Addendum to Lesson 10: Compositions of Rigid Motions

The Classification Theorem for Rigid Motions of a Plane tells us that there are four and only four types of rigid motions of a plane: translations, rotations, reflections and glide reflections. Thus, there are 4 x 4 = 16 different ways to compose two rigid motions. These 16 different compositions are listed in the following grid. The

$T_{A,B}∘T_{D,E}$	$T_{A,B}∘R_{D,b}$	$T_{A,B}∘Z_M$	$T_{A,B}∘G_{D,E}$
$R_{C,a}∘T_{D,E}$	$R_{C,a}∘R_{D,b}$	$R_{C,a}∘Z_M$	$R_{C,a}∘G_{D,E}$
$Z_L∘T_{D,E}$	$Z_L∘R_{D,b}$	$Z_L∘Z_M$	$Z_L∘G_{D,E}$
$G_{A,B}∘T_{D,E}$	$G_{A,B}∘R_{D,b}$	$G_{A,B}∘Z_M$	$G_{A,B}∘G_{D,E}$

Classification Theorem for Rigid Motions implies that each of these 16 compositions must also be either a translation, a rotation, a reflection or a glide reflection. In this Addendum, we will study these 16 compositions through a series of examples and homework problems to determine the type of each composition. Six of the 16 compositions are explained in the text. The analysis of the remaining 10 compositions is relegated to homework problems. Our analysis will depend heavily on the Sliding and Twisting Tricks.

At the end of this lesson on page 177 is a *"Multiplication Table" for the Composition of Two Rigid Motions* where the results of our analysis of these 16 compositions is to be recorded. We have already recorded in this table the results of the analysis of the 6 compositions explained in the text. When you complete a homework problem that analyzes some of the remaining 10 compositions, you should also record your results in this table.

We already have complete information about *two* of the 16 compositions.

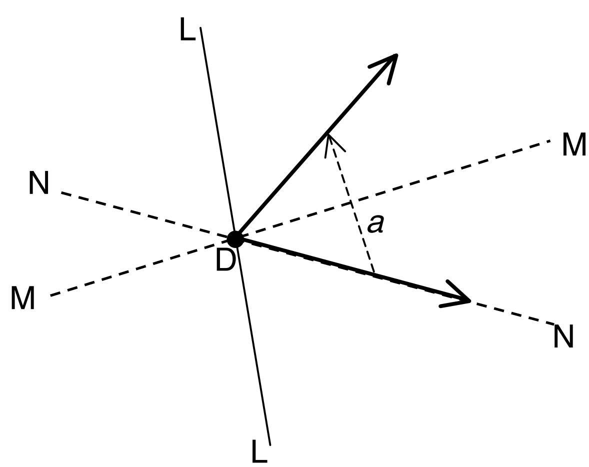
- Activity 2c in Lesson 9 analyzed the composition of two translations – $T_{A,B}∘T_{D,E}$. That activity revealed that $T_{A,B}∘T_{D,E}$ is another translation. Specifically, $T_{A,B}∘T_{D,E} = T_{D,F}$ where $F = T_{A,B}(E)$.
- Activity 2a and 2b in Lesson 9 analyzed the composition of two reflections – $Z_L∘Z_M$. These activities revealed that $Z_L∘Z_M$ is either a translation or a rotation. More precisely, $Z_L∘Z_M$ is a translation if L and M are parallel, while $Z_L∘Z_M$ is a rotation if L and M intersect. Observations 1 and 2 of Lesson 10 give complete information about this situation.

We also have complete information about the composition of *three* reflections – $Z_K \circ Z_L \circ Z_M$. From Activity 2 in Lesson 10, we know that if lines K, L and M are either parallel or concurrent (all passing through the same point), then $Z_K \circ Z_L \circ Z_M$ is a reflection. From the description of Step 2 of the proof of the Classification Theorem for Rigid Motions of a Plane at the end of the exposition in Lesson 10, we know that if the lines K, L and M are neither parallel nor concurrent, then $Z_K \circ Z_L \circ Z_M$ is a glide reflection. We can use this information to analyze certain compositions of *two* reflections. Specifically, we can use our knowledge about the composition of three reflections to analyze the four compositions $Z_L \circ R_{D,b}$, $R_{C,a} \circ Z_M$, $Z_L \circ T_{C,D}$ and $T_{A,B} \circ Z_M$.

