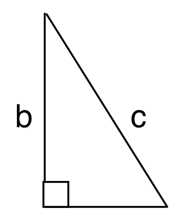
Lesson 13: Proofs in Geometry

Beginning with this lesson and continuing for the next few lessons, we will explore the role of proofs and counterexamples in geometry.

To begin, recall the Pythagorean Theorem.

The Pythagorean Theorem. In a triangle which has sides of lengths a, b and c, if the sides of length a and b form a right angle, then side lengths satisfy the following equation:

$$a^2 + b^2 = c^2.$$



Activity 1. The class as a whole should discuss the following questions.

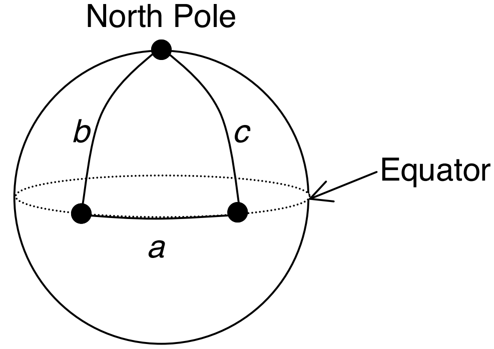
a) *Is the Pythagorean Theorem true?*

b) *In what contexts is the Pythagorean Theorem true?*

c) *How do we decide whether the Pythagorean Theorem is true?*

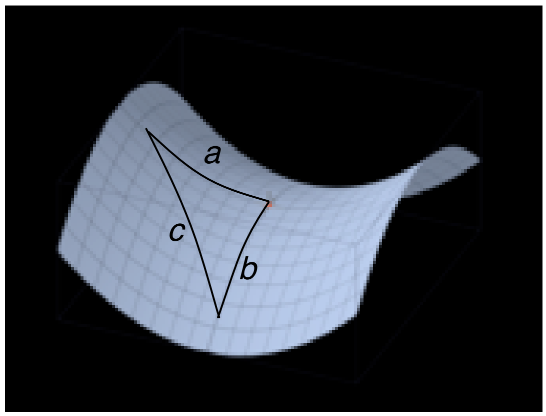
In answering these questions, take into account the following facts.

Consider a “triangle” on the Earth’s surface made from very long tightly stretched



pieces of string. One side which is of length *a* runs along the Equator, and two sides of lengths *b* and *c* running from the North Pole to the Equator. Then the sides of length *a* and *b* form a right angle, as do the sides of length *a* and *c*. Also $b = c = {}^1/_4$ the circumference of the Earth. Hence, $a^2 + b^2 = a^2 + c^2 > c^2$.

Consider a string "triangle" on the surface of a saddle which has sides of length *a, b* and *c* such that the sides of length *a* and *b* form a right angle. Each of the strings follows the shortest path on the saddle surface between its endpoints. In this case it is known that $a^2 + b^2 < c^2$.



It is known that the Earth's gravitational force warps space so that the paths that we think are straight, such as tightly pulled strings or light rays, actually bend a little bit. Furthermore, surfaces that we think are flat planes are actually warped a little bit so that they are either slightly spherical or slightly saddle shaped. Therefore a triangle made from tightly pulled strings that we imagine lies in a flat plane actually lies in a surface that is either slightly spherical or slightly saddle shaped. Hence, if we were able to measure the side lengths of this triangle with a very high degree of accuracy (to within a unit that is the size of the diameter of an atom), we would find that either $a^2 + b^2 > c^2$ or $a^2 + b^2 < c^2$.

There is an "abstract" plane called the *coordinate plane* or the *Cartesian plane* which is a purely mental creation. The points of this plane are *ordered pairs of numbers*, and the distance in this plane between two points (x,y) and (x´,y´) is *defined* to be

$$\sqrt{(x - x')^2 + (y - y')^2}\,.$$

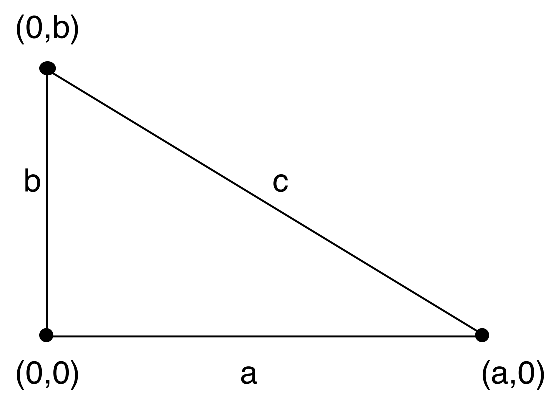
It is not hard to verify that the Pythagorean Theorem holds in this plane.

