

Lesson 16: Various Types of Quadrilaterals

Activity 1. The class as a whole should consider the question:

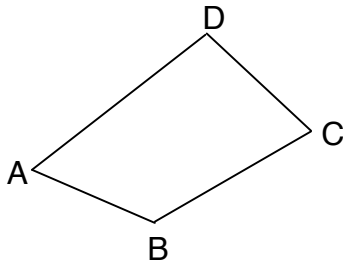
What is the sum of the measures of the angles of a quadrilateral?

Make a conjecture¹ about what the answer might be, and try to create a proof of its conjecture if possible.

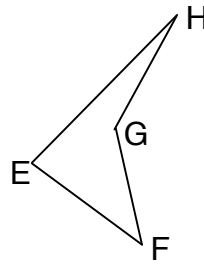
We now introduce a concept that is useful for understanding the problem of determining the sum of the angle measures of a quadrilateral.

Definition. A quadrilateral is *convex* if each vertex of the quadrilateral lies in the interior of the opposite angle. Thus, the quadrilateral $\square ABCD$ is convex if:

- vertex A lies in the interior of angle $\angle C$,
- vertex B lies in the interior of angle $\angle D$,
- vertex C lies in the interior of angle $\angle A$, and
- vertex D lies in the interior of angle $\angle B$.



convex
quadrilateral



non-convex
quadrilateral

Observe that in the quadrilateral $\square EFGH$ pictured here, the vertices E, F and H do *not* lie in the interiors of the opposite angles. In other words, vertex E lies outside the interior of angle $\angle G$, vertex F lies outside the interior of angle $\angle H$, and vertex H lies outside the interior of angle $\angle F$.

The concept of convexity helps us to formulate a correct theorem about the angle sum of quadrilateral.

¹ Recall that a *conjecture* is a statement which you believe to be true but whose truth is not yet known. Therefore, a conjecture takes the form of a declarative sentence, not a question. If a conjecture can be proved, then it becomes a theorem. On the other hand, if a counterexample to the conjecture can be found, then the conjecture becomes just another false statement.

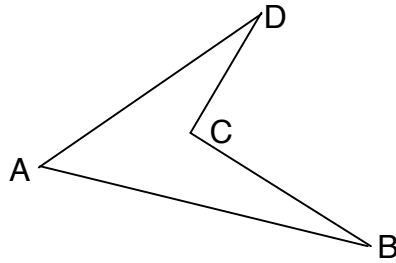
Theorem 11. The sum of the measures of the angles of a convex quadrilateral is 360° . In other words, if $\square ABCD$ is a convex quadrilateral, then

$$m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) = 360^\circ.$$

Remark. If the quadrilateral $\square ABCD$ is not convex, then the equation

$$m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) = 360^\circ$$

will be false. Indeed, in the non-convex quadrilateral $\square ABCD$ pictured below, the



term $m(\angle C)$ would need to be replaced by the term $(360^\circ - m(\angle C))$ to make the equation correct. In other words, the correct equation for the sum of the measures of the angles that applies to the quadrilateral shown here is

$$m(\angle A) + m(\angle B) + (360^\circ - m(\angle C)) + m(\angle D) = 360^\circ.$$

Activity 2. The class as a whole should create a proof of Theorem 11.

Recall that in Lesson 14 we stated and proved Theorem 7 which says that if a quadrilateral is a parallelogram, then opposite angles are congruent. However, we never stated or proved the converse of this theorem. Now with the help of Theorem 11, we are ready to establish the converse of Theorem 7. Unfortunately, since Theorem 11 applies only to convex quadrilaterals, we must add “convexity” to the hypothesis of our converse to Theorem 7. We now state this converse (with the added “convexity” hypothesis.)

Theorem 12. If the opposite angles of a *convex* quadrilateral are congruent, then the quadrilateral is a parallelogram. In other words, if $\square ABCD$ is a *convex* quadrilateral such that $\angle A \cong \angle C$ and $\angle B \cong \angle D$, then $\square ABCD$ is a parallelogram.

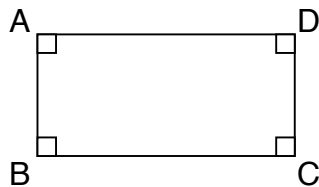
Activity 3. The class as a whole should create a proof of Theorem 12.

Hint: Use Theorem 11.

