## 1) If $\overline{A B}$ is congruent to $\overline{A C}$, then $\angle B$ is congruent to $\angle C$.

## Proof of 1).

1) Assume $\overline{A B} \cong \overline{A C}$. (We must prove that $\angle B \cong \angle C$.)
2) $\angle A \cong \angle A$, because the identity is a rigid motion that moves $\angle A$ to $\angle A$.
3) Therefore, $\triangle A B C \cong \triangle A C B$ by the $\qquad$ Axiom. (The correspondence between the vertices of $\triangle A B C$ and the vertices of $\triangle A C B$ that we are using here is: $A \rightarrow A, B \rightarrow C, C \rightarrow B$.)
4) The rigid motion that moves $\triangle A B C$ to $\triangle A C B$, moves $\angle B$ to $\angle C$. Hence, $\angle B \cong \angle C$. This completes the proof of 1 ).

Remark. The congruence relation $\triangle A B C \cong \triangle A C B$ which is established in this proof is associated with a correspondence between the vertices of $\triangle A B C$ and the vertices of $\triangle A C B$. This correspondence sends vertex $A$ to itself and interchanges vertices $B$ and $C$. Thus, although the two triangles $\triangle A B C$ and $\triangle A C B$ are identical, the correspondence between their vertices is not the identity correspondence. In proving the congruence relation $\triangle \mathrm{ABC} \cong \triangle \mathrm{ACB}$, we are tacitly demonstrating that there is a rigid motion of the plane containing $\triangle A B C$ that keeps point $A$ fixed and interchanges points $B$ and C , although the existence of this rigid motion doesn't arise explicitly in the proof. Since this rigid motion fixes A while interchanging B and $C$, then it can't be the identity and, in fact, must be a reflection in the perpendicular bisector of the line segment $\overline{B C}$.

## 2) If $\angle B$ is congruent to $\angle C$, then $\overline{A B}$ is congruent to $\overline{A C}$.

## Proof of 2).

1) Assume $\angle B \cong \angle C$. (We must prove $\overline{A B} \cong \overline{A C}$.)
2) $\overline{B C} \cong \overline{B C}$, because the identity is a rigid motion that moves $\overline{B C}$ to itself.
3) Therefore, $\triangle A B C \cong \triangle A C B$ by the $\qquad$ Axiom.
(The correspondence between the vertices of $\triangle A B C$ and the vertices of $\triangle A C B$ that we are using here is: $A \rightarrow A, B \rightarrow C, C \rightarrow B$.)
4) The rigid motion that moves $\triangle A B C$ to $\triangle A C B$, moves $\overline{A B}$ to $\overline{A C}$. Hence, $\overline{A B} \cong \overline{A C}$.

This completes the proof of 2 ).
Remark. In this proof, as in the previous proof, when we establish the congruence relation $\triangle \mathrm{ABC} \cong \triangle \mathrm{ACB}$, we are tacitly (but not explicitly) demonstrating the existence of a rigid motion of the plane containing $\triangle A B C$ that fixes vertex $A$ and interchanges vertices $B$ and $C$. This rigid motion must be the reflection in the perpendicular bisector of the line segment $\overline{B C}$.

Our final group of axioms for geometry involve the concept of parallel lines and some related ideas. We now define these notions.

Definition. Two lines in space are parallel if they both lie in a plane and don't intersect. Two lines in space which don't lie in a single plane are called skew lines.


Parallel Lines


Skew Lines

Definition. Let $\mathrm{L}, \mathrm{M}$ and T be lines in the same plane such that T intersects both $L$ and $M$, as in the picture below. Then line $T$ is called a transversal to lines $L$ and $M$. In the figure below, the pair of angles labeled a and $\mathrm{a}^{\prime}$ are called corresponding angles. (The pair of angles labeled b and $\mathrm{b}^{\prime}$ are also called corresponding angles, as are the pair $c$ and $c^{\prime}$ and the pair $d$ and $d^{\prime}$.) Also in this figure, the pair of angles labeled $c$ and $\mathrm{a}^{\prime}$ are called alternate interior angles or simply alternate angles. (The pair of angles labeled $b$ and d' are also called alternate interior angles.) Also in this figure the pair of angles labeled b and $\mathrm{a}^{\prime}$ are called supplementary angles. (The pair of angles labeled c and d' are also called supplementary angles.)


We now state four basic axioms about parallel lines. Again we assume these axioms to be true.