Below we show a triangle in the coordinate plane with sides of length a, b and c such that the sides of length a and b form a right angle. In this triangle,

$$c = \sqrt{(a-0)^2 + (0-b)^2} = \sqrt{a^2 + b^2}.$$

Therefore,

$$a^2 + b^2 = c^2.$$



In answering the questions in Activity 1, consider the question:

How do we decide which statements about the coordinate plane are true?

According to the prevailing opinion about how to decide whether a mathematical statement is true, the statement can't be considered in isolation. A mathematical statement belongs to a certain subject area (such as geometry). To determine its truth, the statement must be viewed in relation to the other statements of its subject area. In each subject area of mathematics, one must decide which statements are the fundamental principles of the subject area, and these statements must be *assumed* to be true. They require no further verification. These fundamental principles are called the *axioms* or *postulates* of the subject area. (Often the axioms of a subject are simple obviously true statements. In the language of the *Declaration of Independence*, they are *self-evident.*)

If a statement about a certain mathematical subject area is not an axiom, then it is not regarded as self-evident. It may be either true or false. To demonstrate that such a statement is true, one must present a *proof* of it from the axioms of the subject. We call a mathematical statement which is not an axiom but which can be proved from the axioms of a subject a *theorem.*

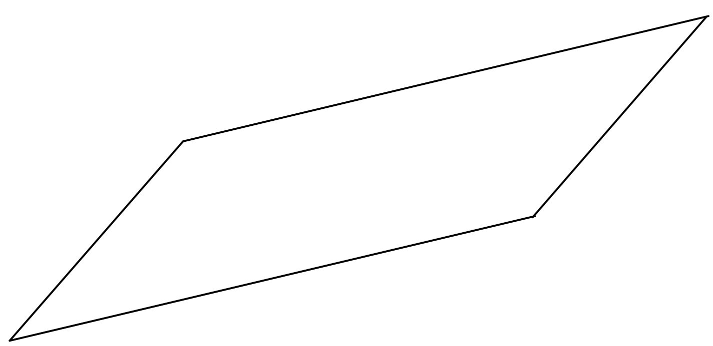
A *proof* of the statement from a set of axioms is a convincing logical argument that the statement follows from the axioms. Since every proof has to build on fundamental principles that are assumed to be true, we see why every mathematical subject area requires axioms. You can't prove something from nothing.

On the other hand, a statement about a mathematical subject area that is not an axiom may be false. To demonstrate that such a statement is false, one must present a *counterexample* to the statement. A *counterexample* to a mathematical statement is a description or picture of a situation in which all the axioms of the subject are true, but the statement in question is false.

In a mathematical subject area, a *conjecture* is a statement that you believe to be true but are not yet sure that it is true. You can try to verify your conjecture by giving a proof of it. If you are unsuccessful in your efforts to prove the conjecture, then it may be false (or it may be true but you have not been clever enough to find a proof of it). To show that your conjecture is false, you must present a counterexample to it, if you can discover one. It can be very hard to discover a proof of a true statement, and it can be equally hard to find a counterexample to a false statement.

Definition. Two lines are *parallel* if they lie in a plane and do not intersect.

