Lesson 14: An Axiom System for Geometry

We are now ready to present an *axiomatic development* of geometry. This means that we will list a set of *axioms* for geometry. These axioms will be simple fundamental facts about geometry which we will assume to be true. We will not require further justification for the axioms.

The statements of the axioms will involve certain basic geometric words like "point" and "line". We will not define these words; we will assume that you have an intuitive idea of their meaning. Your intuitive understanding of these words may help you to visualize the concepts they represent; however, all the hard information you will need about these words is expressed in the axioms. You do not need to bring any extra information about these geometric words to the process of developing geometry from its axioms. In fact, it would be wrong to bring any information to this task that is not contained in the axioms or their logical consequences.

In addition to these basic words, we will find it convenient to introduce other geometric words which we will define in terms of the basic words.

Finally, we will state and prove a number of *theorems* of geometry. A theorem is a true geometric statement that is not an axiom. The axioms are foundational statements that we assume to be true without proof. However, the theorems require proof to establish their truth. For each theorem that we state, we will either present its complete proof or present a partial proof or present no proof and leave some aspect of the proof or all of the proof as a class activity or homework problem.

Of course, many geometric statements are false. There is no hope of proving such statements. If we suspect that a statement is false, we need a technique for demonstrating its falsity. The usual method for showing that a geometric statement is false is to present a *counterexample* to the statement. A *counterexample* to a geometric statement is a description or picture of a situation in which all the axioms of geometry are true, but the statement in question is false.

To summarize: in an axiomatic development of geometry, a true geometric statement or theorem is shown to be true by presenting a proof of it from the axioms or from previously proved theorems. A false geometric statement is shown to be false by presenting a counterexample in which the statement is false but all the axioms are true.

Historically, the earliest known axiomatic development of geometry is the book *The Elements* by the Greek mathematician Euclid. Euclid lived in Alexandria, Egypt about 300 B.C. Egypt was conquered by Alexander the Great in 331 B.C. He created the city of Alexandria and made it the capital of Egypt. When Alexander died at age 33 in 323 B.C., his empire was divided among his generals. The name of the general who took control of Egypt was Ptolemy. Ptolemy and his descendants ruled Egypt until it was conquered by the Romans in 80 B.C. (The last member of the Ptolemaic dynasty to rule Egypt was Cleopatra.) Alexander and his general Ptolemy who were of Greek heritage made Alexandria the primary center of Greek culture and learning, and the greatest Greek thinkers of that era found their way to Alexandria, including Euclid.

In writing *The Elements,* Euclid codified the previous three centuries of work of Greek mathematicians including Thales, Pythagorus and Plato. (Thales first learned of geometry in Egypt and brought it to ancient Greece around 600 B.C.) The contribution of these mathematicians was to view geometry as a subject to be developed axiomatically. They did not view geometry as it had been seen by earlier cultures as simply a large collection of related facts that are useful for measurement and design. They realized that all the facts of geometry can be proved logically from a few basic assumptions or *axioms*, and they regarded this observation as a powerful approach to the subject that allowed them to organize it and generate new geometric facts in a very efficient way. Euclid brought together in one book his own geometric discoveries as well as those of his predecessors and organized them into an axiomatic system. In subsequent centuries, *The Elements* became the primary model for the organization of knowledge in science, philosophy and many other areas of human thought, and has persisted in this role to the present day.

In *The Elements* Euclid based geometry on only five axioms or *postulates.* Our development of geometry is much less efficient than Euclid's; we will base our axiomatic approach to geometry on 14 axioms. The basic reason that we tolerate this glut of axioms is to save time. We could pare our list of axioms down to a much smaller list and then use the axioms on the smaller list to prove all the others. But this would be a complicated undertaking that would consume much thought and time and would be inappropriate for this course. Also with a larger list of axioms, it becomes easier to prove theorems. We have chosen for our axioms statements that are reasonably familiar and self-evident and which are numerous enough to lead to relatively straightforward, short and transparent proofs of the theorems we want to establish.

We begin our axiomatic development of geometry by introducing the basic geometric terms *point, line, plane* and *space* (meaning *3-dimensional space*). We assume that you know the meaning of these terms. We will not define them.[1] We also assume that you understand that for any two points in space, there is a *distance* between the two points that is a specific number which is ≥ 0.