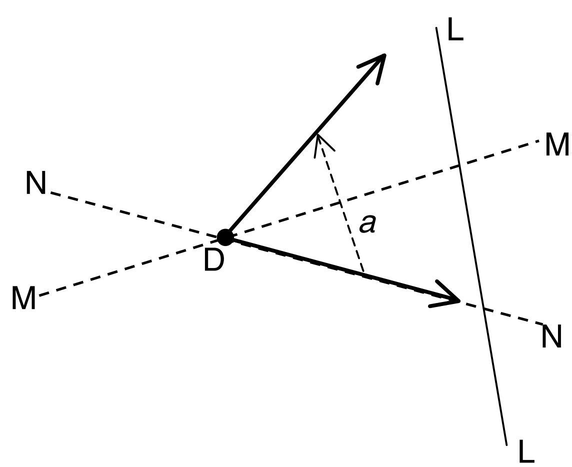
We now study the two compositions – $Z_L \circ R_{D,b}$ and $R_{C,a} \circ Z_M$. We make the analysis of the other two composition – $Z_L \circ T_{C,D}$ and $T_{A,B} \circ Z_M$ – homework problems. Our analysis of $Z_L \circ R_{D,b}$ and $R_{C,a} \circ Z_M$ will yield the following information.

- If the line L passes through the point D, then $Z_L \circ R_{D,b}$ is a reflection. Similarly, if the line M passes through the point C, then $R_{C,a} \circ Z_M$ is a reflection.
- If the line L does not pass through the point D, then $Z_L \circ R_{D,b}$ is a glide reflection. Similarly, if the line M does not pass through the point C, then $R_{C,a} \circ Z_M$ is a glide reflection.

Here is how we arrive at these conclusions. Consider the composition $Z_L \circ R_{D,b}$. We can express $R_{D,b}$ as the composition of two reflections: $R_{D,b} = Z_M \circ Z_N$. (The lines M and N intersect at the point D and the (smaller) oriented angle from line N to line M has oriented angle measure (1/2)*b*.) Then $Z_L \circ R_{D,b} = Z_L \circ Z_M \circ Z_N$. Hence, $Z_L \circ R_{D,b}$ and can be expressed as the composition of three reflections, a situation we already understand.



L passes through D



L does not pass through D

Either the line L passes through the point D or it does not.

- If L passes through D, then L, M and N are concurrent. In this case, we know that the composition $Z_L \circ Z_M \circ Z_N$ is a reflection. Thus, if L passes through D, then $Z_L \circ R_{D,b}$ is a reflection.
- If L does not pass through D, then L, M and N are neither concurrent nor parallel. In this case, we know that the composition $Z_L \circ Z_M \circ Z_N$ is a glide reflection. Thus, if L does not pass through D, then $Z_L \circ R_{D,b}$ is a glide reflection.

This analysis applies with minor changes to the composition $R_{C,a} \circ Z_M$. (Verify this.)

Next we state a homework problem that asks you to analyze the two compositions $Z_L \circ T_{C,D}$ and $T_{A,B} \circ Z_M$.

Homework Problem 12. Analyze the two compositions $Z_L \circ T_{D,E}$ and $T_{A,B} \circ Z_M$. Then express the result of your analysis by filling in each blank in each of the following two statements with the name of one of the four types of rigid motions – “translation”, “rotation”, “reflection” or “glide reflection”.

- If the line L is perpendicular to the line $\overleftrightarrow{DE}$, then $Z_L \circ T_{D,E}$ is a ___________. Similarly, if the line M is perpendicular to the line $\overleftrightarrow{AB}$, then $T_{A,B} \circ Z_M$ is a ___________.
- If the line L is not perpendicular to the line $\overleftrightarrow{DE}$, then $Z_L \circ T_{D,E}$ is a ___________. Similarly, if the line M is not perpendicular to the line $\overleftrightarrow{AB}$, then $T_{A,B} \circ Z_M$ is a ________.