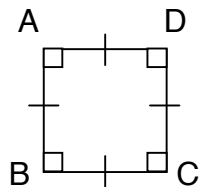
Remark. The “convexity” hypothesis in Theorem 12 is there to make the proof simpler. Its presence makes it easy to apply Theorem 11 which only holds for convex quadrilaterals. However, Theorem 12 remains true even if we remove the “convexity” hypothesis. In other words, any quadrilateral in which opposite angles are congruent must be a parallelogram. We don’t have to assume that the quadrilateral is convex to reach the conclusion that congruency of opposite angles implies that the quadrilateral is a parallelogram. The reason that we don’t need to assume convexity of the quadrilateral is that it is possible to prove that every quadrilateral in which opposite angles are congruent must be convex. Thus, we can omit the convexity of the quadrilateral as a hypothesis because we can prove that it is convex as long as we know that its opposite angles are congruent. However, we won’t investigate this proof here because it is complicated and would require ideas that would diverge from the main focus of this course.

We have already studied one special type of quadrilateral – *parallelograms*. We now introduce four other types of quadrilaterals – *rectangles*, *squares*, *rhombuses* and *kites* and explore the relationships between the various types of quadrilaterals.

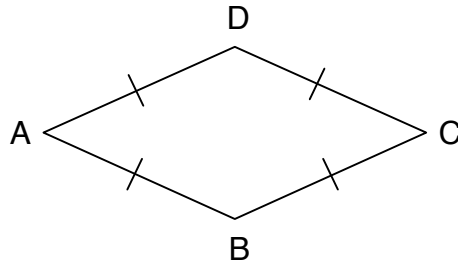
Definition. A quadrilateral is a *rectangle* if all of its angles are right angles. Thus, the quadrilateral $\square ABCD$ is a rectangle if $m(\angle A) = m(\angle B) = m(\angle C) = m(\angle D) = 90^\circ$.



Definition. A quadrilateral is a *square* if all of its angles are right angles and all of its sides are congruent. Thus, the quadrilateral $\square ABCD$ is a square if $m(\angle A) = m(\angle B) = m(\angle C) = m(\angle D) = 90^\circ$ and $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{AD}$.

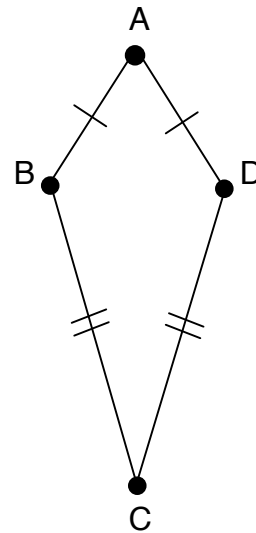
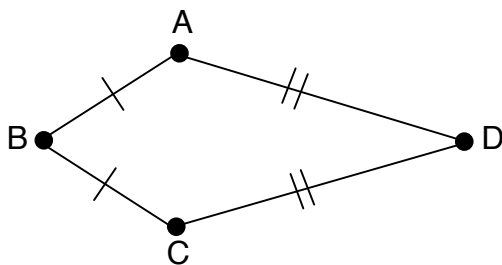


Definition. A quadrilateral is a *rhombus* if all of its sides are congruent. Thus, the quadrilateral $\square ABCD$ is a rhombus if $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{AD}$.



Definition. Two sides of a quadrilateral are *adjacent* if they intersect in a single common vertex. Thus, two sides of a quadrilateral are adjacent if and only if they are not opposite. In the quadrilateral $\square ABCD$, there are four pairs of adjacent sides: \overline{AB} and \overline{BC} , \overline{BC} and \overline{CD} , \overline{CD} and \overline{AD} , \overline{AD} and \overline{AB} .

Definition. A quadrilateral is a *kite* if its four sides can be grouped into two pairs of congruent adjacent sides. Thus, the quadrilateral $\square ABCD$ is a *kite* if either $\overline{AB} \cong \overline{BC}$ and $\overline{CD} \cong \overline{AD}$, or $\overline{AB} \cong \overline{AD}$ and $\overline{BC} \cong \overline{CD}$.



We have defined five kinds of quadrilaterals:

parallelograms rectangles squares rhombuses kites.

We now explore the relations holding between these quadrilateral types.

Activity 4. The class as a whole should carry out the following two activities.

- a)** Create all the possible conjectures that you believe to be true that are of the form:

“Every _____ is a _____”

where the two blanks are filled in with two distinct words from the following list:

parallelogram rectangle square rhombus kite.

- b)** Try to create a proof of each conjecture if possible. If you are unable to create a proof of the conjecture, assume it is false and try to create a counterexample.

Activity 5. The class as a whole should carry out the following two activities.

- a)** Create all the possible conjectures that you believe to be true that are of the form:

“A quadrilateral is a _____ if and only if it is a _____ and a _____”

where the three blanks are filled in with three distinct words from the following list:

parallelogram rectangle square rhombus kite.