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[1] In every mathematical theory, there must be some *undefined terms.* This is because every *defined term* has a definition which explains the term by relating it to terms that are already understood. So the very first terms of a subject can't be defined because there are no previously understood terms to which they can be related. *Point, line, plane* and *space* are some of the undefined terms in our development of geometry.

We recall two logical concepts that are useful in all areas of mathematics, not just geometry.

- Two objects are *distinct* if they are not identical. Thus, A and B are distinct if $A \neq B$.
- An object is the *unique* object with certain properties, if the object has the properties and it is the only object that has these properties.

Now we state our first axiom for geometry.

The Incidence Axiom. Any two distinct points in space lie on a unique line. In other words, if A and B are distinct points in space ($A \neq B$), then there is one and only one line that passes through both A and B.

Sometimes we paraphrase this principle by saying that any two distinct points in space *determine* a unique line. Here, the line determined by the two distinct points is the unique line that passes through the two points.[2]

We will use the same basic notation for lines, line segments, rays and distance that we introduced earlier.

Notation. Let A and B be points in space.

- If $A \neq B$, let $\overleftrightarrow{AB}$ denote the unique line determined by A and B.
- Let $\overline{AB}$ denote the *line segment* with endpoints A and B. Thus, $\overline{AB}$ consists of all the points of the line $\overleftrightarrow{AB}$ that lie between A and B, together with the points A and B themselves.
- If $A \neq B$, let $\overrightarrow{AB}$ denote the *ray* that emanates from A and passes through B. Thus, $\overrightarrow{AB}$ consists of all the points of the line $\overleftrightarrow{AB}$ that lie either between A and B or on the opposite side of B from A, together with the points A and B themselves.
- Let AB denote the distance between A and B.

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[2] The Incidence Axiom which we have just stated is sufficient for a development of 2-dimensional or plane geometry. We are going to concentrate on 2-dimensional geometry in this axiomatic development of geometry. So this Incidence Axiom is all we need. However, if we were going to pursue an axiomatic development of 3-dimensional geometry, we would require the following three additional Incidence Axioms.

Incidence Axiom 2. Any three points in space that do not lie on a single line lie in a unique plane.

Incidence Axiom 3. If two distinct points in space lie in the plane P in space, then the line determined by the two points also lies in the plane P.

Incidence Axiom 4. If two distinct planes in space intersect, then their intersection is a line.

We recall our earlier observations about this notation. First note that if A and B are points in space, then AB is a *number* which is ≥ 0, while $\overleftrightarrow{AB}$, $\overline{AB}$ and $\overrightarrow{AB}$ are *sets* in space. Indeed, AB is the *length* of $\overline{AB}$. **Warning:** Don't write one of these notations when you mean the other. Also, we recall that distance has the following three properties:

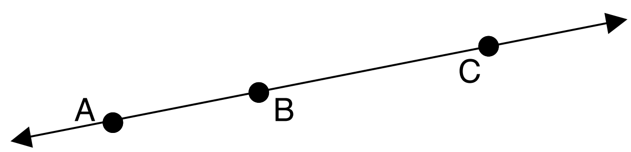
$$AB \geq 0, \quad AB = 0 \text{ if and only if } A = B, \quad \text{and} \quad AB = BA.$$

Definition. Let A, B and C be three points in a plane or in space. If A, B and C all lie on the same line, we say they are *collinear.* If A, B and C do not lie on a single line, then we say they are *noncollinear.*

Here is our second axiom for geometry.

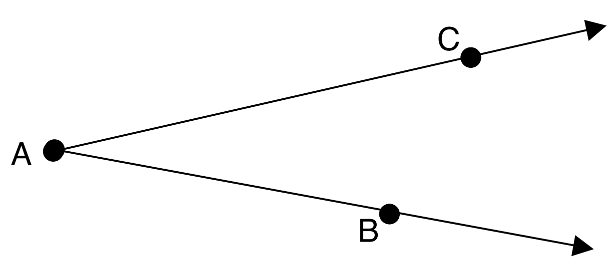
The Length Addition Axiom. If A, B and C are collinear points and B lies between A and C, then

$$AC = AB + BC.$$



Next we recall the definition of *angle.*

Definition. Let A, B and C be three points in space such that A ≠ B and A ≠ C. The union of the two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is called the *angle* with *vertex* A and *sides* $\overrightarrow{AB}$ and $\overrightarrow{AC}$. This angle can be denoted either ∠BAC or ∠CAB.



