Lesson 17: Diagonals of Quadrilaterals

We have introduced five types of quadrilaterals:

parallelograms, rectangles, rhombuses, squares and *kites*.

Previously we characterized parallelograms in terms of properties of their diagonals. Specifically Theorems 8 and 10 tell us that a quadrilateral is a parallelogram if and only if its diagonals bisect each other. In this lesson we will search for characterizations of the remaining four types of quadrilaterals in terms of properties of their diagonals.

Activity 1. The class as a whole should speculate about what properties of the diagonals of a quadrilateral will determine that the quadrilateral is either a *rectangle,* a *square,* a *rhombus* or a *kite*, and they should express their speculations in the form of four conjectures. These four conjectures, if correct, will characterize *rectangles, squares, rhombuses* and *kites* in terms properties of their diagonals. Specifically, the class should fill in the blanks in the following four incomplete conjectures with appropriate properties.

Conjecture A. A quadrilateral □ABCD is a *rectangle* if and only if its diagonals $\overline{AC}$ and $\overline{BD}$ have the following properties:

____

Conjecture B. A quadrilateral □ABCD is a *rhombus* if and only if its diagonals $\overline{AC}$ and $\overline{BD}$ have the following properties:

____

Conjecture C. A quadrilateral □ABCD is a *square* if and only if its diagonals $\overline{AC}$ and $\overline{BD}$ have the following properties:

____

Conjecture D. A quadrilateral □ABCD is a *kite* if and only if its diagonals $\overline{AC}$ and $\overline{BD}$ have the following properties:

____

Each of the preceding four conjectures is an "if and only if" statement. Hence, each conjecture can be broken into two statements:

1) If the quadrilateral □ABCD is a -----------, then its diagonals $\overline{AC}$ and $\overline{BD}$ have the following properties: ------------------------.

2) If the diagonals $\overline{AC}$ and $\overline{BD}$ of the quadrilateral □ABCD have the following properties: ------------------------, then the quadrilateral is a -----------.

For convenience, we call statement **1)** the *forward form* of the conjecture, and we call statement **2)** the *converse form* of the conjecture.

Activity 2. Four theorems (numbered 13 through 16) are stated below with blanks. The blanks must be filled in so that these theorems become the *forward forms* of the four conjectures formulated in Activity 1. Then the class should break into 8 groups. Two groups should be assigned to each of these four theorems. Each of the two groups assigned to a particular theorem should write up an initial proof of that theorem. The two groups should then compare their initial proofs, and based on these proofs they should jointly write up a single final proof of their theorem. One member of the two groups will present the final proof of their theorem to the class. The final proofs should be written in a format that is easy for the entire class to read (for example, on a classroom chalkboard, or on portable whiteboards or large paper tablets, or on overhead projector transparencies).

Theorem 13. If the quadrilateral □ABCD is a *rectangle,* then its diagonals $\overline{AC}$ and $\overline{BD}$ have the following properties:

____

Theorem 14. If the quadrilateral □ABCD is a *rhombus,* then its diagonals $\overline{AC}$ and $\overline{BD}$ have the following properties:

____

Theorem 15. If the quadrilateral □ABCD is a *square,* then its diagonals $\overline{AC}$ and $\overline{BD}$ have the following properties:

____

Theorem 16. If the quadrilateral □ABCD is a *kite,* then its diagonals $\overline{AC}$ and $\overline{BD}$ have the following properties:

____

Activity 3. Four theorems (numbered 17 through 20) are stated below with blanks. The blanks must be filled in so that these theorems become the *converse forms* of the four conjectures formulated in Activity 1. Then the class should break into 8 groups. Two groups should be assigned to each of these four theorems. Each of the two groups assigned to a particular theorem should write up an initial proof of that theorem. The two groups should then compare their initial proofs, and based on these proofs they should jointly write up a single final proof of their theorem. One member of the two groups will present the final proof of their theorem to the class. The final proofs should be written in a format that is easy for the entire class to read (for example, on a classroom chalkboard, or on portable whiteboards or large paper tablets, or on overhead projector transparencies).

Theorem 17. If the diagonals $\overline{AC}$ and $\overline{BD}$ of the quadrilateral □ABCD have the following properties:

____,

then □ABCD is a *rectangle.*

Theorem 18. If the diagonals $\overline{AC}$ and $\overline{BD}$ of the quadrilateral □ABCD have the following properties:

____,

then □ABCD is a *rhombus.*

Theorem 19. If the diagonals $\overline{AC}$ and $\overline{BD}$ of the quadrilateral □ABCD have the following properties:

____,

then □ABCD is a *square.*

Theorem 20. If the diagonals $\overline{AC}$ and $\overline{BD}$ of the quadrilateral □ABCD have the following properties:

____,

then □ABCD is a *kite.*

Homework Problems 1, 2 and 3. In Activity 2 in class, you were a member of a group that wrote up a proof of one of the four theorems numbered 13 through 16. For Homework Problems 1 through 3, write out the proofs of the remaining three theorems among Theorems 13 through 16 that your group did not write up.