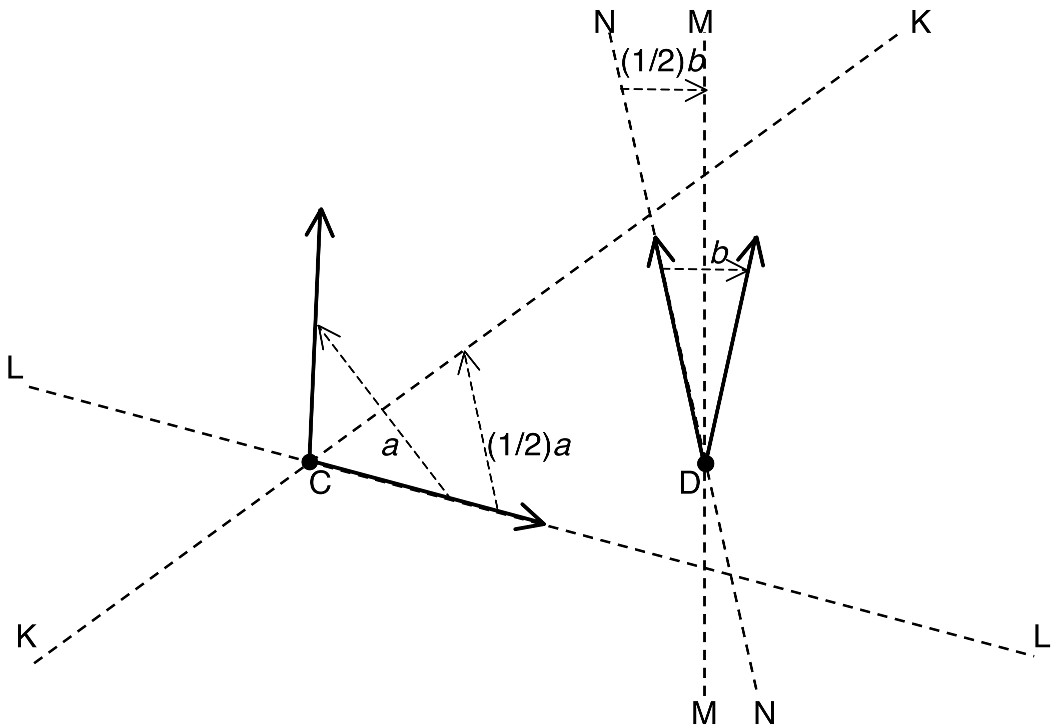
Hint. Express each of $T_{D,E}$ and $T_{A,B}$ as the composition of two reflections in parallel lines. (Don’t forget to record your results in the table on page 177.)

So far we have determined the nature of 6 of the 16 compositions of two rigid motions. Specifically, we understand the compositions $T_{A,B} \circ T_{D,E}$, $Z_L \circ Z_M$, $Z_L \circ R_{D,b}$, $R_{C,a} \circ Z_M$, $Z_L \circ T_{D,E}$ and $T_{A,B} \circ Z_M$. Next we turn to the study of the three compositions $R_{C,a} \circ R_{D,b}$, $R_{C,a} \circ T_{D,E}$ and $T_{A,B} \circ R_{D,b}$. We will analyze one of these composition – $R_{C,a} \circ R_{D,b}$ – and leave the analysis of the other two compositions as homework.

Our analysis of $R_{C,a} \circ R_{D,b}$ will yield the following conclusions.

- If $b = -a$, then $R_{C,a} \circ R_{D,b}$ is a translation.
- If $b \neq -a$, then $R_{C,a} \circ R_{D,b}$ is a rotation.

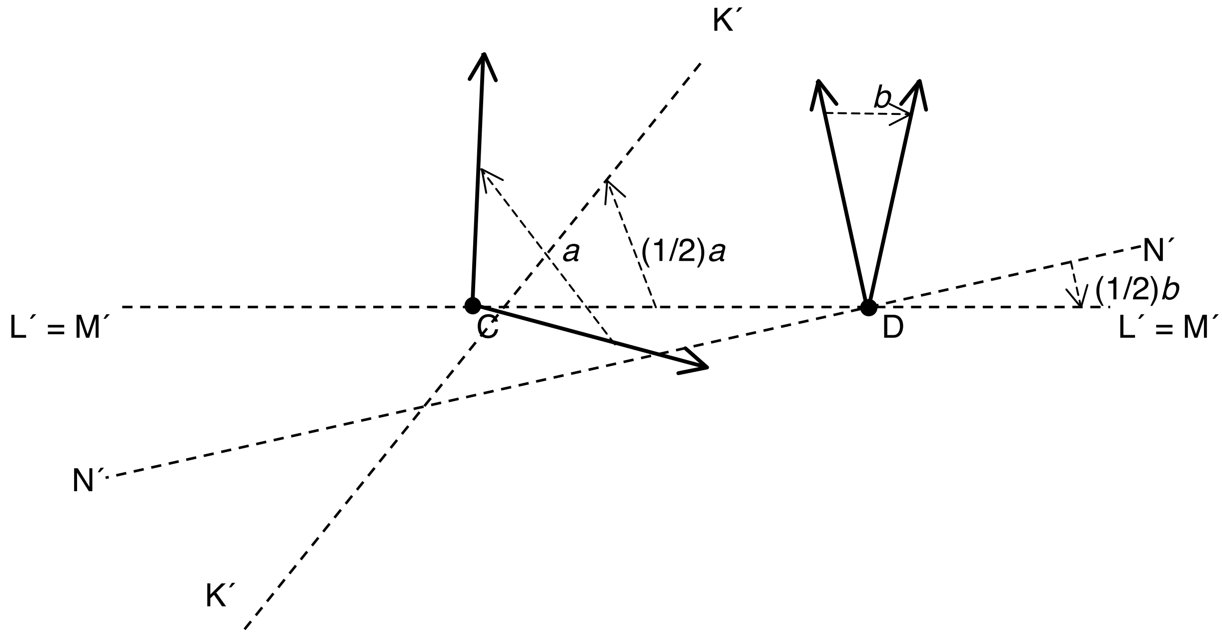
Here is how we arrive at these conclusions. We express each of $R_{C,a}$ and $R_{D,b}$ as the composition of two reflections: $R_{C,a} = Z_K \circ Z_L$ and $R_{D,b} = Z_M \circ Z_N$. (The lines K and L intersect at the point C and the (smaller) oriented angle from line L to line K has oriented angle measure (1/2)*a*. The lines M and N intersect at the point D and the (smaller) oriented angle from line N to line M has oriented angle measure (1/2)*b*.) Then $R_{C,a} \circ R_{D,b} = Z_K \circ Z_L \circ Z_M \circ Z_N$. Thus, $R_{C,a} \circ R_{D,b}$ can be expressed as the composition of four reflections.