The Parallel Postulate. If $L$ is a line and $P$ is a point not on $L$, then there is exactly one line that passes through $P$ and is parallel to $L$.

The Corresponding Angles Axiom. Let $\mathrm{L}, \mathrm{M}$ and T be lines in the same plane such that $T$ intersects both $L$ and $M$, as in the figure above. Then $L$ and $M$ are parallel if and only if a pair of corresponding angles are congruent.

The Alternate Interior Angles Axiom. Let L, M and T be lines in the same plane such that $T$ intersects both $L$ and $M$, as in the figure above. Then $L$ and $M$ are parallel if and only if a pair of alternate interior angles are congruent.

The Supplementary Angles Axiom. Let $\mathrm{L}, \mathrm{M}$ and T be lines in the same plane such that $T$ intersects both $L$ and $M$, as in the figure above. Then $L$ and $M$ are parallel if and only if the angle measures of a pair of supplementary angles add up to $180^{\circ}$.

We have just stated four parallel line principles as axioms. As with the congruence axioms, we could have taken a more economical approach: we could have stated only one parallel line axiom and proved the other three principles as theorems from this axiom. Indeed, Euclid took this approach in The Elements where he stated a single parallel line principle as an axiom and proved the other parallel line principles from it. (Euclid's basic parallel line axiom was similar to the Supplementary Angles Axiom.) As we did with the congruence axioms, we avoid this more economical approach because the process of proving three parallel line principles from one is long and complicated and is not consistent with the goals of this course. We prefer to assume all four parallel line principles as axioms and have the immediate use of them to give simple proofs of other theorems

We now use the axioms we have assumed so far to prove two very important theorems of geometry.

Theorem 3: The Exterior Angle Theorem. In a triangle, the measure of the exterior angle at a vertex equals the sum of the measures of the interior angles at the other two vertices. In other words, in triangle $\triangle A B C$ pictured below,

$$
\mathrm{m}(\angle B C D)=\mathrm{m}(\angle A)+\mathrm{m}(\angle B) .
$$



Activity 3. The class as a whole should carry out the following activity. A proof of the Exterior Angle Theorem is given below. However, in six lines of the proof there are blanks where the name of one of the previously stated axioms or the numbers of earlier lines of the proof need to be entered to justify the line. Fill in the name of an appropriate axiom or the appropriate line numbers to justify each of the six lines of the proof in which a blank occurs.

## Proof of the Exterior Angle Theorem.

1) Draw a ray $C E$ emanating from the point $C$ parallel to the line $A B$ so that $E$ is interior to the angle $\angle B C D$, as in the figure below. (This ray exists by the Parallel Postulate.)

2) $\mathrm{m}(\angle \mathrm{BCD})=\mathrm{m}(\angle \mathrm{DCE})+\mathrm{m}(\angle \mathrm{BCE})$, by $\qquad$ .
3) $\angle A$ is congruent to $\angle D C E$ by $\qquad$ .
4) Therefore, $m(\angle A)=m(\angle D C E)$ by $\qquad$ .
5) $\angle B$ is congruent to $\angle B C E$ by $\qquad$ .
6) Therefore, $m(\angle B)=m(\angle B C E)$ by
7) Hence, $m(\angle B C D)=m(\angle A)+m(\angle B)$, by lines $\qquad$ -.

This completes the proof.
We now introduce some well known terminology that will be used in the homework problems.

Definition. The midpoint of a line segment $\overline{\mathrm{AB}}$ is a point C on $\overline{\mathrm{AB}}$ such that $\overline{A C} \cong \overline{B C}$. Thus, the Congruence Axiom for Line Segments implies that $C$ is the midpoint of $\overline{A B}$ if and only if $C$ is a point on $\overline{A B}$ such that $A C=B C$.


Definition. Let $S$ be a line segment and let $L$ be a line or a ray or a line segment. We say that $L$ bisects $S$ and we call $L$ a bisector of $S$ if $L$ crosses $S$ at the midpoint of $S$.

Definition. An angle is a right angle if its angle measure is $90^{\circ}$.
Definition. Two lines $L$ and $M$ are perpendicular if they intersect and if any of the four angles between the two lines is a right angle.


Definition. A line $L$ is a perpendicular bisector of a line segment $\overline{A B}$ if $L$ bisects $\overline{A B}$ (i.e., $L$ crosses $\overline{A B}$ at the midpoint $C$ of $\overline{A B}$ ) and $L$ is perpendicular to the line $\overleftrightarrow{A B}$.


