## Lesson 11: The Symmetry of Bounded Plane Figures

In this lesson we begin the study of symmetry of figures in a plane.

Activity 1. The class should discuss the following two questions. The eight figures on this and the next two pages exhibit symmetry.

- What makes each figure symmetric?
- In what way is the symmetry of one figure the same as or different from that of the other figures?
You may find it helpful to use patty paper. In searching for symmetry in a figure, ignore fine distinctions in color and shading that would disappear if you traced the figure on patty paper with a pencil.

b)

c)

d)

e)

(Imagine that this figure extends to the right and left forever.)

g)

(Imagine that this figure fills its plane, extending right, left, up and down forever.)
h)


Mathematicians associate the symmetry of a figure in a plane $\Pi$ with the rigid motions of $\Pi$ that carry the figure onto itself. For example, in the figure $\mathbf{S}$ pictured below, there is $180^{\circ}$ rotation about the point C that carries the $\mathbf{S}$ onto itself. On the

other hand, in the figure $\mathbf{T}$ below there is a reflection in the line $L$ that carries the $\mathbf{T}$ onto itself. We might express these facts by saying that the figure $\mathbf{S}$ has rotational symmetry

and the figure $\mathbf{T}$ has reflectional symmetry.
Some figures have both rotational and reflectional symmetry. Consider the figure $\mathbf{H}$ below. There are two different reflections (in lines $L$ and $M$ ) as well as a $180^{\circ}$ rotation (about the point C ) that carry the H to itself.


To differentiate the symmetry of the figure $\mathbf{H}$ from the figures $\mathbf{S}$ and $\mathbf{T}$, mathematicians associate the symmetry of a figure with the collection of all of the rigid motions that carry the figure onto itself, and they call this collection the symmetry group of the figure.

Definition. Let $X$ be a figure in a plane $\Pi$. If $M$ is a rigid motion of $\Pi$ that carries the set X to itself, then M is called a symmetry of the figure X . The set of all symmetries of the figure X is called the symmetry group of the figure X and is denoted $\operatorname{Sym}(X)$.

Thus, if $X$ is a figure in a plane $\Pi$, then each element of the symmetry group $\operatorname{Sym}(X)$ is a rigid motion $M$ of $\Pi$ such that $M(X)=X$. To list the elements of the symmetry group of the figure $X$, one simply lists all the rigid motions of the plane $\Pi$ that carry X to itself.

We now list three fundamental properties of symmetry groups of plane figures. Let $X$ be a figure in a plane $\Pi$. Then the symmetry group $\operatorname{Sym}(X)$ has the following three properties.

- The identity motion $I_{\Pi}$ is an element of $\operatorname{Sym}(X)$.
- If $M$ is an element of $\operatorname{Sym}(X)$, then so is $\mathrm{M}^{-1}$. In other words, $\operatorname{Sym}(X)$ is closed under inversion.
- If $M_{1}$ and $M_{2}$ are elements of $\operatorname{Sym}(X)$, then so is $M_{2} \circ M_{1}$. In other words, $\operatorname{Sym}(X)$ is closed under composition.

We will explain these three fundamental properties of symmetries a little more.
First, recall that the identity motion $I_{\Pi}$ of the plane $\Pi$ is a rigid motion of $\Pi$ that carries every subset of $\Pi$ to itself. Thus, if $X$ is a figure in $\Pi$, then $I_{\Pi}(X)=X$. Thus, $I_{\Pi}$ is a symmetry of $X$. In other words, $I_{\Pi}$ is an element of $\operatorname{Sym}(X)$.

Second, suppose that $X$ is a figure in a plane $\Pi$ and $M$ is a symmetry of $X$. (In other words, M is an element of $\operatorname{Sym}(\mathrm{X})$.) We will prove that $\mathrm{M}^{-1}$ is a symmetry of X . Since $M$ is a symmetry of $X$, then $M$ is a rigid motion of the plane $\Pi$ such that

$$
X=M(X)
$$

We apply $\mathrm{M}^{-1}$ to both sides of this e quation to obtain

$$
M^{-1}(X)=M^{-1}(M(X))
$$

Now focus on the right hand side of this equation. Since $\mathrm{M}^{-1} \circ \mathrm{M}=I_{\Pi}$, then

$$
M^{-1}(M(X))=M^{-1} \circ M(X)=I_{\Pi}(X)=X
$$

Combining the preceding two equations, we get

$$
M^{-1}(X)=X
$$

Therefore, $\mathrm{M}^{-1}$ is a symmetry of X . (In other words, $\mathrm{M}^{-1}$ is an element of $\operatorname{Sym}(\mathrm{X})$.) We have proved that if $M$ is an element of $\operatorname{Sym}(X)$, then so is $M^{-1}$. Mathematicians express this fact by saying that $\operatorname{Sym}(\mathrm{X})$ is closed under inversion.