Next we use the Twisting Trick twice. We twist lines K and L around C to lines K´ and L´ so that L´ passes through the point D. Also we twist lines M and N around D to lines M´ and N´ so that M´ passes through C. Then the Twisting Trick implies $R_{C,a} = Z_K \circ Z_L = Z_{K'} \circ Z_{L'}$ and $R_{D,b} = Z_M \circ Z_N = Z_{M'} \circ Z_{N'}$. Hence, $R_{C,a} \circ R_{D,b} = Z_{K'} \circ Z_{L'} \circ Z_{M'} \circ Z_{N'}$. Since the lines L´ and M´ pass through point C and point D, then $L' = \overleftrightarrow{CD} = M'$. Therefore, $Z_{L'} = Z_{M'}$. It follows (by substitution) that $R_{C,a} \circ R_{D,b} = Z_{K'} \circ Z_{L'} \circ Z_{L'} \circ Z_{N'}$

We now introduce another "trick" which we exploit to finish our analysis. If L is a line in a plane Π, then the reflection Z_L is its own inverse. Hence, $Z_L \circ Z_L = I_\Pi$. Therefore, if M_1 and M_2 are rigid motions of Π, then $M_1 \circ Z_L \circ Z_L \circ M_2 = M_1 \circ I_\Pi \circ M_2 = M_1 \circ M_2$. Similarly, $M_1 \circ Z_L \circ Z_L = M_1 \circ I_\Pi = M_1$ and $Z_L \circ Z_L \circ M_2 = I_\Pi \circ M_2 = M_2$. These remarks prove:

The Cancelling Trick. If L is a line in a plane ∏ and M_1 and M_2 are rigid motions of ∏, then $M_1 \circ Z_L \circ Z_L = M_1$ and $Z_L \circ Z_L \circ M_2 = M_2$ and $M_1 \circ Z_L \circ Z_L \circ M_2 = M_1 \circ M_2$.



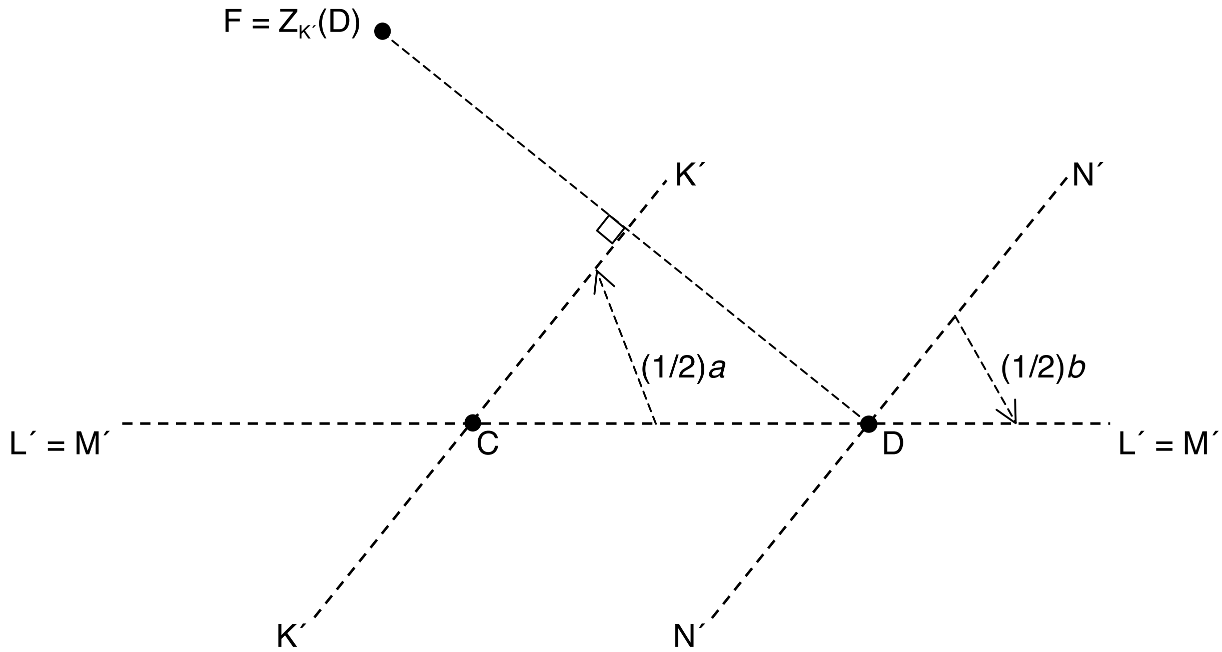
Returning to the analysis of $R_{C,a} \circ R_{D,b}$: the Cancelling Trick implies that $Z_{K'} \circ Z_{L'} \circ Z_{L'} \circ Z_{N'} = Z_{K'} \circ Z_{N'}$. Hence, $R_{C,a} \circ R_{D,b} = Z_{K'} \circ Z_{N'}$. Thus, $R_{C,a} \circ R_{D,b}$ is the composition of two reflections, a situation we already understand.

