Lesson 21: Eratosthenes' Measurement of the Earth

The most important parallel of latitude is the equator; it is the only parallel of latitude that is a great circle. However, in the previous lesson we learned about four other parallels of latitude which are of great importance because of the relative geometry of the Earth and the Sun. These are the Tropics of Cancer and Capricorn and the Arctic and Antarctic Circles. We briefly review the basic facts about these four parallels.

At each moment on the day of the summer solstice, there is a single point on the surface of the Earth that is closet to the Sun. As the Earth rotates through the day of the summer solstice, this closest point to the Sun moves around the surface of the Earth and traces a parallel of latitude called the Tropic of Cancer. Another description of the Tropic of Cancer is that it is the set of all points on the Earth's surface where the Sun is directly overhead at solar noon on the day of the summer solstice. The latitude coordinate of the Tropic of Cancer equals the measure of the angle that the Earth's axis makes with the perpendicular to the ecliptic plane: approximately 23.5º N.

On the day of the summer solstice, as the Earth rotates about its axis, the terminator (the division between daytime and nighttime) moves, and its northernmost point traces out a parallel of latitude called the Arctic Circle. Similarly, as the Earth rotates, the southernmost point of the terminator traces out a parallel of latitude called the Antarctic Circle. The latitude coordinate of the Arctic circle is 66.5º N. (66.5º = 90º – 23.5º.) The latitude coordinate of the Antarctic circle is 66.5º S. On the day of the summer solstice, every point north of the Arctic Circle has 24 hours of daylight: the sun never sets. By contrast, on the day of the summer solstice, every point south of the Antarctic circle has 24 hours of night: the sun never rises.

On the day of the winter solstice, the set of points on the Earth's surface that are closest to the sun trace out a parallel of latitude called the Tropic of Capricorn. A point lies on the Tropic of Capricorn if and only if the point has the property that the Sun appears to be directly overhead at solar noon on the day of the summer solstice. The latitude coordinate of the Tropic of Capricorn is 23.5º S. On the day of the summer solstice, every point north of the Arctic Circle has 24 hours of night, while every point south of the Antarctic Circle has 24 hours of daylight.

Eratosthenes' Measurement of the Circumference of the Earth

A recent poll of physicists asked them to name the ten most "beautiful" science experiments of all time.[1] Ranked seventh on the list was Eratosthenes' estimate of the circumference of the Earth in the third century B.C.

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[1] "Here They Are, Science's 10 Most Beautiful Experiments", The New York Times, September 24, 2002.

Eratosthenes was the chief librarian of the great Library at Alexandria, Egypt from about 245 BC to 204 BC. Alexander the Great had conquered a great swath of the civilized world from the Egypt of the pharaohs to western India about 333 BC. Alexander died at the age of 24 in 332 BC, and his generals split up his empire among themselves. The general who took over Egypt was named Ptolemy. Ptolemy and his eleven descendants (Ptolemy I through Ptolemy XII) ruled Egypt until it was conquered by Julius Caesar in 30 BC and became part of the Roman Empire. Ptolemy built the city of Alexandria to honor Alexander and to provide a city in which the Greek bureaucracy which ran Egypt could live. Alexandria was modeled on city states of ancient Greece and eventually replaced them as the center of Greek culture. Ptolemy II built the great Library at Alexandria to rival the libraries of ancient Athens. Eventually the Library became the greatest repository of learning in the ancient world, housing between 400,000 and 700,000 scrolls. The Library was part of the larger institution called the Museum at Alexandria which was not a museum in the modern sense. Rather it was the greatest government supported research institute of its time. The most important thinkers of that era in that part of the world gathered there. Presumably, Eratosthenes was one of the head scientists of the institution.

Eratosthenes (276 BC – 194 BC) was apparently the third chief librarian of the great Library. He was an expert in geography, geometry and other forms of mathematics. Knowledge of several of his geographic and mathematical discoveries has survived to the present day including his remarkably accurate estimate of the Earth's circumference.[2]

Ancient Greek scholars including Aristotle, the most influential of the Greek philosophers, had marshaled various arguments in favor of the hypothesis that the Earth is a sphere, and they believed their arguments. Aristotle cited as evidence for a spherical Earth the fact that during a lunar eclipse, the shadow of the Earth on the Moon's face is circular. Another observation that supported the spherical Earth hypothesis was the fact that when a ship sailed away from port observers would see the hull of the ship vanish below the horizon while the top of the sail was still visible. This observation is consistent with a spherical Earth but not a flat Earth. Once it is commonly believed that the Earth is spherical, the next obvious question is: how big is it? Eratosthenes answered this question.