Definition. Let $\angle B A C$ be an angle and let $\overrightarrow{A D}$ be a ray originating at the point A. We say that the ray $\overrightarrow{\mathrm{AD}}$ bisects the angle $\angle \mathrm{BAC}$ and we call $\overrightarrow{\mathrm{AD}}$ the bisector of $\angle B A C$ if the point $D$ lies in the interior of the angle $\angle B A C$ and $\angle B A D \cong \angle D A C$. Thus, the Congruence Axiom for Angles implies that $\overrightarrow{A D}$ is the bisector of $\angle B A C$ if and only if the point D lies in the interior of the angle $\angle B A C$ and $m(\angle B A D)=m(\angle D A C)$.


We close this lesson with:
Some General Remarks on the Philosophy of Proof. The method of establishing geometric facts by proof from axioms is also known as the axiomatic method. It is a completely reliable procedure for growing our geometric knowledge. It is completely reliable because it starts from basic axioms that we know are true (along with some basic terms and definitions), and it accepts new statements as true only if their truth can be argued logically from the basic axioms or from other previously proved truths. This procedure is very conservative. It does not allow a statement to be
asserted on a line of a proof simply because it looks like its true in all the pictures that you can draw or because you think you remember the truth of a similar statement being asserted in another class. A statement is acceptable as a line in a proof only if it follows logically from an axiom or a previously proved statement. The very restrictiveness of the procedure is what guarantees its reliability. It prevents the process of accumulating geometric facts from being contaminated by dubious statements.

A metaphor for the process of proving new geometric facts from the axioms is the growth of a tree. The roots represent the basic terms, definitions and axioms that we assume to be true. Moving to a new fact through proof is like growing a new branch of the tree. Moving from one step of a proof to the next is like growing a new twig out of a joint or node of the tree. The new twig can grow only if it is securely founded on the part of the tree that lies beneath it: the roots together with the joints of the tree that lie between the new twig and the roots. The roots represent the axioms that we assume to be true, and the joints of the tree between the new twig and the roots represent previously proved facts. Like a new twig of a tree, each new step in a proof must be securely based on the structure beneath it. The roots of this structure are, as we said earlier, the axioms we have assumed to be true. The nodes of the structure between the axioms and the new step of the proof consist of previously proved truths that either occur in earlier theorems or appear on lines of the proof that precede the new step.

To be justified, each new line in a proof must be related to the mathematical structure that precedes it (axioms, previously proved theorems, and earlier lines of the proof) in a specific way. Namely, the new line of the proof must fall into one of the following five categories.

- It is a logical or mathematical truth like $x+y=y+x$.
- It is one of basic principles or axioms that we have assumed to be true.
- It is a previously proved theorem.
- It is a hypothesis (the if... part) of an if..., then... statement that you are trying to prove.
- It follows logically from previous lines in the proof.

When you write a proof, you should check that each line of your proof falls into one of these five categories.

Homework Problem 1. In parts a) - j) of this problem, decide whether the two given triangles must necessarily be congruent. If they must be congruent, express this by writing a correct equation of the form " $\triangle \mathrm{ABC} \cong \triangle \mathrm{DEF}$ ", and write down the acronym for the congruence axiom (SAS, ASA, SSS or AAS) that justifies your conclusion. If they are not necessarily congruent, then write "not congruent".

b)

c) In $\triangle \mathrm{NPQ}, \mathrm{m}(\angle \mathrm{N})=80^{\circ}, \mathrm{m}(\angle \mathrm{P})=60^{\circ}$ and $\mathrm{m}(\angle \mathrm{Q})=40^{\circ}$. In $\triangle$ RST, $m(\angle R)=60^{\circ}, m(\angle S)=80^{\circ}$ and $m(\angle \mathrm{~T})=40^{\circ}$.

e) In $\triangle A B C, m(\angle A)=55^{\circ}, m(\angle C)=81^{\circ}$ and $A B=4 \mathrm{~cm}$.

In $\triangle X Y Z, m(\angle X)=81^{\circ}, m(\angle Y)=55^{\circ}$ and $X Y=4 \mathrm{~cm}$.
f) In $\triangle \mathrm{DEF}, \mathrm{m}(\angle \mathrm{E})=105^{\circ}, \mathrm{m}(\angle \mathrm{F})=39^{\circ}$ and $\mathrm{EF}=12 \mathrm{in}$.