∠BAC

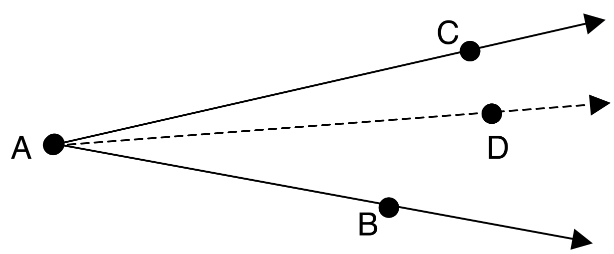
We assume that you understand that every angle $\angle BAC$ has a *measure* that is a number between 0 and 180. The measure of the angle $\angle BAC$ is denoted $m(\angle BAC)$. To indicate that angles are being measured in *degrees,* we attach the symbol "$^\circ$" to the numerical measure of each angle. Thus, $0^\circ \leq m(\angle BAC) \leq 180^\circ$.

We observe that if $\angle BAC$ is an angle in space, then $\angle BAC$ is a *set* in space, while $m(\angle BAC)$ is a *number* between 0° and 180°. **Warning:** Don't write one of these notations when you mean the other.

Here is our third axiom for geometry.

The Angle Addition Axiom. If D is a point that lies in the interior of the angle $\angle BAC$, then

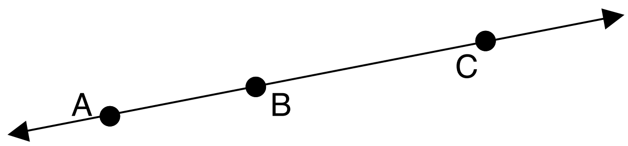
$$m(\angle BAC) = m(\angle BAD) + m(\angle DAC).$$



We remark that the equation "$m(\angle BAC) = m(\angle BAD) + m(\angle DAC)$" makes sense because $m(\angle BAC)$, $m(\angle BAD)$ and $m(\angle DAC)$ are *numbers* and *we know how to add numbers.* On the other hand, the equation "$\angle BAC = \angle BAD + \angle DAC$" does not make sense because $\angle BAC$, $\angle BAD$ and $\angle DAC$ are sets in space, and, in general, *we **don't** know how to add sets.*

(We observe that a similar remark applies to the two equations: "$AC = AB + BC$" and "$\overline{AC} = \overline{AB} + \overline{BC}$". The first equation makes sense because we know how to add numbers, but the second equation doesn't make sense because, in general, we don't know how to add sets.)

We recall some geometry terminology about angles. If A, B and C are collinear points, and B lies between A and C, then the angle $\angle ABC$ is often called a *straight angle.*



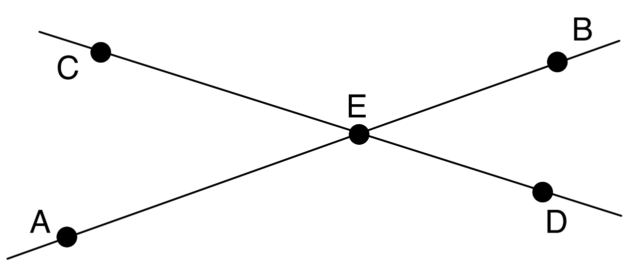
We now state our fourth axiom for geometry.

The Straight Angle Axiom. If A, B and C collinear points and B is between A and C, then

$$m(\angle ABC) = 180^\circ.$$

Up to this point we have stated four axioms of geometry. We have assumed that these four statements are true. Hopefully, their truth is obvious to you. We now come to our first theorem. In other words, the truth of the following geometric statement will *not* simply be assumed. It will be demonstrated by proving it from the previously stated axioms. Before stating this theorem, we need to introduce the concept of *vertical angles.*

Definition. If lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at a point E such that E lies between A and B on line $\overleftrightarrow{AB}$ and E lies between C and D on line $\overleftrightarrow{CD}$, then the angles $\angle AEC$ and $\angle BED$ are called *vertical angles* or *opposite angles*. The angles $\angle AED$ and $\angle BEC$ are also called vertical angles or opposite angles.



Theorem 1: The Vertical Angles Theorem. Vertical angles have equal measures. In other words, if $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are lines that intersect at a point E such that E lies between A and B on line $\overleftrightarrow{AB}$ and E lies between C and D on line $\overleftrightarrow{CD}$, then $m(\angle AEC) = m(\angle BED)$. (Also $m(\angle AED) = m(\angle BEC)$.)