Eratosthenes' method for estimating the circumference of the Earth relied on two convenient occurrences. Alexandria lies on the Mediterranean Sea near the mouth of the Nile River. Eratosthenes knew that in a city south of Alexandria along the Nile called Syene (the modern day city of Aswan, site of a massive dam on the Nile), the sun would be directly overhead at noon on the day of the summer solstice (June 21). We would say that Eratosthenes knew that Syene lies on the Tropic of Cancer. Apparently Eratosthenes knew this fact about Syene because he had heard a report that at noon

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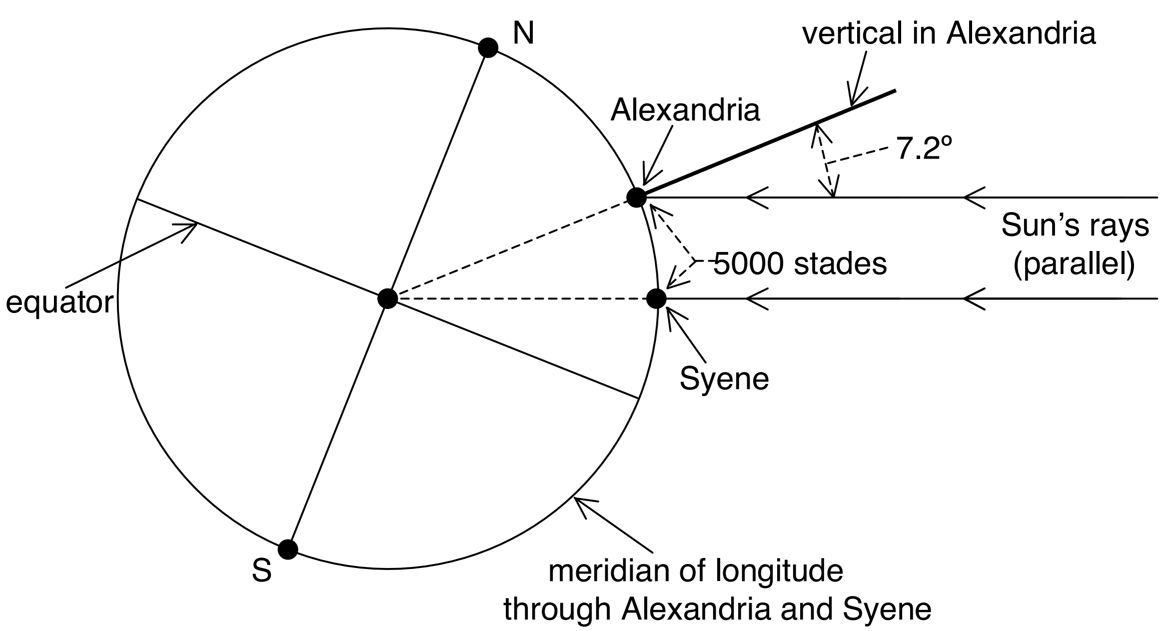
[2] We strongly recommend an excellent book *for young readers* about Eratosthenes' life, times and work: *The librarian who measured the Earth* by Kathryn Lasky.

on the day of the summer solstice the Sun would shine all the way down an empty well in Syene creating no shadow on the bottom or sides of the well. The second piece of knowledge that helped Eratosthenes perform his estimate was that Syene lay almost due south of Alexandria. We would observe that Alexandria and Syene essentially lie on the same meridian of longitude.

