Lesson 12: The Symmetry of Unbounded Plane Figures

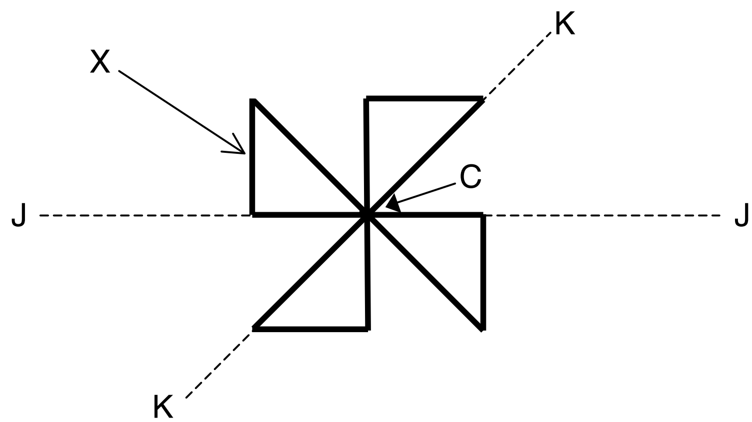
In this lesson we will explore the symmetry groups of unbounded figures. Unbounded figures often have infinite symmetry groups. Hence, we frequently can't list *all* the elements of the symmetry group of an unbounded figure. Instead we can try to identify all the different types of rigid motions (translations, rotations, reflections, glide reflections) that belong to the symmetry group of an unbounded figure and perhaps notice some relationships between these rigid motions.

Also in this lesson we will investigate ways of distinguishing the symmetry groups of two different figures. If we can discover how to tell the difference between the symmetry groups of two figures, then we will have identified a way in which the two figures have different kinds of symmetry. For example, if the symmetry group of one figure contains rotations while the symmetry group of another figure does not, then we will say that the two figures have different kinds of symmetry: the first figure has rotational symmetry while the second figure does not.

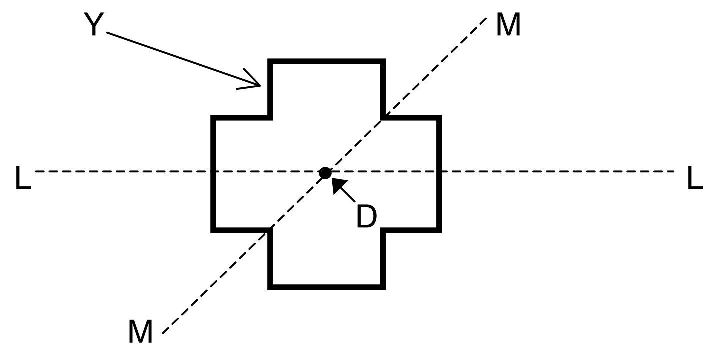
We now study a concept which is very useful for distinguishing symmetry groups: the notion of a *primitive element* of a symmetry group.

Definition. Let $R_{C,a}$ be a rotation that is an element of the symmetry group of a figure X. We say that $R_{C,a}$ is a *primitive rotation* in Sym(X) if $R_{C,a}$ *cannot be expressed* as the composition of two reflections that belong Sym(X).

For example, $R_{C,90}$ is a primitive rotation in the symmetry group Sym(X) of the figure X shown below. $R_{C,90}$ can't be written as the composition of two reflections that belong to this symmetry group, because this symmetry group contains no reflections. Indeed, there are no reflections that take this figure onto itself. Although $R_{C,90}$ can be expressed as the composition of two reflections, such as the reflections Z_J and Z_K in the lines J and K shown in the figure, these reflections don't belong to the symmetry group of the figure.



