up") at the upper end of this line.
2. Draw a line up the middle of a piece of paper and write " $N$ " (for "north") at the top end of the line. Put the piece of paper in a clipboard and place the clipboard on the ground.
[^2]Use a magnetic directional compass to align the clipboard so that the line points to magnetic north with the " N " at the north end of the line. Tie a weight to one end of a piece of string and hold the other end to create a vertical line that casts a shadow across the clipboard, and trace this shadow on the piece of paper. Write " S "" (for "away from the Sun") at the end of this line that is farthest from the sun.

Important: Create the figures in Steps 1 and 2 at the same time and write the date and the time on each piece of paper.
3. On the figure created in Step 2, use a protractor to carefully measure the angle between the line pointing toward magnetic north ("N") and the line pointing away from the sun (" $\mathrm{S}^{* " \text { "). (If the lines you drew are "wiggly", use a ruler to approximated them by }}$ straight lines before measuring the angle.) Since $S^{*}$ lies to the east of north, then to transform this angle measure into the measure of the angle between true north (not magnetic north) and $\mathrm{S}^{*}$, we must subtract $3^{\circ} 33^{\prime}$ from it. (In Milwaukee in 2010, the magnetic north pole lies about $3^{\circ} 33^{\prime}$ west of the true north pole. So the initial measure of the angle must be corrected by moving the north pole $3^{\circ} 33^{\prime}$ to the east. Since the angle we are measuring lies to the east of north, this has the effect of decreasing the angle measure by $3^{\circ} 33^{\prime}$. Let a denote this corrected angle measure. ( $a=$ the initial angle measure minus $3^{\circ} 33^{\prime}$.)
4. On the figure created in Step 1, use a protractor to carefully measure the angle between the vertical line pointing up toward $U$ and the line pointing toward the Sun at $S$. (Again, if the lines you drew are "wiggly", use a ruler to approximated them by straight lines before measuring the angle.) Subtract this angle measure from $90^{\circ}$ to get the measure of the angle that the Sun makes with the horizon. Call this angle measure $z$. ( $z=90^{\circ}$ minus the initial angle measure.)
5. A map entitled Equidistant Azimuthal Projection Centered on City of Milwaukee, can be found on the final page of this addendum. Make a copy of this map (enlarged, if possible). On this copy use a protractor and ruler to draw a straight ray emanating northeast from Milwaukee and making an angle of a degrees with the meridian of longitude through Milwaukee. You are going to search for the point on this ray where sunset occurred at the exact moment that you created the figures in Steps 1 and 2. To accomplish this, go online to the website
http://www.usno.navy.mil/USNO/astronomical-applications/data-services/rs-one-dayworld
Fill in the date on which you performed Steps 1 and 2 (if it differs from the current date.) Scroll down to the blank titled Time Zone and fill in 6 if daylight savings time was not in effect when these measurements were taken, but fill in 5 if daylight savings was in effect when the measurements were taken. Then click the west of Greenwich button. Now choose a point on the ray you drew where it is easy to estimate the latitude and longitude coordinates. (The edges of the rectangles on the map are each $1^{\circ}$ long. If
you choose a point where the ray crosses the edge of a rectangle, then you will know either the latitude coordinate or the longitude coordinate of that point to the nearest degree. Then you can estimate the other coordinate of the point to the nearest degree and 10 minutes.) Fill these coordinates into the blanks entitled Latitude and Longitude. Then click the Get Data button. A page will then appear that contains information including the time of sunset at the point whose coordinates you inputted on the date you specified.

If this sunset time agrees precisely with the time you created the figures in Steps 1 and 2, you're done. However, if the time you created the figures is earlier (later) than this sunset time, then you must move along your ray to a new point that is east (west) of your first point, and input the latitude and longitude coordinates of the new point, and check the sunset time of the new point. Repeat this process until you find the point on your line where sunset occurred at exactly the same time as you created the figures. Record the longitude and latitude coordinates of this point.

To find a point on the ray where the sunset time agrees precisely with the time you performed Steps 1 and 2, you may have to choose a point on your ray that lies in the interior of a rectangle on the map. In this case, you will need to "interpolate" the longitude and latitude coordinates of such points. For example, suppose that you choose a point that lies in the interor of the rectangle which is bounded by the $66^{\circ} \mathrm{W}$ and $67^{\circ} \mathrm{W}$ meridians and by the $49^{\circ} \mathrm{N}$ and $50^{\circ} \mathrm{N}$ parallels. Measure the distance between the sides of this rectangle lying on the $66^{\circ} \mathrm{W}$ and $67^{\circ} \mathrm{W}$ meridians, and measure the distance of your point from the $66^{\circ} \mathrm{W}$ meridian. If the distance between the $66^{\circ} \mathrm{W}$ and $67^{\circ} \mathrm{W}$ meridians is 1.8 cm and the distance from your point to the $66^{\circ} \mathrm{W}$ meridian is 1.3 cm , then your point has longitude coordinate

$$
66^{\circ}+\left(\frac{1.3}{1.8}\right) 60^{\prime} W=66^{\circ} 43^{\prime} W
$$

