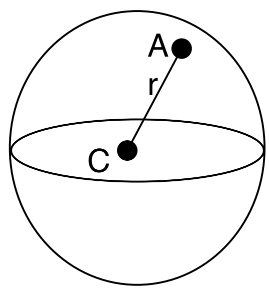
Lesson 18: Spheres and Globes

With this lesson we begin a unit that will explore the geometry of spheres. The surface of the Earth is one of the spheres to which we will pay particular attention. As part of this exploration, we will study the geometry of the relationship between the Earth and the Sun and the way in which this geometry effects the Earth. For example, the seasons – spring, summer, fall and winter – are a direct consequence of this geometry.

Let C be a point in 3-dimensional space and let $r > 0$. The *sphere* with *center C* and *radius r* is the 2-dimensional surface in 3-dimensional space consisting of all points A such that $CA = r$. In other words, a point is on this sphere if and only if its distance from C is exactly r. (This sphere is **not** 3-dimensional. It is the *skin* of the 3-dimensional *ball* that consists of all points A such that $CA \leq r$.)



Activity 1. The class will discuss the following question. Let S be a sphere with center C and radius r and let L be a line, and consider the intersection $S \cap L$ of S with L. Which of the following types of sets can occur as the intersection $S \cap L$?

1) a set consisting of one point,

2) a set consisting of two points whose distance is equal to 2r,

3) a set consisting of two points whose distance is $> 2r$,

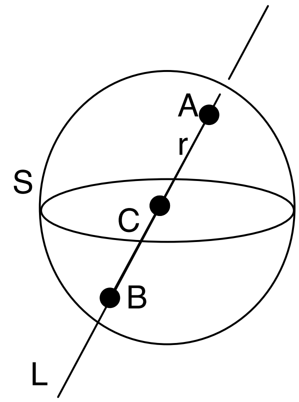
4) a set consisting of two points whose distance is $< 2r$,

5) a set consisting of n points where $3 \leq n < \infty$,

6) a straight line segment,

7) an empty set.

If the line L passes through the center C of the sphere S, then the intersection L ∩ S of L with S is a pair of points that we call *poles* or *antipodal points* of S.



A and B are poles of S

Activity 2. The class will discuss the following two questions. Suppose that A and B are two points on a sphere S with center C and radius r.

1) What can be said about the distance AB in the case that A and B are poles of S?

2) What can be said about AB in the case that A and B are **not** poles of S?

Activity 3. The class will discuss the following question. Let S be a sphere with center C and radius r and let P be a plane, and consider the intersection S ∩ P of S with P. Which of the following types of sets can occur as the intersection S ∩ P?

1) a circle of radius r,

2) a circle of radius > r,

3) a circle of radius < r,

4) a non-circular ellipse,

5) a set consisting of one point,

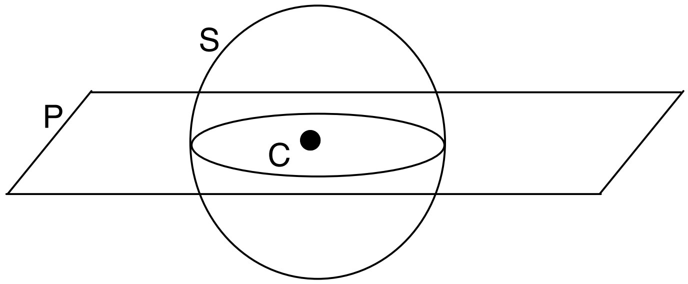
6) a set consisting of n points where $2 \leq n < \infty$,

7) a straight line segment,

8) an empty set.

As an aid to analyzing part 4) of this question, each group will be given an orange and a knife. By using the knife to make straight cuts through the orange, you can simulated a plane intersecting a sphere.

If a plane P passes through the center C of a sphere S, then the intersection P ∩ S of P with S is a circle that is called a *great circle* of S.