Third, suppose $X$ is a figure in a plane $\Pi$ and $M_{1}$ and $M_{2}$ are symmetries of $X$. (In other words, $M_{1}$ and $M_{2}$ are elements of Sym(X).) We will prove that the composition $M_{2} \circ M_{1}$ is a symmetry of $X$. Since $M_{1}$ and $M_{2}$ are symmetries of $X$, then $M_{1}$ and $M_{2}$ are rigid motions of $\Pi$ such that

$$
M_{1}(X)=X \text { and } M_{2}(X)=X
$$

We apply $M_{2}$ to both sides of the equation $M_{1}(X)=X$ to obtain

$$
M_{2}\left(M_{1}(X)\right)=M_{2}(X)
$$

Combining this equation with the equation $M_{2}(X)=X$, we get

$$
\mathrm{M}_{2}\left(\mathrm{M}_{1}(\mathrm{X})\right)=\mathrm{X}
$$

Since $M_{2}{ }^{\circ} M_{1}(X)=M_{2}\left(M_{1}(X)\right)$, we conclude

$$
\mathrm{M}_{2} \mathrm{O}_{1}(\mathrm{X})=\mathrm{X}
$$

Therefore, $M_{2} \circ M_{1}$ is a symmetry of $X$. (In other words, $M_{2} \circ M_{1}$ is an element of Sym $(X)$.) We have proved that if $M_{1}$ and $M_{2}$ are elements of $\operatorname{Sym}(X)$, then so is $M_{2} \circ M_{1}$. Mathematicians express this fact by saying that $\operatorname{Sym}(X)$ is closed under compostion.

Two planar figures have different symmetry if their symmetry groups differ in some essential way. For example, the figures $\mathbf{S}, \mathbf{T}$ and $\mathbf{H}$ have different symmetry because their symmetry groups are essentially different:

- $\operatorname{Sym}(\mathbf{S})$ contains a rotation but no reflection.
- $\operatorname{Sym}(T)$ contains a reflection but no rotation.
- $\operatorname{Sym}(\mathbf{H})$ contains both a rotation and a reflection.

We will study the symmetry groups of both bounded and unbounded figures. Figures a), b), c), d), f) and h) in Activity 1 as well as the figures $\mathbf{S}, \mathbf{T}$ and $\mathbf{H}$ are bounded. Figures e) and g) in Activity 1 are unbounded.

Definition. A figure in a plane or in 3-dimensional space is bounded it can be enclosed in a circle or sphere of finite radius. A figure which is not bounded is said to be unbounded. Thus, a figure which extends infinitely in any direction, like figures e) and $g$ ) in Activity 1, are unbounded.

We now explore the symmetry groups of various bounded figures.

Activity 2. Each group should carry out the following activities and report its results to the class. List all the elements of the symmetry group of each of the figures below and on the next page. You may draw and label points and lines in these figures to help you name the elements of the symmetry groups. (Don't forget the identity motion.)
a)

b)

c)

d)

e)

f)

g)

h)

i)

j)


The symmetry groups of the bounded figures we just studied in Activity 2 fall into two categories. The symmetry groups of bounded figures that contain no reflections are called cyclic groups, while the symmetry groups of bounded figures that contain reflections are called dihedral groups. We can be more explicit about the forms of these groups.

First observe that if $C$ is a point in a plane $\Pi$, then the rotations $R_{C, 0}$ and $R_{C, 360}$ around the point $C$ through angles of measure 0 degrees and 360 degrees are both the same as the identity motion $I_{\Pi}$ of $\Pi$. In other words, $R_{C, 0}=R_{C, 360}=I_{\Pi}$.