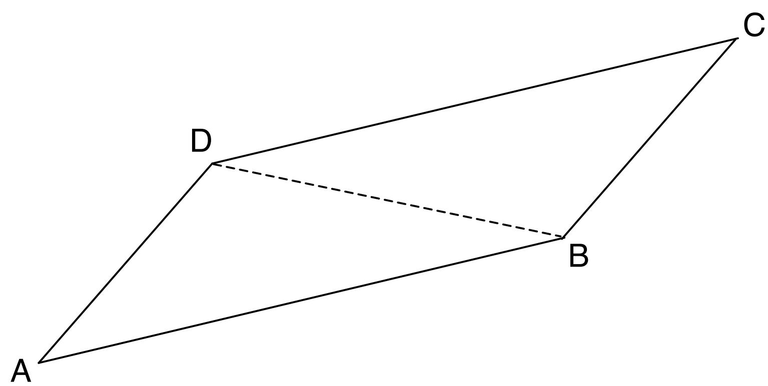
Definition. A 4-sided figure is called a *quadrilateral.* A quadrilateral is a *parallelogram* if each pair of opposite sides lies in a pair of parallel lines.



Activity 2. The class as a whole should carry out these activities. Let ▢ABCD be a parallelogram, and draw the diagonal $\overline{BD}$.

a) Make a conjecture about the relationship between the triangles ΔABD and ΔCDB.

b) Prove your conjecture. Since we have not yet made a list of axioms for geometry, part of the process of constructing this proof will be to decide which fundamental principles of geometry that are needed for this proof should be taken to be axioms. The statements which are taken to be axioms should satisfy the following criteria: they should be *simple* and *self-evident* and should *not be easily proved from other simpler statements*. If you propose that a certain statement be declared an axiom, and the instructor does not feel that your statement satisfies these criteria, he may veto your proposal and force you to show that your statement can be proved from other statements that are more appropriate choices for axioms.



Activity 3. The class as a whole should carry out these activities. Let ▢ABCD be a parallelogram.

a) Make a conjecture about the relationship between the opposite sides of the parallelogram. In other words, make a conjecture about the relationship between the opposite sides $\overline{AB}$ and $\overline{CD}$ that also applies to the relationship between the opposite sides $\overline{AD}$ and $\overline{BC}$.

b) Prove your conjecture.

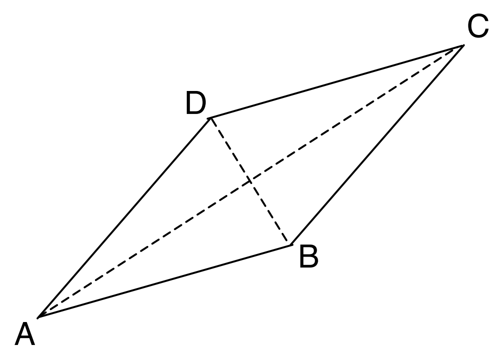
Hint: Use the knowledge gained from Activity 2.

Activity 4. The class as a whole should carry out these activities. Let ◻ABCD be a parallelogram.

a) Make a conjecture about the relationship between the opposite angles of the parallelogram. In other words, make a conjecture about the relationship between the opposite angles $\angle BAD$ and $\angle DCB$ that also applies to the relationship between the opposite angles $\angle ABC$ and $\angle CDA$.

b) Prove your conjecture.

Hint: Again use the knowledge gained from Activity 2.



Activity 5. The class as a whole should carry out these activities. Let ◻ABCD be a quadrilateral, and draw the diagonals $\overline{AC}$ and $\overline{BD}$.

a) Consider the following statement. "If ◻ABCD is a parallelogram, then the diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular." Do you think this statement is true or false?

b) If you think the statement is true, prove it.

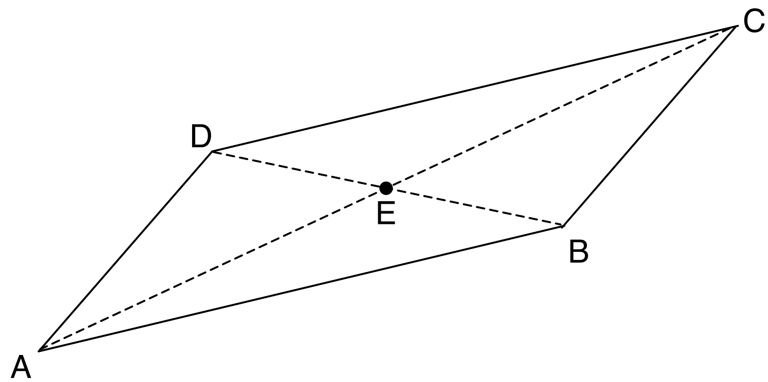
c) If you think the statement is false, draw a counterexample to it.