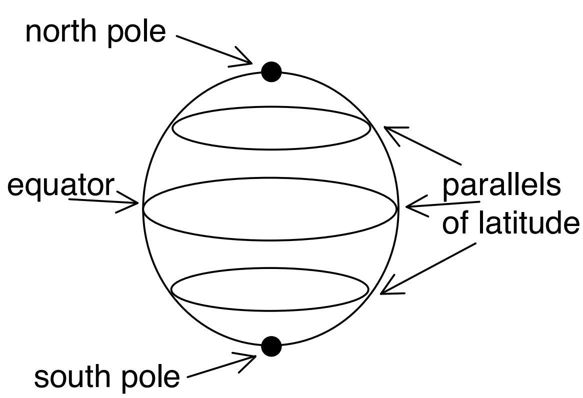


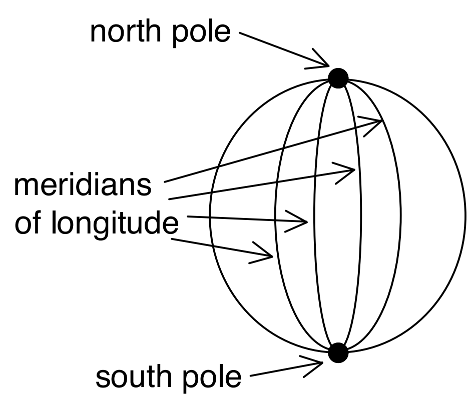
Activity 4. The class will discuss the following two questions. Suppose that J is a circle on a sphere S of radius r.

1) What can be said about the radius of J in the case that J is a great circle of S?

2) What can be said about the radius of J in the case that J is **not** a great circle of S?

We will use the word *"globe"* to mean a spherical map of the Earth. (The word "globe" has other meanings as well.) On the globe a special pair of poles are marked off and called the *north pole* and the *south pole.* Also there are special circles marked off. One family of parallel circles that don't contain the north and south poles are known variously as *parallels of latitude* or simply *parallels* or *latitude curves* or simply *latitudes.* The parallel of latitude which is half way between the north and south poles is called the *equator.* There is also a family of semicircles joining the north and south poles known variously as *meridians of longitude* or simply *meridians* or *longitude curves* or simply *longitudes.*





Activity 5. The class will discuss the following question. Which of the parallels of latitude and meridians of longitude are great circles or arcs of great circles and which are not?

Activity 6. The class will discuss the following two questions. Let C be the center of a globe and let L be the line that passes through the north and south poles.

1) Consider the planes that contain the various parallels of latitude. Describe in as much detail as you can the relationship between the point C, the line L and these various planes.

2) Consider the planes that contain the various meridians of longitude. Describe in as much detail as you can the relationship between the point C, the line L and these various planes.

Activity 7. Globes and pieces of string should be distributed to the class. With the help of these objects, the class should discuss the following two questions. Choose two points A and B on a globe.

1) How can we find the shortest path on the surface of the globe between the points A and B?

2) How can we find the distance on the surface of the globe (not through space) from A to B?

Activity 8. Each group will be assigned one of the following six pairs of cities. Each group should locate the two points on its globe that represent its pair of assigned cities. Each such pair of points is located on (or close to) a single parallel of latitude or meridian of longitude, and these parallels of latitude and meridians of longitude are indicated in parentheses to help you locate the two points on the globe.

1) Chicago, Illinois and Madrid, Spain (41° N parallel of latitude)

2) Cape Town, South Africa and Sidney, Australia (34º S parallel of latitude)

3) Santiago, Chile and Boston, Massachusetts (71° W meridian of longitude)

4) Quito, Equador and Singapore (0º N parallel of latitude)

5) San Francisco, California and Tokyo, Japan (37º N parallel of latitude)

6) Toronto, Canada and Quito, Equador (80º W meridian of longitude)

Stretch a string ***tightly*** between each pair of points on the globe. Notice that sometimes the tightly stretched string follows the parallel of latitude or meridian of longitude that passes through the pair of points, and sometimes it doesn't. Now try to answer the following question.

- What determines whether or not the tightly stretched string follows the parallel of latitude or meridian of longitude?

Each group should report its answer to the class, and the class should discuss and analyze these answers.

Remark concerning Activity 8: A tightly stretched string joining two points on the surface of a globe traces the shortest path on the globe between these two points. Hence, in Activity 8, we are really trying to answer the question:

- What determines whether or not the shortest path between two points that lie on the same parallel of latitude or meridian of longitude follows that parallel/meridian?

In other words, we are trying to discover a property of a parallel of latitude or meridian of longitude that will determine whether or not it contains the shortest path between any pair of its points.

Recall that Activity 5 revealed that the equator is the only parallel of latitude that is a great circle. On the other hand, every meridian of longitude is half of a great circle.

Activity 9. The class should discuss the following two questions.

1) What is the relationship between the answers to the question in Activity 8 and the issue of which parallels of latitude and meridians of longitude are great circles?

2) What is the relationship between the shortest path between two points on the surface of a globe and the great circle on the globe that passes through these two points?

If A and B are two points on a sphere, then the shortest path joining A to B on the sphere is called the *geodesic* from A to B. The geodesic joining two points on the surface of a globe is the path of the tightly stretched string between the two points. This geodesic is not a straight line because the straight line joining the two points in space would not stay on the surface of the globe; it would go through the globe's interior. The geodesic must be curved to hug the surface of the globe.

The objective of the previous three activities in this lesson is the following fact.