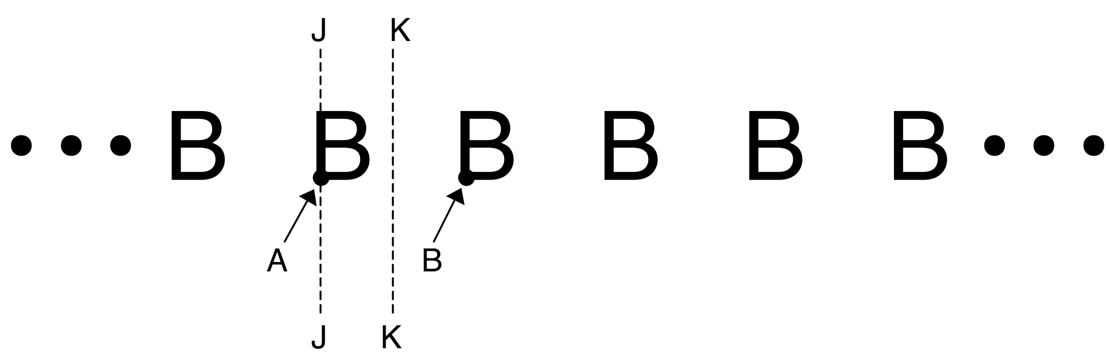
On the other hand, in the symmetry group Sym(Y) of the figure Y shown below, $R_{D,90}$ is *not* a primitive rotation in Sym(Y) because it can be written as the composition $Z_M \circ Z_L$ of the two reflections in the lines L and M, and these reflections are elements of Sym(Y).



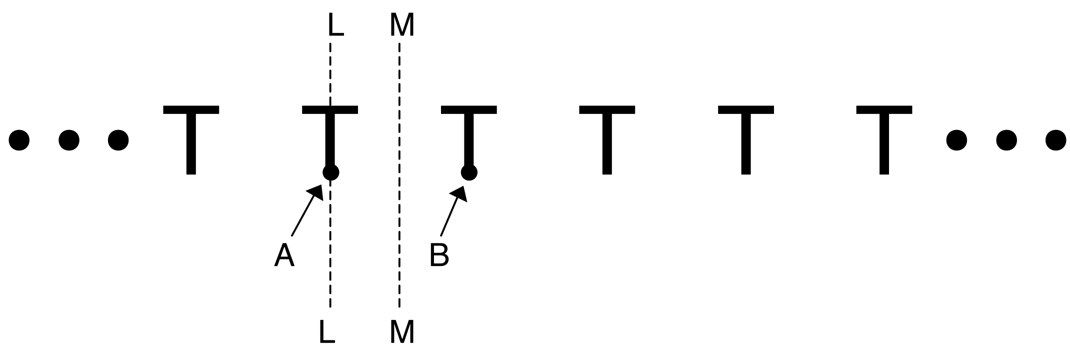
In the preceding two figures: the symmetry group of X is a cyclic group of rotations and the symmetry group of Y is a dihedral group of reflections and rotations. In a cyclic group of rotations, every rotation is a primitive rotation because the symmetry contains no reflections. On the other hand, in a dihedral group of reflections and rotations, no rotation is primitive, because every rotation in a dihedral group can be expressed as the composition of two reflections in the group. Thus, we can distinguish the symmetry of X from the symmetry of Y by observing that Sym(X) contains a primitive rotation while Sym(Y) does not.

Definition. Let $T_{A,B}$ be a translation that is an element of the symmetry group of a figure X. We say that $T_{A,B}$ is a *primitive translation* in Sym(X) if $T_{A,B}$ *cannot be expressed* as the composition of two reflections that belong Sym(X).

For example, if the figure X is the infinite sequences of **B**'s shown below (dots indicating that the figure extends forever to the right and left), then $T_{A,B}$ is a primitive translation in Sym(X). This is because $T_{A,B}$ can't be expressed as the composition of two reflections that are elements of Sym(X). ($T_{A,B}$ can be expressed as the composition $Z_K \circ Z_J$ of the two reflections is the lines J and K; but these reflections don't belong to symmetry group of X. Indeed, Sym(X) contains no reflections in vertical lines.)