In $\triangle U V W, m(\angle W)=39^{\circ}, U W=12$ in and $m(\angle U)=105^{\circ}$.
g)

h) In $\triangle \mathrm{KLM}, \mathrm{LM}=25 \mathrm{~cm}, \mathrm{~m}(\angle \mathrm{~L})=21^{\circ}$ and $\mathrm{KM}=14 \mathrm{~cm}$. In $\triangle N P Q, m(\angle Q)=21^{\circ}, N P=14 \mathrm{~cm}$ and $N Q=25 \mathrm{~cm}$.
i) In $\triangle A C E, A C=23$ in, $C E=38$ in and $A E=19$ in. In $\triangle B D F, B D=19 \mathrm{in}, \mathrm{BF}=23 \mathrm{in}$ and $\mathrm{DF}=38 \mathrm{in}$.
j)


Homework Problem 2. Theorem 4 - The Angle Sum Theorem - is stated below, and a proof is given. However, on five lines of the proof there are blanks where the name of an axiom or a previously proved theorem or the numbers of earlier lines of the proof need to be entered to justify the line. In each of these blanks, fill in the name of the appropriate axiom or theorem or the appropriate line numbers.

Theorem 4: The Angle Sum Theorem. The measures of the angles of a triangle add up to $180^{\circ}$. In other words, in triangle $\triangle A B C$ pictured below,

$$
m(\angle A)+m(\angle B)+m(\angle C)=180^{\circ} .
$$



## Proof of the Angle Sum Theorem.

1) Extend the line segment $\overline{A C}$ to a ray $\overline{A D}$ as in the figure below.

2) Then $m(\angle A C D)=180^{\circ}$ by $\qquad$ -
3) $\mathrm{m}(\angle A C D)=\mathrm{m}(\angle B C D)+\mathrm{m}(\angle C)$ by $\qquad$ .
4) Therefore, $m(\angle B C D)+m(\angle C)=180^{\circ}$, by lines $\qquad$ .
5) $m(\angle B C D)=m(\angle A)+m(\angle B)$ by $\qquad$ .
6) Therefore, $m(\angle A)+m(\angle B)+m \angle C)=180^{\circ}$, by lines $\qquad$ .

This completes the proof.

Homework Problem 3. Here we present an alternative proof of
Theorem 2: The Isosceles Triangle Theorem. Let $\triangle A B C$ be a triangle. Then $\overline{\mathrm{AB}}$ is congruent to $\overline{\mathrm{AC}}$ if and only if $\angle \mathrm{B}$ is congruent to $\angle \mathrm{C}$.

Recall that because this theorem asserts an if and only if statement, then it splits into two separate assertions each of which conveys independent information and each of which must be proved separately. The two assertions conveyed by this statement are:

1) if $\overline{A B}$ is congruent to $\overline{A C}$, then $\angle B$ is congruent to $\angle C$, and
2) if $\angle B$ is congruent to $\angle C$, then $\overline{A B}$ is congruent to $\overline{A C}$.

The proofs of assertions 1) and 2) which are given below contain blanks. Each blank must filled in with the name of an axiom or a statement to justify the line in which the blank appears. Fill in each blank with the appropriate statement or axiom name.

## Proof of 1).

1) Assume $\overline{A B} \cong \overline{A C}$. (We must prove that $\angle B \cong \angle C$.)
2) Let $\overrightarrow{A D}$ be the ray emanating from the point $A$ that bisects $\angle A$, and let $E$ be the point where the ray $\overrightarrow{A D}$ intersects the line segment $\overrightarrow{B C}$.

3) Then $\angle B A E \cong \angle C A E$. (This is what " $\overrightarrow{A D}$ bisects $\angle A$ " means.)
4) $\overline{\mathrm{AE}} \cong \overline{\mathrm{AE}}$ because $\qquad$ .
5) Therefore, $\triangle \mathrm{ABE} \cong \triangle \mathrm{ACE}$ by the $\qquad$ Axiom.
6) The rigid motion that moves $\triangle A B E$ to $\triangle A C E$, moves $\angle B$ to $\angle C$. Hence, $\angle B \cong \angle C$.

This completes the proof of 1 ).

## Proof of 2).

1) Assume that $\angle B \cong \angle C$. (We must prove $\overline{A B} \cong \overline{A C}$.)
2) Let $\overrightarrow{A D}$ be the ray emanating from the point $A$ that bisects $\angle A$, and let $E$ be the point where the ray $\overrightarrow{A D}$ intersects the line segment $\overrightarrow{B C}$.