The ***geodesic*** *joining two points A and B on a sphere is the* ***shorter arc*** *of the* ***great circle*** *of the sphere that passes through the two points A and B.*

If A and B are two points on a sphere, then the *distance* from A to B *(along the sphere)* is the length of the shortest path or geodesic from A to B. Thus, if A and B are two points on a globe, then the distance from A to B is the length of the stretched string joining A and B along the surface of the globe.

Activity 10. The class will discuss the following two questions. Let A and B be two points on a sphere S.

1) Is there always a geodesic joining A to B on S? (If so, this geodesic is an arc of a great circle on S.)

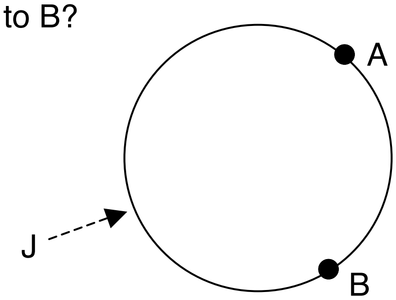
As an aid to analyzing this question, each group will be given an orange and a knife. Mark two points A and B on the surface of the orange and consider the following question. Can you use the knife to make a straight cut through the orange that passes through the points A and B and the center of the orange?

2) Can there be more than one great circle on S that passes through A to B? If so, how many?

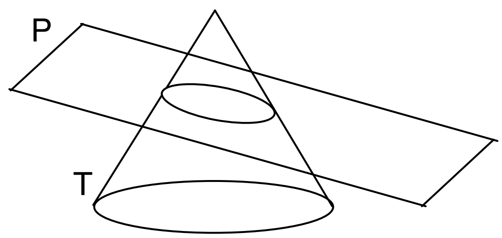
In analyzing this question, it is useful to consider two cases:

- A and B are not antipodal points.
- A and B are antipodal points.

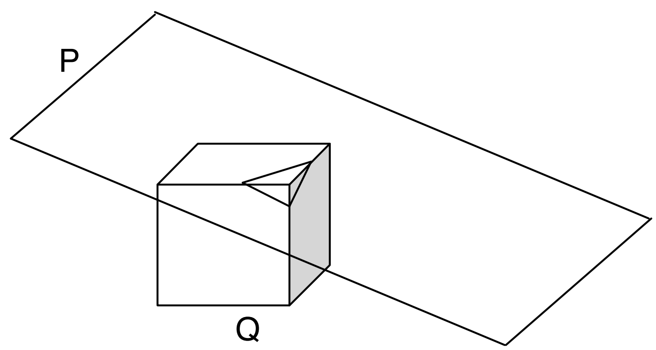
3) If J is a great circle on S that passes through the points A and B, then J is the *union* of two arcs that join A to B. Which of these two arcs is a geodesic joining A to B? Can both arcs be geodesics joining A to B? Under what circumstances are both arcs geodesics joining A to B?



Homework Problem 1. Let T be a truncated cone and let P be a plane in 3- dimensional space. List all the different kinds of sets can result from the intersection of T with P?



Homework Problem 2. Let Q be a cube and let P be a plane in 3-dimensional space. List all the different kinds of sets can result from the intersection of Q with P?



Homework Problem 3. Let P and Q be two antipodal points on a sphere S. (Think of P and Q as the north and south poles on a globe.) Then the distance from P to Q along the sphere S is the length of any geodesic joining P to Q. Let A and B be two other points on the sphere S.

a) How does the distance from A to B compare to the distance from P to Q in each of the following cases:

- A and B are antipodal points.
- A and B are not antipodal points.

b) Stretch a string tightly between the north and south poles of a globe and cut it so that its length is exactly the distance between the north and south poles. Let A be any point on the globe. Use the information in part a) of this problem to devise a strategy that employs the string to find the point on the globe that is antipodal to A.

Homework Problem 4. Suppose that L and L´ are two parallels of latitude on a globe and L is closer to the equator than L´. Also suppose that M and M´ are two meridians of longitude on this globe. Let:

- A be the point where L meets M,
- B be the point where L meets M´,
- C be the point where L´ meets M,
- D be the point where L´ meets M´.

a) Draw a picture of this situation.

b) Compare the distance between A and B to the distance between C and D.

c) Compare the distance between A and C to the distance between B and D.

Homework Problem 5. Suppose that A and B are points on a globe that both lie south of the equator and north of the south pole. Then A lies on a meridian of longitude that crosses the equator at a point we will call C, and B lies on a meridian of longitude that crosses the equator at a point we will call D.

a) Draw a picture of this situation.

b) Fill in the blank in the following statement in a way that makes the statement true.

The geodesic joining A to B passes through the south pole if and only if C and D are

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