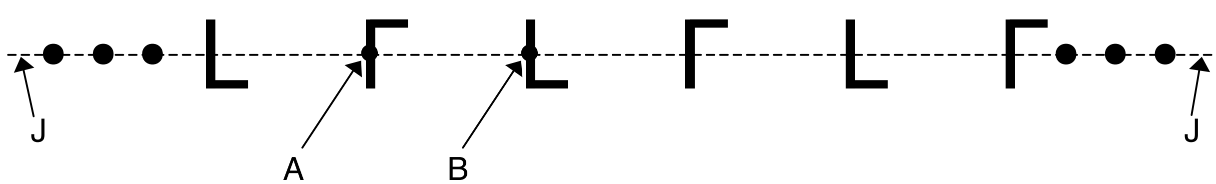
On the other hand, if the figure Y is the infinite string of **T**'s shown below (dots indicating that the figure extends forever to the right and left), then $T_{A,B}$ is *not* a primitive translation in Sym(Y) because it can be written as the composition $Z_M \circ Z_L$ of the two reflections in the lines L and M, and these reflections are elements of Sym(Y).



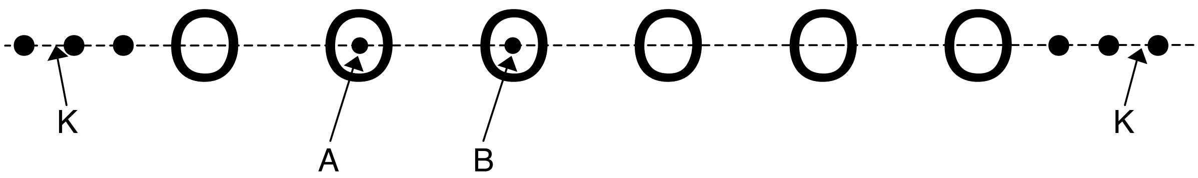
We can distinguish the symmetry of the preceding two figures X and Y by observing that Sym(X) contains a primitive translation while Sym(Y) does not.

Definition. Let $G_{A,B}$ be a glide reflection that is an element of the symmetry group of a figure X. We say that $G_{A,B}$ is a *primitive glide reflection* in Sym(X) if $G_{A,B}$ *cannot be expressed* as the composition of a translation and a reflection that belong Sym(X).

For example, if the figure X is the infinite sequences of **L**'s and upside down **L**'s shown below (dots indicating that the figure extends forever to the right and left), then $G_{A,B}$ is a primitive glide reflection in Sym(X). This is because $G_{A,B}$ can't be expressed as the composition of a translation and a reflection that are elements of Sym(X). ($G_{A,B}$ can be expressed as the composition $Z_J \circ T_{A,B}$ of the translation $T_{A,B}$ and the reflection Z_J; but the translation $T_{A,B}$ and the reflection Z_J don't belong to symmetry group of X.)



On the other hand, if the figure Y is the infinite string of **O**'s shown below (dots indicating that the figure extends forever to the right and left), then $G_{A,B}$ is *not* a primitive glide reflection in Sym(Y) because it can be written as the composition $Z_K \circ T_{A.B}$ of the translation $T_{A,B}$ and the reflection Z_K, and the translation $T_{A,B}$ and the reflection Z_K are elements of Sym(Y).



We can distinguish the symmetry of the preceding two figures X and Y by observing that Sym(X) contains a primitive glide translation while Sym(Y) does not.

Activity 1. Each group should carry out the following activities and report its results to the class. Identify as many as you can of the different types of rigid motions that belong to the symmetry groups of the unbounded figures in parts a) through f) of this activity on this and the next two pages. Look for primitive elements of these symmetry groups. You may draw and label points and lines in these figures to help you name the elements of the symmetry groups. (Don't forget the identity motion.)

a)

··· H H H H H H ···

(The dots indicate that this figure extends to the right and left forever.)

b)

··· S S S S S S ···

(The dots indicate that this figure extends to the right and left forever.)