We already know that $Z_{K'} \circ Z_{N'}$ is a translation if K´ and N´ are parallel, while $Z_{K'} \circ Z_{N'}$ is a rotation if K´ and N´ intersect. Hence, $R_{C,a} \circ R_{D,b}$ is a translation if K´ and N´ are parallel, while $R_{C,a} \circ R_{D,b}$ is a rotation if K´ and N´ intersect.

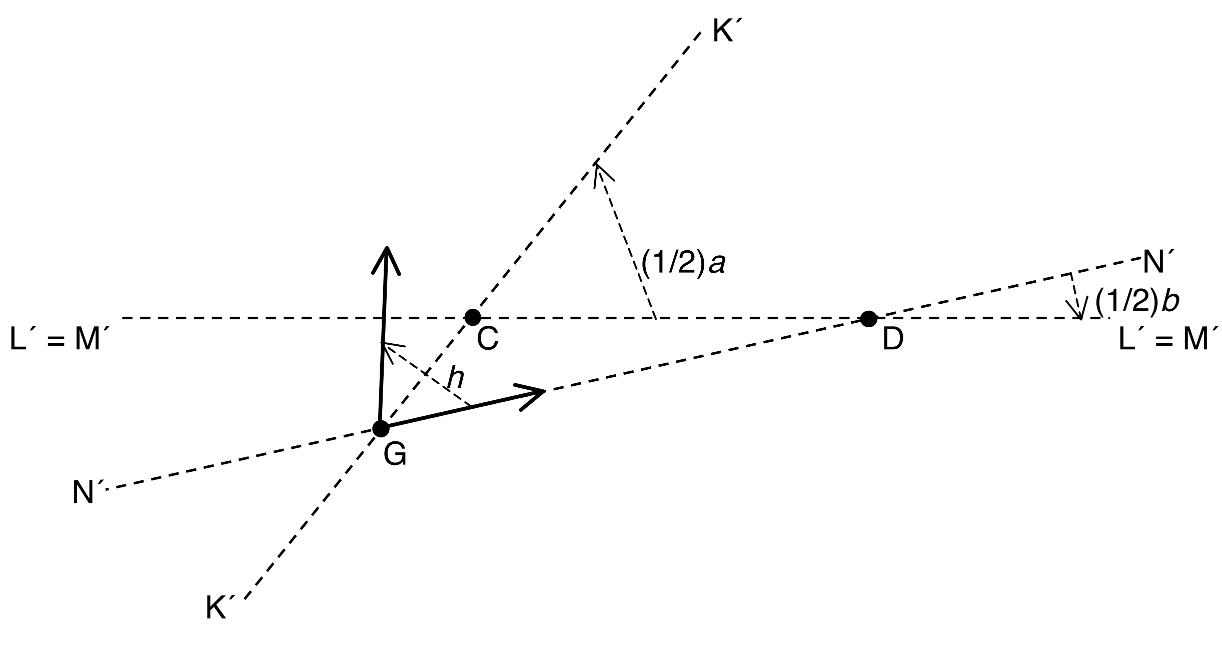
From our picture, we can see that if $b = -a$, then $(1/2)b = -(1/2)a$, which makes K´ and N´ parallel. In this case, $Z_{K'} \circ Z_{N'}$ is a translation. In particular, $Z_{K'} \circ Z_{N'} = T_{D,F}$ where $F = Z_{K'}(D)$. Thus, if $b = -a$, then $R_{C,a} \circ R_{D,b} = T_{D,F}$.