Homework Problems 4, 5 and 6. In Activity 3 in class, you were a member of a group that wrote up a proof of one of the four theorems numbered 17 through 20. For Homework Problems 4 through 6, write out the proofs of the remaining three theorems among Theorems 17 through 20 that your group did not write up.

Homework Problem 7. Consider the following conjecture:

A quadrilateral $\square ABCD$ is a rhombus if and only if its diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other and the ray $\overrightarrow{AC}$ bisects the angle $\angle BAD$.

If this conjecture is true, prove it. If this conjecture is false, draw a counterexample.

Homework Problem 8. Consider the following conjecture:

A quadrilateral $\square ABCD$ is a kite with $\overline{AB} \cong \overline{AD}$ and $\overline{BC} \cong \overline{CD}$ if and only if $\overline{AB} \cong \overline{AD}$ and the ray $\overrightarrow{AC}$ bisects the angle $\angle BAD$.

If this conjecture is true, prove it. If this conjecture is false, draw a counterexample.

Homework Problem 9. Consider the following conjecture:

A quadrilateral $\square ABCD$ is a kite with $\overline{AB} \cong \overline{AD}$ and $\overline{BC} \cong \overline{CD}$ if and only if $\overline{AB} \cong \overline{AD}$ and the ray $\overrightarrow{CA}$ bisects the angle $\angle BCD$.

If this conjecture is true, prove it. If this conjecture is false, draw a counterexample.

Homework Problem 10. Consider the following conjecture:

A quadrilateral $\square ABCD$ is a kite with $\overline{AB} \cong \overline{AD}$ and $\overline{BC} \cong \overline{CD}$ if and only if the ray $\overrightarrow{AC}$ bisects the angle $\angle BAD$ and the ray $\overrightarrow{CA}$ bisects the angle $\angle BCD$.

If this conjecture is true, prove it. If this conjecture is false, draw a counterexample.

Homework Problem 11. Consider the following conjecture:

A quadrilateral $\square ABCD$ is a rhombus if and only the ray $\overrightarrow{AC}$ bisects the angle $\angle BAD$, the ray $\overrightarrow{CA}$ bisects the angle $\angle BCD$, the ray $\overrightarrow{BD}$ bisects the angle $\angle ABC$, and the ray $\overrightarrow{DB}$ bisects the angle $\angle ADC$.

If this conjecture is true, prove it. If this conjecture is false, draw a counterexample.

Homework Problem 12. Each of the following sentences has two blanks. Make the sentence true by filling in the first blank with the with the name of one of the quadrilaterals we have studied:

parallelograms *rectangles* *squares* *rhombuses* *kites.*

Then, depending on whether the sentence is a definition or a theorem, fill in the second blank with either *definition* or *theorem.*

a) A quadrilateral is a ___________________ if and only if its diagonals bisect each other. ___________________

b) A quadrilateral is a ___________________ if and only if its diagonals are congruent and bisect each other. ___________________

c) A quadrilateral is a ___________________ if and only if its diagonals are perpendicular and bisect each other. ___________________

d) A quadrilateral is a ___________________ if and only if its opposite angles are congruent. ___________________

e) A quadrilateral $\square ABCD$ is a ___________________ if and only if either $\overline{AB} \cong \overline{CB}$ and $\overline{CD} \cong \overline{AD}$ or $\overline{AB} \cong \overline{AD}$ and $\overline{CB} \cong \overline{CD}$. ___________________

f) A quadrilateral is a ___________________ if and only if its angles are all right angles and its sides are all congruent. ___________________

g) A quadrilateral is a ___________________ if and only if its opposite sides are parallel. ___________________

h) A quadrilateral is a ___________________ if and only if its diagonals are congruent, perpendicular and bisect each other. ___________________

i) A quadrilateral is a ___________________ if and only if its opposite sides are congruent. ___________________

j) A quadrilateral is a ___________________ if and only if its diagonals are perpendicular and one bisects the other. ___________________

k) A quadrilateral is a ___________________ if and only if all its sides are congruent. ___________________

l) A quadrilateral is a ___________________ if and only if its angles are all right angles. ___________________