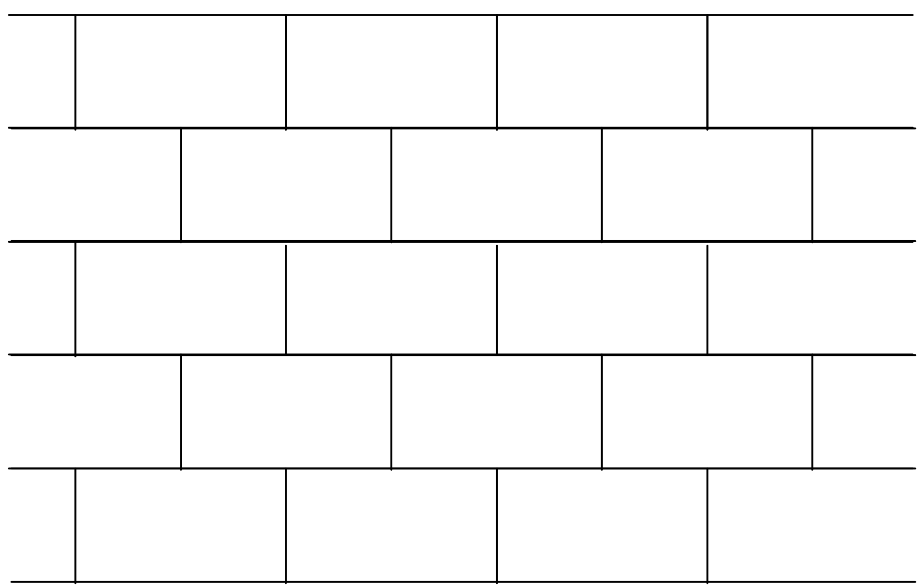
c)

(The dots indicate that this figure extends to the right and left forever.)

d)

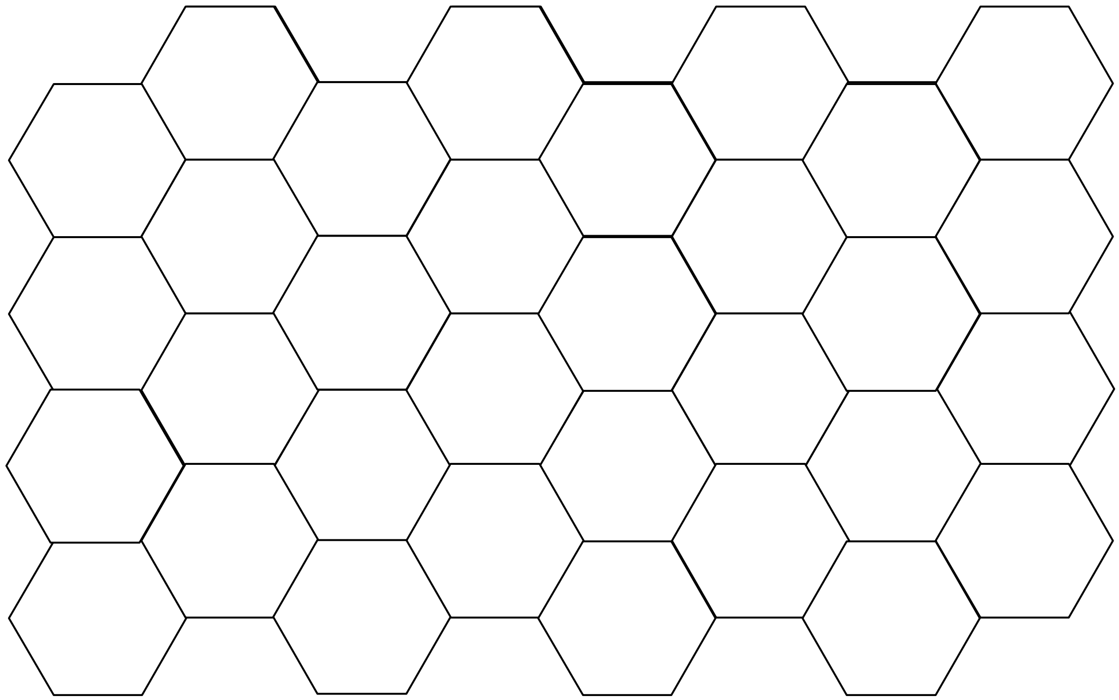
(The dots indicate that this figure extends to the right and left forever.)

e)



(Imagine that this figure fills its plane, extending right, left, up and down forever.)

f)



(Imagine that this figure fills its plane, extending right, left, up and down forever.)

The symmetry groups of the unbounded figures we studied in Activity 1 fall into two categories. The symmetry groups of the unbounded figures in parts a) through d) of Activity 1 contain translations that move in parallel directions. They don't contain translations that move in two non-parallel directions. These figures are called friezes and their symmetry groups are called *frieze groups.* The symmetry groups of the unbounded figures in parts e) and f) of Activity 1 contain translations that move in two (or more) non-parallel directions. These figures are called wallpaper patterns and their symmetry groups are called *wallpaper groups*.