However, if $b \neq -a$, then K´ and N´ are not parallel. In this case, K´ and N´ intersect at a point we will call “G”. In this case, $Z_{K'} \circ Z_{N'}$ is a rotation. In particular, $Z_{K'} \circ Z_{N'} = R_{G,h}$ where h is twice the oriented angle measure of the smaller oriented angle from N´ to K´. Thus, in this case $R_{C,a} \circ R_{D,b} = R_{G,h}$.

This completes our analysis of the composition of two rotations.



$b = -a$, K´ and N´ are parallel, and $R_{C,a} \circ R_{D,b} = T_{D,F}$.



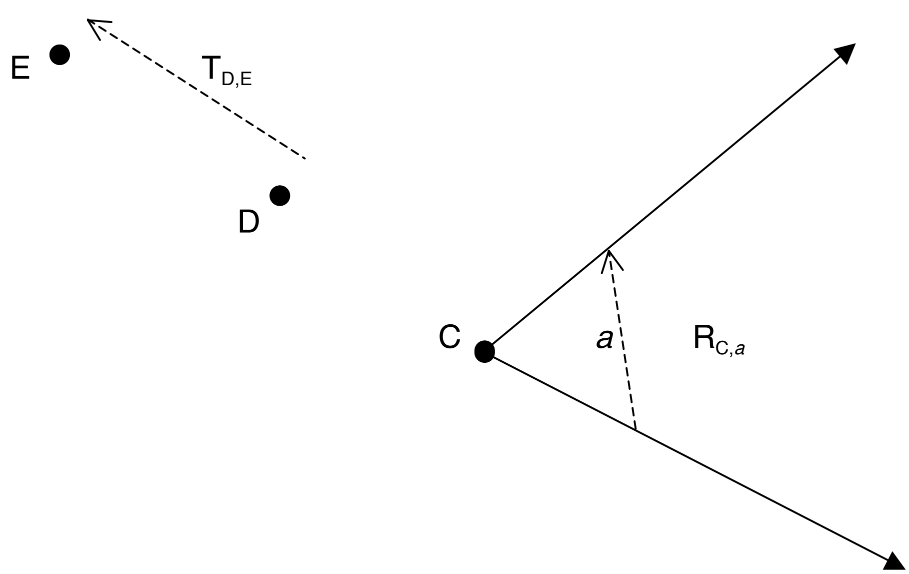
$b \neq -a$, K´ and N´ intersect at E, and $R_{C,a} \circ R_{D,b} = R_{G,h}$.

Next we state a homework problem that asks you to analyze the two compositions $R_{C,a}∘T_{D,E}$ and $T_{A,B}∘R_{D,b}$.

Homework Problem 13. Analyze the two compositions $R_{C,a}∘T_{D,E}$ and $T_{A,B}∘R_{D,b}$. Then express the result of your analysis by filling in the blank in each of the following two statements with the name of one of the four types of rigid motions – “translation”, “rotation”, “reflection” or “glide reflection”.

- $R_{C,a}∘T_{D,E}$ is a ______________________.
- $T_{A,B}∘R_{D,b}$ is a ______________________.

Hint. To analyze the composition $R_{C,a}∘T_{D,E}$, use the figure below. Begin by finding lines K, L, M and N so that $R_{C,a} = Z_K∘Z_L$ and $T_{D,E} = Z_M∘Z_N$. Then use the Twisting and Sliding Tricks to twist lines K and L to lines K´ and L´ and slide lines M and N to lines M´ and N´ so that L´ = M´. Finally, use the Cancelling Trick to conclude that $R_{C,a}∘T_{D,E} = Z_{K´}∘Z_{M´}$ and analyze what this means about $R_{C,a}∘T_{D,E}$. Treat the composition $T_{A,B}∘R_{D,b}$ similarly. (Don’t forget to record your results in the table on page 177.)



We have now studied 9 of the 16 compositions of two rigid motions. We have have analyzed all the compositions that occur in the first three rows and columns of the table. We have yet to consider the compositions that lie in the bottom row and right hand column of the table. There are exactly 7 of these compositions and they all involve glide reflections. Here is the list of these 7 remaining compositions: $G_{A,B}\circ T_{D,E}$, $T_{A,B}\circ G_{D,E}$, $G_{A.B}\circ R_{D,b}$, $R_{C,a}\circ G_{D,E}$, $G_{A.B}\circ Z_M$, $Z_L\circ G_{D,E}$ and $G_{A,B}\circ G_{D,E}$. Fortunately, the analysis of these 7 compositions is made easier by the work we have already done; and we will exploit our knowledge of the 9 compositions we've already studied when we analyze these remaining 7 compositions.