To complete his estimate of the circumference of the Earth, Eratosthenes needed two more pieces of information. First he needed to know the distance between Alexandria and Syene. He was able to estimate this distance to be 5000 stades. (A stade was a unit of length equal to the length of an ancient Greek stadium. It was approximately one tenth of a mile or .16 km.) Eratosthenes needed one more measurement to complete his estimate. At noon on the day of the summer solstice, at the moment that the Sun was directly overhead in Syene, Eratosthenes measured the angle that the rays of the Sun made with a vertical stick in Alexandria. He found this angle to be about 1/50 of a circle or roughly 7.2º. With this information, Eratosthenes estimated the circumference of the Earth. We now know that his estimate was remarkably accurate.

Activity 1. Each group should find Eratosthenes' estimate of the circumference of the Earth in stades, miles and kilometers, and report its results to the class. As an aid in this activity, consult the following figure and the explanation that follows it.



Solar Noon in Alexandria and Syene
on the Day of the Summer Solstice

To understand the relation of the preceding figure to the problem of estimating the Earth's circumference, consider the moving plane that contains the meridian of longitude that passes through Syene and Alexandria. This plane cuts the surface of the Earth in a great circle. Hence, the circumference of the Earth equals the circumference of this great circle. The figure on the previous page shows this plane and great circle. Also at noon on the day of the summer solstice, this moving plane passes through the center of the Sun. Hence, at that moment the rays from the Sun to Syene and Alexandra appear as parallel lines in this plane.

Eratosthenes' Calculation. To calculate the circumference of the Earth, Eratosthenes actually calculated the circumference of the great circle on the Earth's surface that contains the meridian of longitude that passes through Alexandria and Syene. (This is the circle pictured in the figure accompanying Activity 1.) Let c denote the circumference of the Earth. Then c is also equal to the circumference of this great circle. Eratosthenes observed that the ratio between the circumference c of the great circle and the length of the arc between Alexandria and Syene (5000 stades) is equal to the ratio between 360º (the total angle measure of the great circle) and the angle measuring 7.2º. This yields the equation

$$\frac{c}{5000} = \frac{360^\circ}{7.2^\circ}.$$

(Or perhaps Eratosthenes observed that the ratio between the circumference c of the great circle and 360º (the total angle measure of the great circle) is equal to the ratio between the length of the arc between Alexandria and Syene (5000 stades) and the angle measuring 7.2º. This observation yields the equation

$$\frac{c}{360^\circ} = \frac{5000}{7.2^\circ}.)$$

In either case, we get

$$c = \left(\frac{360}{7.2}\right)5000 = 50 \times 5000 = 250{,}000 \text{ stades.}$$

Since 1 stade = .1 miles, then arrive at Eratosthenes' estimate of the circumference of the Earth:

$$c = 250{,}000 \times .1 = 25{,}000 \text{ miles.}$$

Modern estimates of the Earth's circumference place it at 24,860 miles. (This is the circumference of a great circle that passes through the north and south poles. The circumference of the equator is 24,900 miles.) Hence, Eratosthenes' estimate is off by 140 miles out of 25,000 miles, an error of about one half of 1%. (140/(25,000) = .0056 = .56%.) Considering how rudimentary the technology available to Eratosthenes was by modern standards, his estimate seems astoundingly accurate.

Today we know that some of the measurements and assumptions that Eratosthenes used in the process of calculating his estimate were slightly inaccurate. It turns out that Syene is not directly south of Alexandria[3] and it's not precisely on the Tropic of Cancer.[4] Also Eratosthenes underestimated the distance from Alexandria to Syene by 29 miles.[5] These inaccuracies are somewhat beside the point. Of more

[3] Syene lies 3.39º east of the meridian of longitude that passes through Alexandria which implies that Syene lies 214 miles east of this meridian.
[4] Syene lies .64º north of the Tropic of Cancer which implies that Syene lies 44.5 miles north of the Tropic of Cancer.
[5] The true distance from Alexandria to Syene is 529 miles = 5290 stades, not 500 miles = 5000 stades.