Activity 2. Each group should carry out the following activities and report its results to the class. In parts a) through m) of this activity, try to show that the two figures have different symmetry by distinguishing the symmetry group of the figure on the left from the symmetry group of the figure on the right. In most parts of this activity, you can accomplish this by finding a type of rigid motion that belongs to one of the symmetry groups but not to the other. However, in a few of the parts of this activity, the situation is more complicated and may require you to look for primitive elements of symmetry groups.

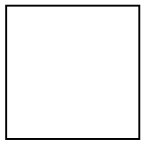
a)

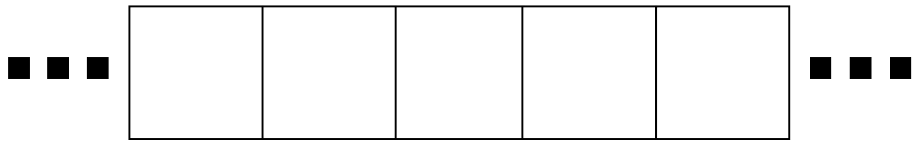
b)

c)

d)

e)





(The dots indicate that this figure extends to the right and left forever.)

f)

…S S S S S… **…I I I I I…**

(The dots indicate that these figures extends to the right and left forever.)

g)

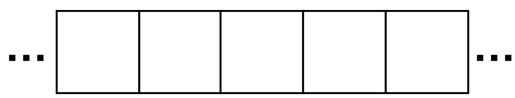
…F F F F F… …Γ L Γ L Γ…

(The dots indicate that these figures extends to the right and left forever.)

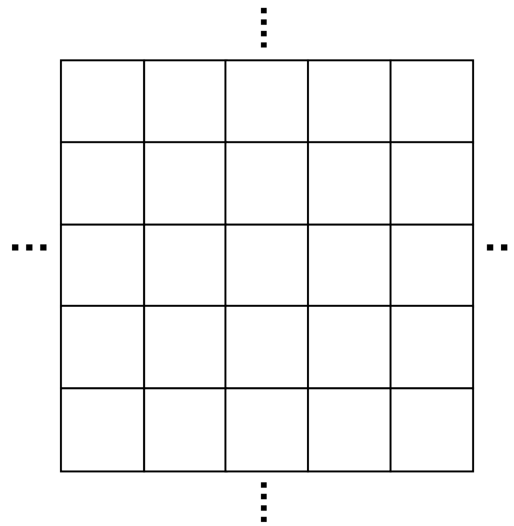
h)

(The dots indicate that these figures extends to the right and left forever.)

i)

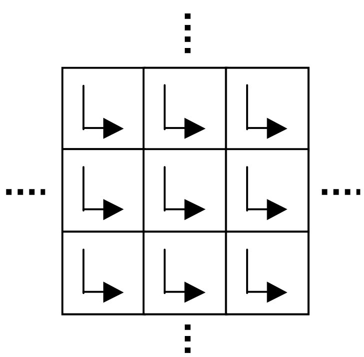


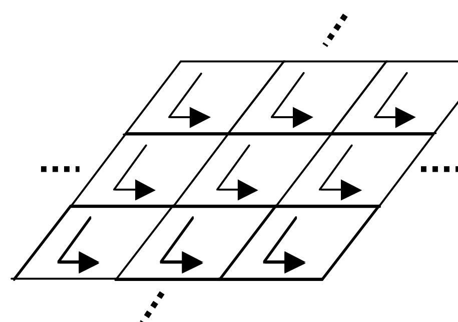
(The dots indicate that this figure extends to the right and left forever.)



(The dots indicate that the figure extends right, left, up and down and covers the entire plane.)