3) Then $\angle B A E \cong \angle C A E$. (This is what " $\overrightarrow{A D}$ bisects $\angle A$ " means.)
4) $\overline{\mathrm{AE}} \cong \overline{\mathrm{AE}}$ because $\qquad$ .
5) Therefore, $\triangle A B E \cong \triangle A C E$ by the $\qquad$ Axiom.
6) The rigid motion that moves $\triangle A B E$ to $\triangle A C E$, moves $\overline{A B}$ to $\overline{A C}$. Hence, $\overline{A B} \cong \overline{A C}$. This completes the proof of 2 ).

Homework Problem 4. Theorem 5 is stated below, and a proof is given. However, there are blanks in several lines of the proof where the name of an axiom or a previously proved theorem or the numbers of earlier lines of the proof or a statement need to be entered to justify the line. Fill in the name of the appropriate axiom or theorem or the appropriate line numbers or the appropriate statement in each of these blanks.

Theorem 5. Let $\triangle \mathrm{ABC}$ be an isosceles triangle with $\overline{\mathrm{AB}}$ congruent to $\overline{\mathrm{AC}}$, and let $D$ be a point of $\overline{B C}$. Then $\overline{A D}$ bisects $\overline{B C}$ if and only if $A D$ is perpendicular to $B C$.


Remark. Since this theorem asserts an if and only if statement, then it splits into two separate assertions each of which conveys independent information and each of which must be proved separately. The two assertions conveyed by this statement are:

1) if $\overline{A D}$ bisects $\overline{B C}$, then $\overline{A D}$ is perpendicular to $\overline{B C}$, and
2) if $\overline{A D}$ is perpendicular to $\overline{B C}$, then $\overline{A D}$ bisects $\overline{B C}$.

## Proof of 1):

1) By hypothesis, $\overline{A B} \cong \overline{A C}$, and $D$ is a point of $\overline{B C}$.
2) Assume that $\overline{A D}$ bisects $\overline{B C}$.
3) Then $D$ is the midpoint of $\overline{B C}$.
4) Thus, $\overline{B D} \cong \overline{C D}$.
5) Hence, $\triangle \mathrm{BDA} \cong \triangle \mathrm{CDA}$ by $\qquad$ .
6) Therefore, $\angle B D A \cong \angle C D A$, because $\qquad$ .
7) Hence, $m(\angle B D A)=m(\angle C D A)$ by $\qquad$ .
8) Since B, D and C are collinear and D is between $B$ and $C$, then $m(\angle B D C)=180^{\circ}$, by $\qquad$ -.
9) $m(\angle B D C)=m(\angle B D A)+m(\angle C D A)$ by
10) Therefore, $m(\angle B D A)+m(\angle C D A)=180^{\circ}$ by $\qquad$ .
11) Thus, $2 m(\angle B D A)=180^{\circ}$ by $\qquad$ .
12) Dividing both sides of the equation in line 11) by 2 gives: $m(\angle B D A)=90^{\circ}$.
13) Therefore, $\angle B D A$ is a right angle.
14) Hence, if $A D$ is perpendicular to $B C$.

This completes the proof of 1 ).

## Proof of 2).

1) By hypothesis, $\overline{A B} \cong \overline{A C}$, and $D$ is a point of $\overline{B C}$.
2) Assume $A \bar{D}$ is perpendicular to $B C$.
3) Therefore, $\angle B D A$ and $\angle C D A$ are right angles.
4) Hence, $m(\angle B D A)=90^{\circ}=m(\angle C D A)$.
5) Thus, $\angle \mathrm{BDA} \cong \angle \mathrm{CDA}$ by
6) Since $\triangle A B C$ is an isosceles triangle with $\overline{\mathrm{AB}}$ congruent to $\overline{\mathrm{AC}}$, then $\angle \mathrm{B} \cong \angle \mathrm{C}$ by $\qquad$ .
7) Therefore, $\triangle \mathrm{BDA} \cong \Delta \mathrm{CDA}$ by $\qquad$ .
8) Hence, $\overline{\mathrm{BD}} \cong \overline{\mathrm{CD}}$, because $\qquad$ .
9) Thus, $D$ is the midpoint of $\overline{B C}$.
10) Therefore, $\overline{A D}$ bisects $\overline{B C}$.

This completes the proof 2).