Activity 1. The class as a whole should carry out the following activity. A proof of Theorem 1 is given below. However, in seven lines of the proof there are blanks where the name of one of the previously stated axioms or the numbers of earlier lines of the proof need to be entered to justify the line. Fill in the name of an appropriate axiom or the appropriate line numbers to justify each of the seven lines of the proof in which a blank occurs.

Proof of the Vertical Angles Theorem.

1) m(∠AEB) = m(∠AEC) + m(∠CEB) by the ____________________ Axiom.

2) m(∠AEB) = 180° by the ____________________ Axiom.

3) Therefore, m(∠AEC) + m(∠CEB) = 180° by lines ____________________ .

4) m(∠CED) = m(∠CEB) + m(∠BED) by the ____________________ Axiom.

5) m(∠CED) = 180° by the ____________________ Axiom.

6) Therefore, m(∠CEB) + m(∠BED) = 180° by lines ____________________ .

7) Hence, m(∠AEC) + m(∠CEB) = m(∠CEB) + m(∠BED) by lines _______________.

8) Subtracting m(∠CEB) from both sides of the equation in line 7) gives:
m(∠AEC) = m(∠BED).

The proof that m(∠AED) = m(∠BEC) is similar.

This completes the proof of the Vertical Angles Theorem.

We continue our axiomatic development of geometry with a review of two of the most important concepts of the subject: *rigid motion* and *congruence*. Recall:

Definition. A *rigid motion* of a plane is a motion of points of the plane to other points of the plane that preserves the distance between any two points. Thus, M is a *rigid motion* of a plane P if:

- M moves each point A of the plane P to another point M(A) of the plane P,

and

- if A and B are any two points of the plane P, then the distance from A to B equals the distance from M(A) to M(B). (In terms of our notation: AB = M(A)M(B).)

A rigid motion of space can be defined similarly. It is a distance preserving motion of the points of space.

Recall that we have identified four different types of rigid motions of a plane: *translations, rotations, reflections* and *glide reflections.* Furthermore, according to the

Classification Theorem for Rigid Motions of a Plane, every rigid motion of a plane is of one of these for types.

Once we have introduced the notion of a *rigid motion*, we are in a position to define the most important idea in geometry: *congruence.*

Definition. Two geometric figures in a plane are *congruent* if and only if there is a rigid motion of the plane that moves one figure to the other.

We remark that we can similarly define the relation of congruence for two geometric figures in space. Two geometric figures in space are *congruent* if there is a rigid motion of space that moves one figure to the other.

Notation. If S and T are geometric figures in a plane, then we will write S $\cong$ T to indicate that S is congruent to T. Thus, S $\cong$ T means there is a rigid motion of the plane containing S and T that moves S to T.

Here are our first two axioms involving the notion of congruence.

The Congruence Axiom for Line Segments. Two line segments are congruent if and only if they have the same length. Thus, $\overline{AB} \cong \overline{CD}$ if and only if AB = CD.

The Congruence Axiom for Angles. Two angles are congruent if and only if they have the same measures. Thus, $\angle ABC \cong \angle DEF$ if and only if $m(\angle ABC) = m(\angle DEF)$.

Observe that each of these Congruence Axioms is an "if and only if" statement. Thus, each of these axioms breaks into two separate assertions which convey independent information.

For example, the Congruence Axiom for Line Segments implies the two assertions:

1) If $\overline{AB} \cong \overline{CD}$, then AB = CD.

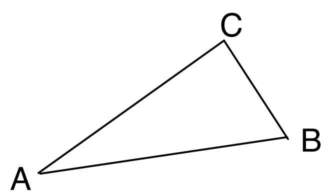
2) If AB = CD, then $\overline{AB} \cong \overline{CD}$.

Assertion 1) says that if there is a rigid motion that moves $\overline{AB}$ to $\overline{CD}$, then AB = CD. This assertion isn't surprising because rigid motions preserve distance. Assertion 2) says that if the distances AB and CD are equal, then there must exist a rigid motion that moves $\overline{AB}$ to $\overline{CD}$. This assertion 2) is a more substantial claim because it promises the existence of a rigid motion based only on the equality of two distances.

Similarly, the Congruence Axiom for Angles breaks into two independent assertions. One assertion says that if there is a rigid motion moving angle $\angle ABC$ to angle $\angle DEF$, then these two angles have the same measure. The other assertion says that if the two angles have the same measure, then there must exist a rigid motion that moves one angle to the other. Again the second assertion is more substantial.