Similarly measure the distance between the sides of this rectangle lying on the $49^{\circ} \mathrm{N}$ and $50^{\circ} \mathrm{N}$ parallels, and measure the distance of your point from the $49^{\circ} \mathrm{N}$ parallel. If the distance between the $49^{\circ} \mathrm{N}$ and $50^{\circ} \mathrm{N}$ parallels in 2.6 cm and the distance from your point to the $49^{\circ} \mathrm{N}$ parallel is .7 cm , then your point has latitude coordinate

$$
49^{\circ}+\left(\frac{.7}{2.6}\right) 60^{\prime} \mathrm{N}=49^{\circ} 16^{\prime} \mathrm{N}
$$

6. Go online to the website

## http://www.chemical-ecology.net/java/lat-long.htm.

Input the latitude and longitude coordinates of Milwaukee ${ }^{5}$ in the blanks under Point one. Input the latitude and longitude coordinates for the point you found in Step 5 in the

[^3]blanks under Point two. Click Distance between button, and record the result (in miles). Call this distance $d$.
7. Let $c$ denote your estimate of the circumference of the Earth. Then $c$ satisfies the equation
$$
\frac{c}{d}=\frac{360}{z}
$$
(Here, z is the angle measure obtained in Step 4 and $d$ is the distance obtained in Step 6.) Thus, $c$ satisfies
$$
c=\left(\frac{360}{z}\right) d
$$

Calculate your estimate $c$ of the circumference of the Earth from this equation. ${ }^{6}$

Homework Problem 8. What changes would need to be made in the procedure for estimating the circumference of the Earth in Milwaukee if:
a) the procedure were performed early in the morning and the sunset point was replaced by the sunrise point on the great circle C ?
b) the procedure were performed just before solar noon and the sunset point was replaced by the point closest to the Sun on the great circle C?
c) the procedure was performed late in the afternoon in Santa Fe, New Mexico?

[^4]!w 008




[^0]:    ${ }^{1}$ In 2010, the magnetic north pole lies in the Arctic Ocean north of Canada at a latitude of $84.97^{\circ} \mathrm{N}$ and $132.35^{\circ} \mathrm{W}$. It is moving north-by-northwest at a rate of about 55 km a year. If it continues its current direction and speed, it will reach Siberia in 50 years. From the perspective of Milwaukee in 2010, the north magnetic pole lies $3^{\circ} 33^{\prime}$ west of true north and is moving west at the rate of $4^{\prime}$ per year. Further information about the magnetic north pole is available at the following web sites:

[^1]:    ${ }^{3}$ This map was created by the UWM Cartography and Geographic Information Systems Center. The UWM Cartography \& GIS Center is one of a few facilities in the U.S. capable of producing this type of map. The Center's website is

[^2]:    ${ }^{4}$ The latitude and longitude coordinates of the plaza outside the UWM Engineering Mathematical Sciences Building are $43^{\circ} 4^{\prime} 32^{\prime \prime} \mathrm{N}$ and $87^{\circ} 53^{\prime} 8^{\prime \prime} \mathrm{W}$. To find latitude and longitude coordinates of other locations go to
    http://itouchmap.com/latlong.html,
    type the name or address of the location in the "Address" blank, and use the map that appears to "zoom in" to get very accurate coordinates. This website allows you to work in the opposite direction: enter latitude and longitude coordinates of a location, and the website supplies a map of the location that can be "zoomed" to high resolution.

[^3]:    ${ }^{5}$ The latitude and longitude coordinates of the plaza outside the UWM Engineering Mathematical Sciences Building are $43^{\circ} 4^{\prime} 30^{\prime \prime} \mathrm{N}$ and $87^{\circ} 53^{\prime} 5^{\prime \prime} \mathrm{W}$.

[^4]:    ${ }^{6}$ You can check the accuracy of your values for $a$ and $z$ online. Go to http://www.usno.navy.mi//USNO/astronomical-applications/data-services/alt-az-us.
    Under Object: click the Sun button. Fill in the date you performed Steps 1 and 2 (if it differs from the current date). Under Tabular Interval (minutes): change 10 to 1. Under City or Town Name: enter Milwaukee. Under State or Territory: choose Wisconsin. Now click Compute Table button. In the table that appears, find the time you performed your measurements in the left hand column. (If Daylight Savings Time was in effect when you performed your measurements, you must subtract 1 hour from your recorded time before matching it with the time in the left hand column of the table.) The Altitude number in the middle column is a precise value for our $z$. Subtract the Azimuth number in the right hand column from $180^{\circ}$ to obtain a precise value for our a. (Azimuth measures the angle from north to the Sun, whereas a measures the angle from north to the direction of the sunset point, which is $180^{\circ}$ away from the Sun.)