Definition. A cyclic group of rotations of a plane $\Pi$ has the following form. Let $n$ be a positive number. (Thus, $\mathrm{n}=1$ or $\mathrm{n}=2$ or $\mathrm{n}=3$ or $\mathrm{n}=4$ or $\mathrm{n}=5 \ldots$..) Let

$$
a=\frac{360}{n} .
$$

(Thus, $a=\frac{360}{1}=360$ if $\mathrm{n}=1, a=\frac{360}{2}=180$ if $\mathrm{n}=2, a=\frac{360}{3}=120$ if $\mathrm{n}=3$,
$a=\frac{360}{4}=90$ if $n=4, a=\frac{360}{5}=72$ if $n=5, \ldots$ ) Let $C$ be a point in a plane $\Pi$.
Then the cyclic group of rotations of the plane $\Pi$ centered at the point $C$ of order $n$ is the
set

$$
\left\{\mathrm{R}_{\mathrm{C}, a}, \mathrm{R}_{\mathrm{C}, 2 a}, \mathrm{R}_{\mathrm{C}, 3 a}, \ldots, \mathrm{R}_{\mathrm{C},(\mathrm{n}-1) a}, \mathrm{R}_{\mathrm{C}, \mathrm{na}}\right\}
$$

Notice that na $=360$. Therefore, $R_{C, n a}=R_{C, 360}=I_{\Pi}$. Thus, the cyclic group of order $n$ can also be expressed as the set

$$
\left\{\mathrm{I}_{\Pi}, \mathrm{R}_{\mathrm{C}, a}, \mathrm{R}_{\mathrm{C}, 2 a}, \ldots, \mathrm{R}_{\mathrm{C},(\mathrm{n}-2) a}, \mathrm{R}_{\mathrm{C},(\mathrm{n}-1) a}\right\}
$$

Therefore the cyclic group of rotations of the plane $\Pi$ centered at the point $C$ of order $n$ is the set consisting of the n rotations of $\Pi$ around the point $C$ through oriented angles whose measures are whole number multiples of $a=360 / n$.

Definition. A dihedral group of reflections and rotations has the following form. Again let n be a positive whole number, and as before let

$$
a=\frac{360}{n}
$$

Let $C$ be a point in a plane $\Pi$. Let $L_{1}, L_{2}, \ldots, L_{n}$ be $n$ lines in the plane $\Pi$ that pass through the point C so that the oriented measure of the oriented angle between two successive lines in this list is equal to $\frac{180}{n}=\frac{a}{2}$ degrees. In other words, the oriented measures of the oriented angles between the lines $L_{1}$ and $L_{2}$, between the lines $L_{2}$ and $L_{3}$, between the lines $L_{3}$ and $L_{4}, \ldots$, between the lines $L_{n-2}$ and $L_{n-1}$, and between the lines $L_{n-1}$ and $L_{n}$ are all equal to $180 / n=a / 2$ degrees. These angle restrictions force the measure of the oriented angle between the lines $L_{n}$ and $L_{1}$ to also equal 180/n =a/2 degrees. (Draw some pictures to verify this.) Now:

- let $Z_{1}$ denote the reflection in the line $L_{1}$,
- let $Z_{2}$ denote the reflection in the line $L_{2}$,
- let $Z_{3}$ denote the reflection in the line $L_{3}$,
- let $Z_{n-1}$ denote the reflection in the line $L_{n-1}$, and
- let $Z_{n}$ denote the reflection in the line $L_{n}$,

Observe that since the measure of the oriented angle between two succesive lines $L_{i}$ and $L_{i+1}$ is $a / 2$ degrees, then composition of any two successive reflections $Z_{i}$ and $Z_{i+1}$ must equal $R_{C, a}$. Thus, $Z_{2} \circ Z_{1}=R_{C, a}, Z_{3} \circ Z_{2}=R_{C, a}, Z_{4} \circ Z_{3}=R_{C, a}, \ldots, Z_{n-1} \circ Z_{n-2}=R_{C, a}$ and $Z_{n} \circ Z_{n-1}=R_{c, a}$. Also since the measure of the oriented angle between the lines $L_{n}$ and $L_{1}$ is $a / 2$, then $Z_{1} \circ Z_{n}=R_{c, a}$. Then the set

$$
\left\{Z_{1}, Z_{2}, Z_{3}, \ldots, Z_{n-1}, Z_{n}, R_{C, a}, R_{c, 2 a}, R_{C, 3 a}, \ldots, R_{C,(n-1) a}, R_{C, n a}\right\}
$$

is a dihedral group of reflections and rotations of the plane $\Pi$ centered at the point $C$ of order 2n. Again since $R_{c, n a}=R_{C, 360}=I_{\Pi}$, then this dihedral group of order 2 n can also be
expressed as the set

$$
\left\{Z_{1}, Z_{2}, Z_{3}, \ldots, Z_{n-1}, Z_{n}, I_{\Pi}, R_{C, a}, R_{C, 2 a}, \ldots, R_{C,(n-2) a}, R_{C,(n-1) a}\right\}
$$