One of the simplest types of geometric objects is a *triangle.* We will study the congruence of *triangles* extensively. We begin by defining this class of geometric objects.

Definition. Let A, B and C be three points in space. The union of the three line segments $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$ is called a *triangle.* We denote this triangle by ΔABC. (Observe that the same triangle can also be denoted by writing ΔBCA, ΔCAB, ΔACB, ΔBAC and ΔCBA.) The points A, B and C are called the *vertices* of the triangle. The line segments $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$ are called the *sides* of the triangle. We write $\angle$A, $\angle$B and $\angle$C as abbreviations for the three angles $\angle$CAB, $\angle$ABC and $\angle$BCA of the triangle.



Observe that if the two triangles ΔABC and ΔDEF that lie in the same plane are congruent, then there is a rigid motion of the plane that moves ΔABC to ΔDEF. This rigid motion will move the vertices A, B and C of the triangle ΔABC to the vertices D, E and F of the triangle ΔDEF. There are potentially six different ways that a rigid motion can create a correspondence between the vertices A, B, C and the vertices D, E, F:

- A $\to$ D, B $\to$ E, C $\to$ F,
- A $\to$ D, B $\to$ F, C $\to$ E,
- A $\to$ E, B $\to$ F, C $\to$ D,
- A $\to$ E, B $\to$ D, C $\to$ F,
- A $\to$ F, B $\to$ D, C $\to$ E, and
- A $\to$ F, B $\to$ E, C $\to$ D.

So the assertion that ΔABC is congruent to ΔDEF tells us only that there are possibly as many as six different rigid motions that move triangle ΔABC to triangle ΔDEF. It does not tell us which of these six possibilities is actually realized by a rigid motion that moves ΔABC to ΔDEF. We now enrich definition of *congruence* and the notation "$\cong$" to overcome this failing.

Definition. When we use the notation "≅" with triangles, we will endow it with extra meaning. If ΔABC and ΔDEF are triangles in a plane, then the writing "$\Delta ABC \cong \Delta DEF$" will mean more than just that the two triangles ΔABC and ΔDEF are congruent. "$\Delta ABC \cong \Delta DEF$" will mean that there is a rigid motion of the plane that moves vertex A to vertex D, vertex B to vertex E, and vertex C to vertex F. Such a rigid motion will then necessarily move edge $\overline{AB}$ to edge $\overline{DE}$, edge $\overline{BC}$ to edge $\overline{EF}$, and edge $\overline{AC}$ to edge $\overline{DF}$. The rigid motion will also move angle $\angle A$ to angle $\angle D$, angle $\angle B$ to angle $\angle E$, and angle $\angle C$ to angle $\angle F$. Furthermore, when $\Delta ABC \cong \Delta DEF$, we will say that ΔABC *is congruent to* ΔDEF *via a rigid motion that extends the vertex correspondence* $A \to D$, $B \to E$, $C \to F$.

The two triangles ΔABC and ΔDEF pictured below are congruent via a rigid motion that extends the vertex correspondence $A \to E$, $B \to F$, $C \to D$. According to the prescription just stated for the use of the notation "≅", all of the following three statements about these two triangles are also true:

$$\Delta ABC \cong \Delta EFD, \quad \Delta BAC \cong \Delta FED, \quad \text{and} \quad \Delta CAB \cong \Delta DEF.$$

On the other hand, each of the following three statements is false:

$$\Delta ABC \cong \Delta DEF, \quad \Delta BCA \cong \Delta FED \quad \text{and} \quad \Delta CBA \cong \Delta FDE.$$

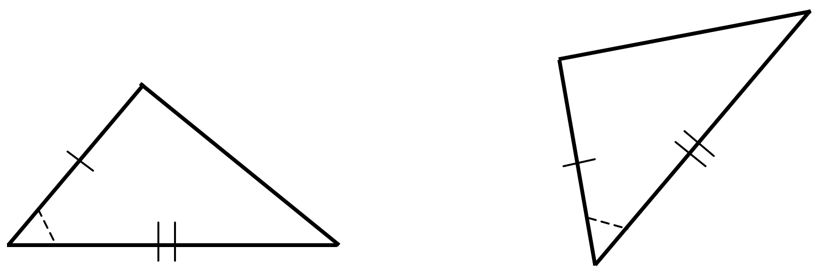
(These are not the only true and false statements expressing congruence relations between these two triangles. For example, $\Delta CBA \cong \Delta DFE$ is true and $\Delta CBA \cong \Delta DEF$ is false.)