- b)** Try to create a proof of each conjecture if possible. If you are unable to create a proof of the conjecture, assume it is false and try to create a counterexample.

Homework Problem 1. Write out a proof of Theorem 11 for yourself.

Homework Problem 2. Write out a proof of Theorem 12 for yourself.

Homework Problem 3. Write out proofs or construct counterexamples for yourself for all the conjectures made in Activity 4.

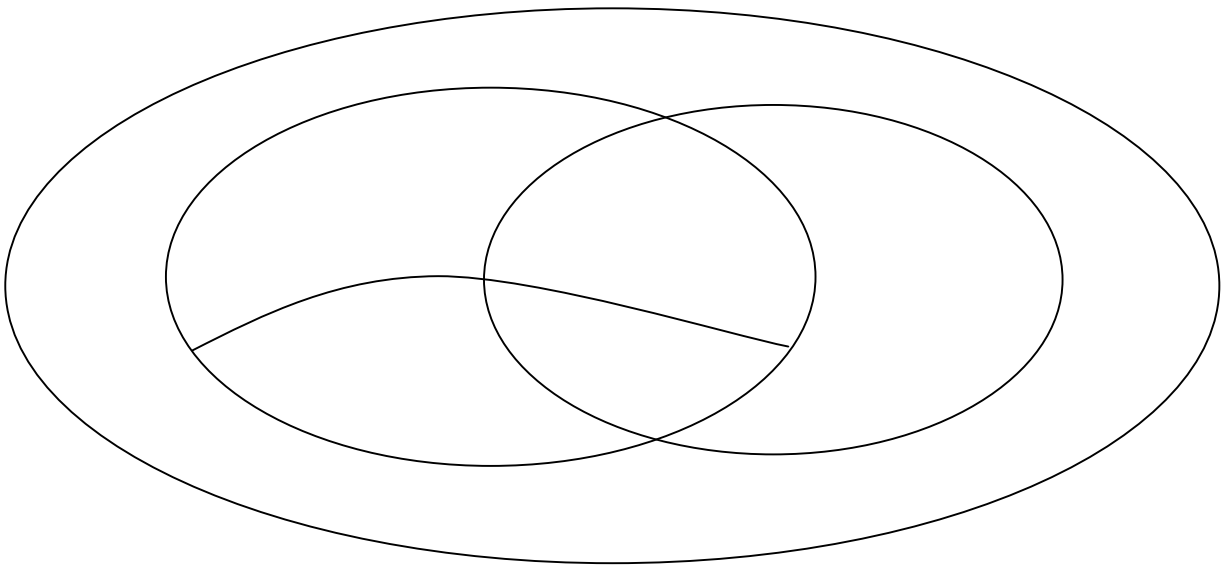
Homework Problem 4. Write out proofs or construct counterexamples for yourself for all the conjectures made in Activity 5.

Homework Problem 5. Here is a list of names of six types of quadrilaterals:

quadrilaterals parallelograms rectangles squares rhombuses kites.

Use the information gathered in Activities 4 and 5 to label six regions in the figure below with the names in the preceding list to create a *Venn Diagram* for the six collections of quadrilaterals named in this list. This means that:

- two of the named collections intersect if and only if the corresponding regions in the figure intersect, and
- one of the named collections is a subset of another named collection if and only if the region in the figure corresponding to the first named collection is a subset of the region in the figure corresponding to the second named collection.



Warning: This figure has six *non-overlapping* regions. The *non-overlapping* regions of the figure are *not* necessarily the ones that should be labeled with the names of the six quadrilateral types in the list. The regions to be labeled with these names may be the union of several of the non-overlapping regions.

Homework Problem 6. Decide whether the following statement is true or false. If it is true, create a proof of it. If it is false, draw a counterexample.

If $\square ABCD$ is a quadrilateral in which \overline{AB} is parallel to \overline{CD} and $\overline{AB} \cong \overline{CD}$, then $\square ABCD$ is a parallelogram.

Homework Problem 7. Decide whether the following statement is true or false. If it is true, create a proof of it. If it is false, draw a counterexample.

If $\square ABCD$ is a quadrilateral in which \overline{AB} is parallel to \overline{CD} and $\overline{BC} \cong \overline{AD}$, then $\square ABCD$ is a parallelogram.

Homework Problem 8. Decide whether the following statement is true or false. If it is true, create a proof of it. If it is false, draw a counterexample.

If $\square ABCD$ is a quadrilateral in which \overline{AB} is parallel to \overline{CD} and $\angle A \cong \angle C$, then $\square ABCD$ is a parallelogram.