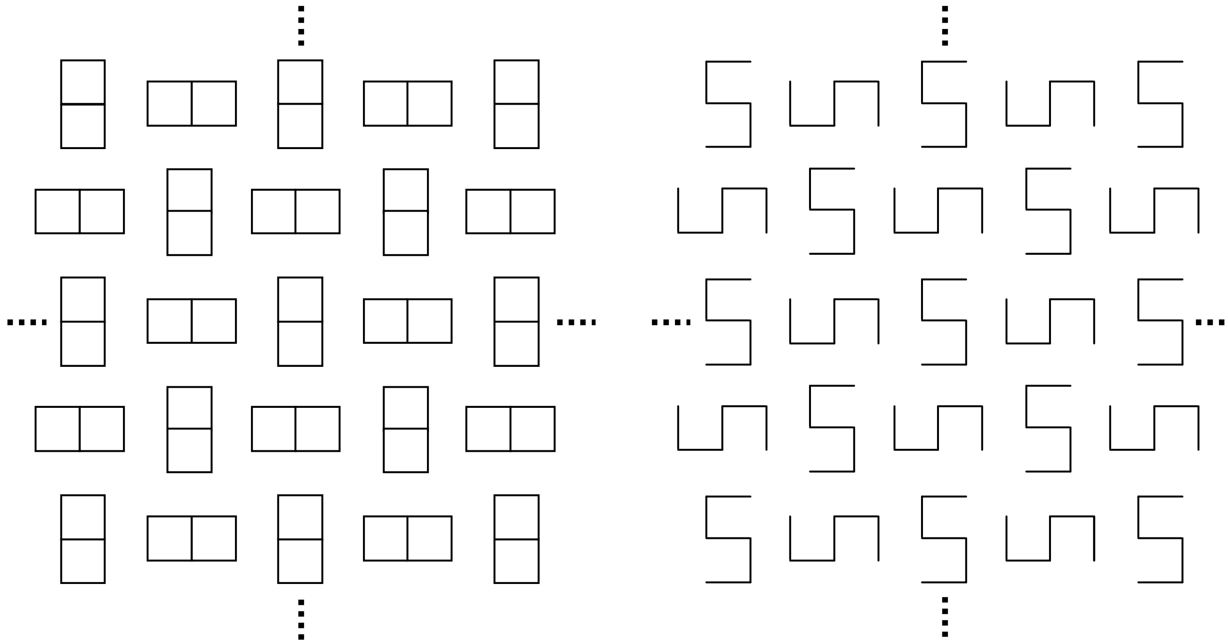
j)





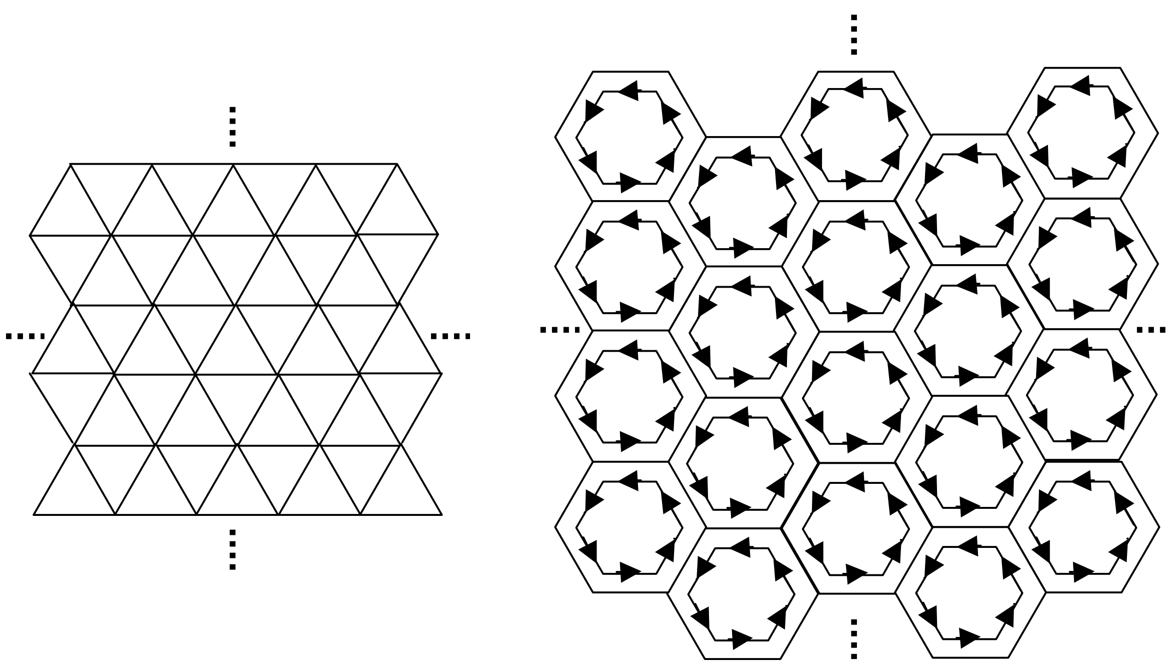
(The dots indicate that the figures extend right, left, up and down and cover the entire plane.)