We now state four congruence axioms for triangles. By declaring these statements to be axioms, we assume that they are true. We do not require proofs to establish their truth. The four congruence axioms stated next tell us that to demonstrate that triangles T and T′ are congruent, it is sufficient to match up *three* appropriately chosen measurements on T to three corresponding measurements on T′.

The Side-Angle-Side Axiom (SAS). If T and T´ are triangles and there is a correspondence between the vertices of T and the vertices of T´ so that:

- two of the sides of T are congruent to the corresponding sides of T´, and
- the included angle between these two sides of T is congruent to the corresponding angle of T´,

then T and T´ are congruent via a rigid motion that extends the given vertex correspondence.

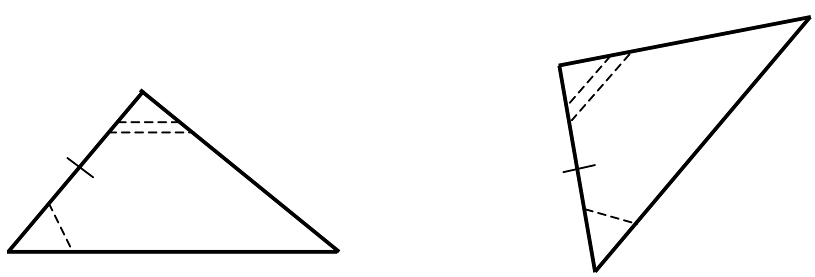


T T´

The Angle-Side-Angle Axiom (ASA). If T and T´ are triangles and there is a correspondence between the vertices of T and the vertices of T´ so that:

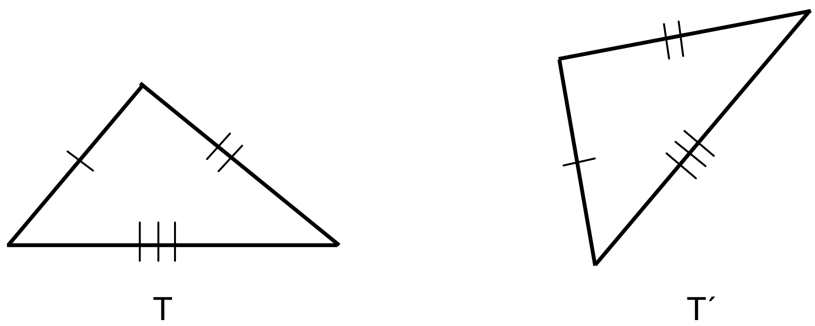
- two of the angles of T are congruent to the corresponding angles of T´, and
- the included side between these two angles of T is congruent to the corresponding side of T´,

then T and T´ are congruent via a rigid motion that extends the given vertex correspondence.



T T´

The Side-Side-Side Axiom (SSS). If T and T´ are triangles and there is a correspondence between the vertices of T and the vertices of T´ so that each side of T is congruent to the corresponding side of T´, then T and T´ are congruent via a rigid motion that extends the given vertex correspondence.



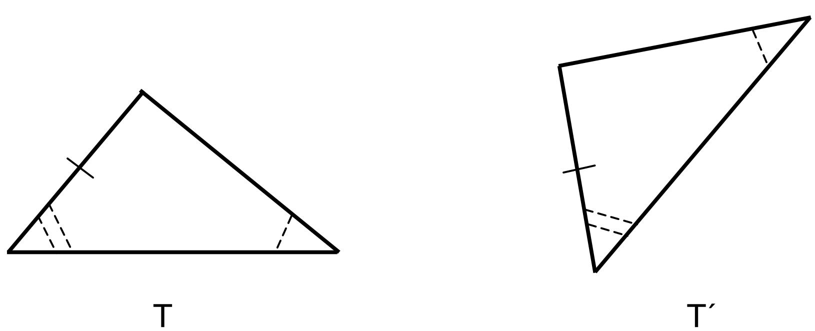
Recall the measures of the three angles of a triangle add up to 180º. Hence, if two angles of a triangle T are congruent to two angles of the triangle T´, then the third angle of T must be congruent to the third angle of T´! It follows that the ASA Axiom can be strengthened: the side doesn't have to be included between the two angles.