First we make an observation about glide reflections. If A and B are points in a plane Π and if L = $\overleftrightarrow{AB}$, then by definition the glide reflection $G_{A,B}$ is the composition of the translation $T_{A,B}$ and the reflection Z_L: $G_{A,B} = Z_L\circ T_{A,B}$. It is a fact about glide reflections that $G_{A,B}$ is also equal to the composition of $T_{A,B}$ and Z_L in the *opposite order:* $G_{A,B} = T_{A,B}\circ Z_L$. We now prove this fact. The proof begins with the observation that we can express $T_{A,B}$ as the composition of two reflections in parallel lines M and N where M and N are perpendicular to the line L = $\overleftrightarrow{AB}$. Thus, $T_{A,B} = Z_N\circ Z_M$. Therefore, $G_{A,B} = Z_L\circ Z_N\circ Z_M$. Since lines M and N are both perpendicular to L, then the Commuting Trick tells us that both Z_M and Z_N *commute* with Z_L. In other words, $Z_L\circ Z_M = Z_M\circ Z_L$ and $Z_L\circ Z_N = Z_N\circ Z_L$. Using these two equations, we have: $G_{A,B} = Z_L\circ T_{A,B} = Z_L\circ Z_N\circ Z_M = Z_N\circ Z_L\circ Z_M = Z_N\circ Z_M\circ Z_L = T_{A,B}\circ Z_L$. This completes our proof. To summarize, we have proved:

- If A and B are points in a plane Π and L = $\overleftrightarrow{AB}$, then $G_{A,B} = Z_L\circ T_{A,B}$ *and* $G_{A,B} = T_{A,B}\circ Z_L$.

The next three homework problems ask you to analyze the 6 compositions $G_{A,B}\circ T_{D,E}$, $T_{A,B}\circ G_{D,E}$, $G_{A.B}\circ R_{D,b}$, $R_{C,a}\circ G_{D,E}$, $G_{A.B}\circ Z_M$ and $Z_L\circ G_{D,E}$. After these three homework problems are stated, we will analyze the one remaining composition – $G_{A,B}\circ G_{D,E}$. If you have difficulty with any of these homework problems, you may find some helpful hints in the analysis of $G_{A,B}\circ G_{D,E}$.

Homework Problem 14. Analyze the two compositions $G_{A.B}\circ Z_M$ and $Z_L\circ G_{D,E}$. Then express the result of your analysis by filling in each blank in each of the following two statements with the name of one of the four types of rigid motions – "translation", "rotation", "reflection" or "glide reflection".

- If the line M is parallel or equal to the line $\overleftrightarrow{AB}$, then $G_{A.B}\circ Z_M$ is a _______. Similarly, if the line L is parallel or equal to the line $\overleftrightarrow{DE}$, then $Z_L\circ G_{D,E}$ is a _______.
- If the line M intersects the line $\overleftrightarrow{AB}$ in a single point, then $G_{A.B}\circ Z_M$ is a _______. Similarly, if the line L intersects the line $\overleftrightarrow{DE}$ in a single point, then $Z_L\circ G_{D,E}$ is a _____.

Hint. To analyze the composition $G_{A,B} \circ Z_M$, first let $K = \overleftrightarrow{AB}$ and write $G_{A,B} = T_{A,B} \circ Z_K$. Then $G_{A,B} \circ Z_M = T_{A,B} \circ Z_K \circ Z_M$. Now under each of the two hypotheses "M is parallel or equal to $K = \overleftrightarrow{AB}$" and "M intersects $K = \overleftrightarrow{AB}$ in a single point", answer the question: "What is $Z_K \circ Z_M$?" Finally, knowing the the answer to the question "What is $Z_K \circ Z_M$?", answer the question "What is $G_{A,B} \circ Z_M = T_{A,B} \circ (Z_K \circ Z_M)$?" Treat the composition $Z_L \circ G_{D,E}$ similarly. (Don't forget to record your results in the table on page 177.)

Homework Problem 15. Analyze the two compositions $G_{A,B} \circ T_{D,E}$ and $T_{A,B} \circ G_{D,E}$. Then express the result of your analysis by filling in each blank in the following statement with the name of one of the four types of rigid motions – "translations", "rotations", "reflections" or "glide reflections".

- $G_{A,B} \circ T_{D,E}$ and $T_{A,B} \circ G_{D,E}$ are either ______________ or _______________.