Therefore a dihedral group of reflections and rotations of the plane $\Pi$ centered at the point $C$ of order $2 n$ is the set consisting of reflections in $n$ lines that pass through the point $C$ and divide the plane $\Pi$ into $n$ congruent sectors of "angular width" $a / 2=180 / n$ together with the $n$ rotations of $\Pi$ around the point $C$ through oriented angles whose oriented measures are whole number multiples of $a=360 / \mathrm{n}$.

We have just described a "cyclic group of order n" and a "dihedral group of order $2 n$ ". Observe that a cyclic group of order $n$ has $n$ elements, and a dihedral group of order 2 n has 2 n elements. In general, the order of a cyclic or dihedral group is the number of elements of the group.

We make an observation about the relationship between cyclic groups of order n and dihedral groups of order 2 n . The list of elements of a dihedral group of reflections and rotations of the plane $\Pi$ centered at the point $C$ of order $2 n$ is

$$
Z_{1}, Z_{2}, Z_{3}, \ldots, Z_{n-1}, Z_{n}, R_{C, a}, R_{C, 2 a}, R_{C, 3 a}, \ldots, R_{C,(n-1) a}, R_{C, n a}
$$

In this list, the sublist

$$
\mathrm{R}_{\mathrm{C}, a}, \mathrm{R}_{\mathrm{C}, 2 a}, \mathrm{R}_{\mathrm{C}, 3 a}, \ldots, \mathrm{R}_{\mathrm{C},(\mathrm{n}-1) a}, \mathrm{R}_{\mathrm{C}, \mathrm{na}}
$$

is a list of the elements of the cyclic group of rotations of the plane $\Pi$ centered at the point $C$ of order $n$. Thus we conclude that the cyclic group of rotations of the plane $\Pi$ centered at the point C of order n is a subgroup of any dihedral group of reflections and rotations of the plane $\Pi$ centered at the point $C$ of order $2 n$.

Next we present examples of cyclic and dihedral groups.
To determine the symmetry group of the following figure X ,

first locate the center point C of the figure. Then we can write


$$
\operatorname{Sym}(X)=\left\{R_{c, 90}, R_{c, 180}, R_{c, 270}, R_{c, 360}\right\} .
$$

Observe that this symmetry group can also be expressed as the set

$$
\left\{I_{\Pi}, R_{c, 90}, R_{c, 180}, R_{c, 270}\right\} .
$$

Hence, the symmetry group of the set $X$ is the cyclic group of rotations centered at $C$ of order 4. Notice that the arrowheads on the sides of $X$ prevent it from having reflectional symmetry. Thus, the symmetry group of X contains no reflections and can't be a dihedral group.

To determine the symmetry group of the following figure Y , first draw the lines K ,


L and M and the point C . Then we can write


Alternatively, we can write

$$
\operatorname{Sym}(Y)=\left\{Z_{K}, Z_{L}, Z_{M}, I_{\Pi}, R_{C, 120}, R_{C, 240}\right\}
$$

Hence, the symmetry group of the set Y is a dihedral group of reflections and rotations centered at C of order 6.

Activity 3. The class should discuss the following problem. For each of the ten figures in Activity 2, fill in the blanks in the following statement:

The symmetry group of this figure is a $\qquad$ group of order $\qquad$
Fill in the first blank with either the word cyclic or the word dihedral, and fill in the second blank with a number.

Activity 4. Each group should carry out the following activities and report its results to the class. In parts a), b), c) and d) of this activity you will create patterns on pieces of patty paper and you will be asked to answer the following three questions about the symmetry groups of these patterns.