k)



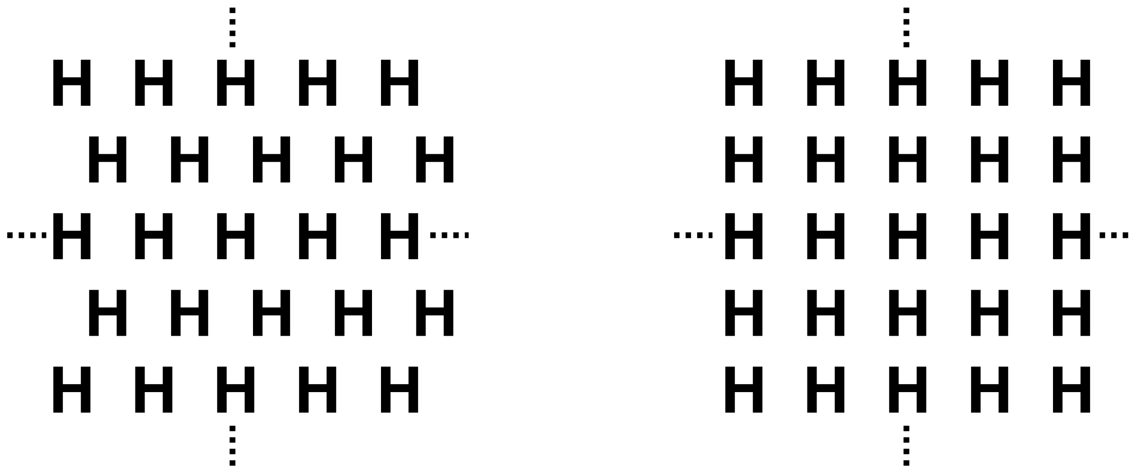
(The dots indicate that the figures extend right, left, up and down and cover the entire plane.)

l)



(The dots indicate that the figures extend right, left, up and down and cover the entire plane.)

m)



(The dots indicate that the figures extend right, left, up and down and cover the entire plane.)

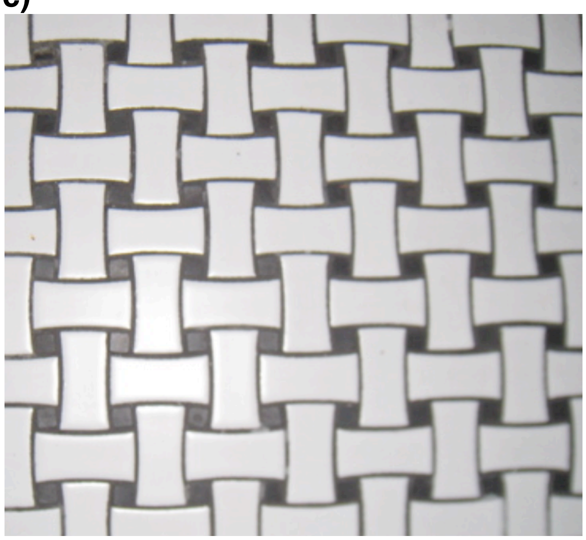
Homework Problem 1. Complete the parts of Activities 1 and 2 that were not finished during the class period.

Homework Problem 2. Can you distinguish the symmetry groups of the the three figures below.

a)

b)

c)



(Figures **a)** and **b)** are photos of the wall and floor tiles in a men's room at the La Fonda Hotel in Sante Fe, New Mexico. Figure **c)** is a photo of floor tiles in a home in Berkeley, California.)