Activity 6. The class as a whole should carry out these activities. Let ◻ABCD be a quadrilateral, and draw the diagonals $\overline{AC}$ and $\overline{BD}$.

a) Consider the following statement. "If ◻ABCD is a parallelogram, then the diagonal $\overline{AC}$ bisects the angles $\angle BAD$ and $\angle DCB$, and the diagonal $\overline{BD}$ bisects the angles $\angle ABC$ and $\angle CDA$." Do you think this statement is true or false?

b) If you think the statement is true, prove it.

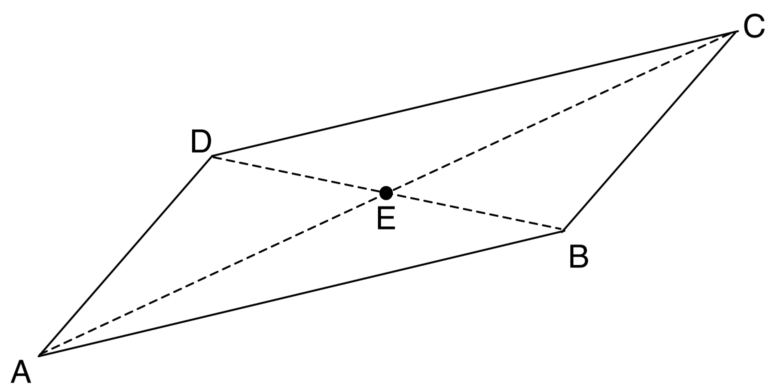
c) If you think the statement is false, draw a counterexample to it.



Activity 7. The class as a whole should carry out these activities. Let ◻ABCD be a parallelogram, and draw the diagonals $\overline{AC}$ and $\overline{BD}$. Let E be the point where the two diagonals $\overline{AC}$ and $\overline{BD}$ intersect.

a) Make a conjecture about the position of the point E on the two line segments $\overline{AC}$ and $\overline{BD}$.

b) Prove your conjecture.



Activity 8. The class as a whole should carry out the following activity. Let ◻ABCD be a quadrilateral, and draw the diagonals $\overline{AC}$ and $\overline{BD}$. Let E be the point where the two diagonals $\overline{AC}$ and $\overline{BD}$ intersect. We say that the two line segments $\overline{AC}$ and $\overline{BD}$ *bisect each other* if E is the midpoint of $\overline{AC}$ and of $\overline{BD}$.

Consider the following statement. "If ◻ABCD is a parallelogram, then the diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other." Formulate the *converse* of this statement. Do you think the converse is true or false?

Homework Problem 1. Write out and hand in the proof discovered in Activity 2 b). Your proof should be carefully written and detailed.

Homework Problem 2. Let □ABCD be a quadrilateral. Consider the following statement. "If □ABCD is a parallelogram, then the opposite sides of □ABCD are congruent (i.e., $\overline{AB}$ is congruent to $\overline{CD}$ and $\overline{AD}$ is congruent to $\overline{BC}$)."

a) Formulate the converse of this statement. Do you think the converse is true or false?

b) If you think the converse is true, prove it.

c) If you think the converse is false, draw a counterexample to it.

Homework Problem 3. Let □ABCD be a quadrilateral. Consider the following statement. "If □ABCD is a parallelogram, then the opposite angles of □ABCD are congruent (i.e., ∠BAD is congruent to ∠DCB and ∠ABC is congruent to ∠CDA)."

a) Formulate the converse of this statement. Do you think the converse is true or false?

b) If you think the converse is true, prove it.

c) If you think the converse is false, draw a counterexample to it.

Homework Problem 4. Let □ABCD be a quadrilateral, and draw the diagonals $\overline{AC}$ and $\overline{BD}$. Let E be the point where the two diagonals $\overline{AC}$ and $\overline{BD}$ intersect. In Activity 8 you formulated the converse of the statement "If □ABCD is a parallelogram, then the diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other" and you formed an opinion about whether this converse is true or false.

a) If you think the converse is true, prove it.

b) If you think the converse is false, draw a counterexample to it.