relevance is the comparison between the true distance from Alexandria to the point directly south of it on the Tropic of Cancer and the figure of 5000 stades which was value that Eratosthenes used for this distance. In fact, the distance from Alexandria to the point on the Tropic of Cancer that is directly south of it is 536 miles = 5360 stades. Hence, Eratosthenes underestimated this distance by 36 miles = 360 stades. This is an underestimate of 6.7%. The other relevant comparison is between the true measure of the angle that the Sun's rays make with a vertical line at solar noon in Alexandria on the day of the summer solstice and the value of 7.2º that Eratosthenes used for this angle measure. In fact, the true measure of the angle that the Sun's rays make with a vertical line at solar noon in Alexandria on the day of the summer solstice is 7.76º. Hence, Eratosthenes underestimated this angle measure by .56º. This is an underestimate of 7.2%. Notice that in the formula for the Earth's circumference c stated above, the distance 5000 is divided by the angle measure 7.2. Since these values represent inaccuracies of the same proportion – about 7%, these inaccuracies roughly cancel each other out. Indeed, replacing the inaccurate value

$\frac{5000}{7.2^\circ} = 694.44$ by the accurate value $\frac{5360}{7.76^\circ} = 690.72$ causes an adjustment of only

3.72 out of 690.72. This is only a .54% difference. So the incredible accuracy of Eratosthenes' estimate – to within about .5% of the true figure – is partially explained by the fact that he used two measurements in his calculations that were both off by about 7%, and these errors canceled each other out. Thus, the spectacular accuracy of Eratosthenes' estimate results in part from the fortuitous cancellation of two larger inaccuracies of the same magnitude. Despite this small irony, it must be said that even if Eratosthenes' estimate of the circumference of the Earth had deviated from the true circumference by 7% (the percentage of error in the measurements that went into his calculation), it would have been a remarkable accomplishment.

Activity 2. Each group should begin working on the Homework Problems together. Each individual member of the class should complete these problems after class.

For the following homework problems: assume that Eratosthenes' measurements were correct:

- Alexandria is 500 miles due north of the Tropic of Cancer,
- at solar noon on the day of the summer solstice in Alexandria, the Sun's rays make an angle of 7.2º with the vertical, and
- the Earth's circumference is 25,000 miles.

Also assume that the latitude coordinate of the Tropic of Cancer is 23.5º N.

Homework Problem 1. a) What is the latitude coordinate of Alexandria?

b) How far is Alexandria from the nearest point on the equator?

c) At solar noon in Alexandria on the day of the **winter** solstice, what is the measure of the angle that the Sun's rays make with the vertical?

Homework Problem 2. At solar noon on the day of the summer solstice, what is the measure of the angle that the Sun's rays make with the vertical at a point that lies:

a) on the equator?

b) at the north pole?

c) on the Arctic Circle?

d) on the Tropic of Capricorn?

e) on the Antarctic Circle?

Homework Problem 3. At solar noon on the day of the summer solstice, what is the measure of the angle that the Sun's rays make with the vertical at a point that lies:

a) 1000 miles due north of Alexandria?

b) 3500 miles due south of Alexandria?

c) 2000 miles north of the equator?

d) 2700 miles south of the equator?

What is the latitude coordinate of each of these points?

Homework Problem 4. What is the distance between a point A on the Tropic of Cancer and:

a) a point on the equator that is due south of the point A?

b) a point on the Arctic Circle that is due north of the point A?

c) the north pole?

d) a point on the Tropic of Capricorn that is due south of the point A?

e) a point of the Antarctic Circle that is due south of the point A?

f) the south pole?

Homework Problem 5. Suppose you are sailing in the Pacific Ocean on the day of the **winter** solstice. At solar noon you measure the angle that the Sun's rays make with the vertical and get 31.4º.

a) What are your possible latitude coordinates?

b) Suppose you can't find the big dipper in the night sky. What can you conclude about your latitude coordinate?

c) How far away is the nearest point on the equator?

Homework Problem 6. Suppose you are sailing in the Pacific Ocean on the day of the **winter** solstice. At solar noon you measure the angle that the Sun's rays make with the vertical and get 38.7º.

a) What are your possible latitude coordinates?

b) Suppose you can see the big dipper in the night sky. What can you conclude about your latitude coordinate?

c) How far away is the nearest point on the equator?

Homework Problem 7. On the day of the summer solstice you climb to the peak of a mountain. At solar noon you measure the angle that the Sun's rays make with the verical and get 39.3º.

a) What are your possible latitude coordinates?

b) Suppose that your gear was carried most of the way up the mountain on the back of a llama. What can you conclude about your latitude coordinate?

c) How far away is the south pole?