The Angle-Angle-Side Axiom (AAS). If T and T´ are triangles and there is a correspondence between the vertices of T and the vertices of T´ so that:

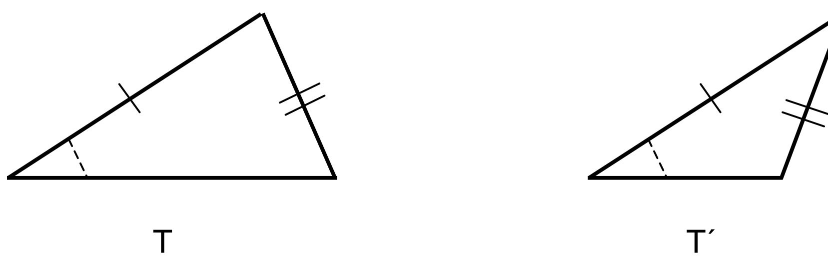
- two of the angles of T are congruent to the corresponding angles of T´, and
- one of the sides of T is congruent to the corresponding side of T´,

then T and T´ are congruent via a rigid motion that extends the given vertex correspondence.

(The selected side of T does not have to be included between the two selected angles of T.)



Warning. Unlike the ASA Principle, the SAS Principle does not generalize. In other words, there is **no ASS principle**. This is because it is possible for two triangles T and T´ to be **non**-congruent and yet have two sides and a non-included angle of T congruent to the corresponding two sides and non-included angle of T´.

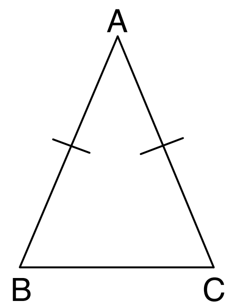


Observe that we have not bothered to formulate the four preceding congruence axioms for triangles as "if and only if" statements. This is because it is obvious that the converses of these four axioms are true, and it is unnecessary to include the assertion of the truth of the converse in the statement of the axiom. Indeed, if the triangles T and T´ are congruent, then there is a rigid motion that carries T to T´. This rigid motion will carry each side or angle of T to the corresponding side or angle of T´. Therefore, the existence of this rigid motion automatically implies that each side or angle of T is congruent to the corresponding side or angle of T´.

We have just stated four congruence principles as axioms. We could have taken a more economical approach: we could have stated only one congruence axiom and proved the other three principles as theorems from this axiom. This is the approach Euclid took in *The Elements* where he stated the SAS principle as an axiom and proved the ASA, SSS and AAS principles from it. We avoid this more economical approach here because the process of proving three congruence principles from one is long and complicated and is not consistent with the focus of this course. We prefer to assume all four congruence principles as axioms and have the immediate use of them to give simple proofs of other theorems.

We now use the congruence axioms to prove a fundamental geometric theorem. First we need a definition.

Definition. A triangle is *isosceles* if two of its sides are congruent.



Theorem 2: The Isosceles Triangle Theorem. Let ΔABC be a triangle. Then $\overline{AB}$ is congruent to $\overline{AC}$ if and only if $\angle B$ is congruent to $\angle C$.

Remark. This theorem asserts an *if and only if* statement. Recall that such statements break into two separate assertions each of which conveys independent information. The two assertions conveyed by this statement are:

1) if $\overline{AB}$ is congruent to $\overline{AC}$, then $\angle B$ is congruent to $\angle C$, and

2) if $\angle B$ is congruent to $\angle C$, then $\overline{AB}$ is congruent to $\overline{AC}$.

To prove this theorem, we will prove assertions 1) and 2) separately.

We will present two different proofs of this theorem, one here and another as Homework Problem 3. Either proof alone is enough to establish the theorem. There are several reasons for presenting two different proofs. One reason is simply to demonstrate that different proofs are possible. Another reason is that different proofs give different insights into the meaning of the theorem. The proof we give here is short but subtle, and perhaps a little hard to understand. The second proof which is appears in Homework Problem 3 is longer but perhaps more obvious. Students may find that it is easier to imagine how someone could come up with the idea for the second proof.

Activity 2. The class as a whole should carry out the following activity. Proofs of parts 1) and 2) of Theorem 2 are given below. However, in one line of each proof there is a blank where the name of one of the previously stated axioms needs to be entered to justify the line. Fill in the name of an appropriate axiom to justify the line of the proof in which a blank occurs.