1) List the elements of the symmetry group. (You may draw and label points and lines to help you name the elements of the symmetry group.)
2) Is the symmetry group cyclic or dihedral?
3) What is the order of the symmetry group?
a) Fold a piece of patty paper in half, and then fold it again along a line that is the perpendicular bisector of the first fold line. Then the two fold lines meet at the center point $C$ of the piece of patty paper. Now make a snowflake by cutting the folded paper along an irregular curve that joins one folded edge to the other and that "cuts off" the two unfolded edges of the folded paper. Unfold the paper to obtain a snowflake. Now answer questions 1), 2) and 3) (above) about the symmetry group of the snowflake.
b) As in part a), fold a piece of patty paper twice along lines that perpendicularly bisect each other and meet at the center $C$ of the piece of patty paper. Next, instead of cutting the folded piece of paper, draw a non-symmetric pattern (e.g., an F) on one exposed side. Then turn the folded piece of paper over and trace this pattern on the other exposed side. Then unfold the paper once and trace the "doubled" pattern on the exposed side of the paper that has no pattern drawn on it yet. Then unfold the remaining fold. Now answer questions 1), 2) and 3) (above) about the symmetry group of the pattern you have created on the unfolded piece of patty paper.
c) This time make three folds in a piece of patty paper. First: fold along a line that goes through the center $C$ of the piece of patty paper. Second: fold along the line through $C$ that is perpendicular to the first fold line. Third: fold along the line through $C$ that bisects the 90 degree angle between the edges created by the first two folds. Next follow either the method in part a) to create a snowflake, or the method in part b) to create a pattern on the piece of patty paper. Finally answer questions 1), 2) and 3) (above) about the symmetry group of the snowflake or pattern that you have created.
d) Discover a way to fold a piece of patty paper along three lines that pass through the center C so that the measures of the angles between adjacent lines are all equal to 180/3 $=60$ degrees. Next follow either the method in part a) to create a snowflake, or the method in part b) to create a pattern on the piece of patty paper. Finally answer questions 1), 2) and 3) (above) about the symmetry group of the snowflake or pattern that you have created.
e) Can you find a way to fold a piece of patty paper along five lines that pass through the center C so that the measures of the angles between adjacent lines are all equal to $180 / 5=36$ degrees? If so, follow either the method in part a) to create a snowflake, or the method in part b) to create a pattern on the piece of patty paper. Finally answer questions 1), 2) and 3) (above) about the symmetry group of the snowflake or pattern that you have created.

Homework Problem 1. Complete the parts of Activiity 4 outside of class that you did not have time to complete in class.

Homework Problem 2. Create 13 patterns on pieces of patty paper that have the following symmetry groups:

- cyclic groups of rotations of orders $2,3,4,5,6,7$ and
- dihedrail groups of reflections and rotations of orders $2,4,6,8.10 .12 .14$.

Make your patterns as artistic as possible, and sign your name. We will mount some of these patterns on a poster board.

Homework Problem 3. Let $C$ be a point in the plane $\Pi$. Let $K$ and $L$ be lines in $\Pi$ that pass through the point C .
a) What is the order of the smallest cyclic group of rotations of $\Pi$ that contains the rotation $\mathrm{R}_{\mathrm{c}, 72}$ ?
b) What is the order of the smallest dihedral group of reflections and rotations of $\Pi$ that contains the rotation $\mathrm{R}_{\mathrm{C}, 72}$ ?
c) What is the order of the smallest cyclic group of rotations of $\Pi$ that contains the rotation $\mathrm{R}_{\mathrm{C}, 144}$ ?
d) What is the order of the smallest cyclic group of rotations of $\Pi$ that contains the rotation $\mathrm{R}_{\mathrm{c}, 80}$ ?
e) If the measure of the smaller angle created by lines $K$ and $L$ is $30^{\circ}$, what is the order of the smallest dihedral group of reflections and rotations of $\Pi$ that contains the reflections $Z_{K}$ and $Z_{L}$ ?
f) If the measure of the smaller angle created by lines $K$ and $L$ is $72^{\circ}$, what is the order of the smallest dihedral group of reflections and rotations of $\Pi$ that contains the reflections $Z_{k}$ and $Z_{L}$ ?
g) If the measure of the smaller angle created by lines K and L is $54^{\circ}$, what is the order of the smallest dihedral group of reflections and rotations of $\Pi$ that contains the reflections $Z_{K}$ and $Z_{L}$ ?

