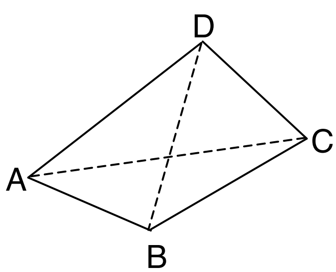
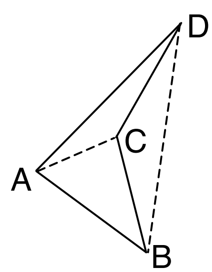
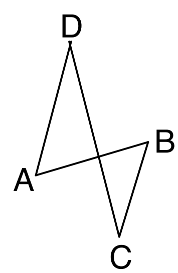
Lesson 15: Properties of Parallelograms

In this lesson we focus our attention on quadrilaterals in general and parallelograms in particular.

Definition. Let A, B, C and D be four points in a plane. We will call the union of the four line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ a *quadrilateral* provided that the opposite sides $\overline{AB}$ and $\overline{CD}$ are disjoint and the opposite sides $\overline{BC}$ and $\overline{DA}$ are disjoint. We denote this quadrilateral by $\square$ABCD. (We can also denote it by $\square$BCDA, $\square$CDAB, $\square$DABC, $\square$DCBA, $\square$CBAD, $\square$BADC and $\square$ADBC.) The points A, B, C and D are called the *vertices* of the quadrilateral. The line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ are called the *sides* of the quadrilateral. The two sides $\overline{AB}$ and $\overline{CD}$ are called *opposite sides* of the quadrilateral, as are the two sides $\overline{BC}$ and $\overline{DA}$. We write $\angle$A, $\angle$B, $\angle$C and $\angle$D as abbreviations for the four angles $\angle$DAB, $\angle$ABC and $\angle$BCD and $\angle$CDA of the quadrilateral. The two angles $\angle$A and $\angle$C are called *opposite angles* of the quadrilateral, as are the two angles $\angle$B and $\angle$D. Also: the angle $\angle$A is called *the angle that is opposite the vertex* C, the angle $\angle$B is called *the angle that is opposite the vertex* D, the angle $\angle$C is called *the angle that is opposite the vertex* A, and the angle $\angle$D is called *the angle that is opposite the vertex* B. The two line segments $\overline{AC}$ and $\overline{BD}$ are called the *diagonals* of the quadrilateral.



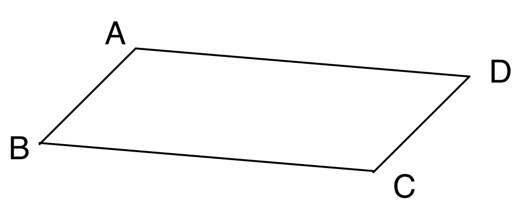




quadrilateral   quadrilateral
(diagonals shown as dashed lines)

not a quadrilateral

Definition. A quadrilateral is a *parallelogram* if the two lines determined by each pair of opposite sides of the quadrilateral are parallel. Thus, the quadrilateral $\square$ABCD is a parallelogram if and only if the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel and the lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$ are parallel.



Activity 1. Five theorems (numbered 6 through 10) are stated below. The class should break into 10 groups. Two groups should be assigned to each of these theorems, and each of the two groups assigned to a particular theorem should write up an initial proof of that theorem. The two groups should then compare their initial proofs, and based on these proofs they should jointly write up a single final proof of their theorem. One member of the two groups will present the final proof of their theorem to the class. The final proofs should be written in a format that is easy for the entire class to read (for example, on a classroom chalkboard, or on portable whiteboards or large paper tablets, or on overhead projector transparencies).

Note. In the proof of any one of these theorems, all the theorems preceding it in the list can be used if needed. For instance, any of Theorems 1 through 8 can be used to justify a line in the proof of Theorem 9.

Theorem 6. If a quadrilateral is a parallelogram, then its opposite sides are congruent. In other words, if the quadrilateral $\square ABCD$ is a parallelogram, then $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{AD}$.

Theorem 7. If a quadrilateral is a parallelogram, then its opposite angles are congruent. In other words, if the quadrilateral $\square ABCD$ is a parallelogram, then $\angle A \cong \angle C$ and $\angle B \cong \angle D$.

Theorem 8. If a quadrilateral is a parallelogram, then its diagonals bisect each other. In other words, if the quadrilateral $\square ABCD$ is a parallelogram, then its diagonals $\overline{AC}$ and $\overline{BD}$ intersect at a point E that is a midpoint of each.

Theorem 9. If the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram. In other words, if $\square ABCD$ is a quadrilateral such that $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{AD}$, then $\square ABCD$ is a parallelogram.

Theorem 10. If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram. In other words, if $\square ABCD$ is a quadrilateral such that its diagonals $\overline{AC}$ and $\overline{BD}$ intersect at a point E that is a midpoint of each, then $\square ABCD$ is a parallelogram.

Observe that Theorem 9 is the *converse* of Theorem 6 and Theorem 10 is the converse of Theorem 8.

Homework Problems 1, 2, 3 and 4. In Activity 1 in class, you were a member of a group that wrote up a proof of one of the five theorems numbered 6 through 10. For Homework Problems 1 through 4, write out the proofs of the remaining four theorems among Theorems 6 through 10 that your group did not write up.