Hint. To analyze the composition $T_{A,B} \circ G_{D,E}$, first let $N = \overleftrightarrow{DE}$ and write $G_{D,E} = T_{D,E} \circ Z_N$. Then $T_{A,B} \circ G_{D,E} = T_{A,B} \circ T_{D,E} \circ Z_N$. Now answer the question: "What is $T_{A,B} \circ T_{D,E}$?" Finally, knowing the the answer to the question "What is $T_{A,B} \circ T_{D,E}$?", answer the question "What is $T_{A,B} \circ G_{D,E} = (T_{A,B} \circ T_{D,E}) \circ Z_N$?" Treat the composition $G_{A,B} \circ T_{D,E}$ similarly. (Don't forget to record your results in the table on page 177.)

Homework Problem 16. Analyze the two compositions $G_{A.B} \circ R_{D,b}$ and $R_{C,a} \circ G_{D,E}$. Then express the result of your analysis by filling in each blank in the following statement with the name of one of the four types of rigid motions – "translations", "rotations", "reflections" or "glide reflections".

- $G_{A.B} \circ R_{D,b}$ and $R_{C,a} \circ G_{D,E}$ are either ______________ or _______________.

We give no hint. Try to adapt the hint for Homework Problem 14 to this situation. (Don't forget to record your results in the table on page 177.)

To complete our study of the 16 compositions of two rigid motions, it remains only to analyze the composition of two glide reflections – $G_{A,B} \circ G_{D,E}$. We now perform this analysis. We assert that:

- If the line $\overleftrightarrow{AB}$ is parallel or equal to the line $\overleftrightarrow{DE}$, then $G_{A,B} \circ G_{D,E}$ is a translation.
- If line $\overleftrightarrow{AB}$ intersects the line $\overleftrightarrow{DE}$ in a single point, then $G_{A,B} \circ G_{D,E}$ is a rotation.

To begin the analysis of $G_{A,B} \circ G_{D,E}$, let $K = \overleftrightarrow{AB}$ and let $N = \overleftrightarrow{DE}$. Then $G_{A,B} = T_{A,B} \circ Z_K$ and $G_{D,E} = Z_N \circ T_{D,E}$. Therefore, $G_{A,B} \circ G_{D,E} = T_{A,B} \circ Z_K \circ Z_N \circ T_{D,E}$.

Now assume $K = \overleftrightarrow{AB}$ is parallel or equal to $N = \overleftrightarrow{DE}$. Then $Z_K \circ Z_N$ is a translation (or the identity if K = N). So we can write $Z_K \circ Z_N = T_{F,G}$ for some points F and G. (F = G if K = N.) Therefore, $G_{A,B} \circ G_{D,E} = T_{A,B} \circ T_{F,G} \circ T_{D,E}$. We know that the composition of two translations is always a translation. Hence, $T_{A,B} \circ T_{F,G} = T_{H,J}$ for some points H and J. Therefore, $G_{A,B} \circ G_{D,E} = T_{H,J} \circ T_{D,E}$. Thus, $G_{A,B} \circ G_{D,E}$ is a composition of two translations. We conclude that $G_{A,B} \circ G_{D,E}$ is a translation in the case that $K = \overleftrightarrow{AB}$ is parallel to $N = \overleftrightarrow{DE}$.

Finally assume $K = \overleftrightarrow{AB}$ intersects $N = \overleftrightarrow{DE}$ in a single point. Then $Z_K \circ Z_N$ is a rotation. So we can write $Z_K \circ Z_N = R_{G,h}$ for some point G and some oriented angle measure *h*. Therefore, $G_{A,B} \circ G_{D,E} = T_{A,B} \circ R_{G,h} \circ T_{D,E}$. We know (by Homework Problem 13) that the composition of a translation and a rotation in either order is always a rotation. Hence, $T_{A,B} \circ R_{G,h} = R_{H,j}$ for some point H and some oriented angle measure *j*. Therefore, $G_{A,B} \circ G_{D,E} = R_{H,j} \circ T_{D,E}$. Thus, $G_{A,B} \circ G_{D,E}$ is a composition of a rotation and a translation. We conclude that $G_{A,B} \circ G_{C,D}$ is a rotation in the case that $K = \overleftrightarrow{AB}$ intersects $N = \overleftrightarrow{DE}$ in a single point.

This concludes our analysis of $G_{A,B} \circ G_{D,E}$.

Now turn to the *"Multiplication Table" for the Composition of Two Rigid Motions* on the next page and make sure that you have recorded all the conclusions you reached from doing the homework problems.

“Multiplication Table” for the Composition of Two Rigid Motions

	$T_{D,E}$	$R_{D,b}$	Z_M	$G_{D,E}$
$T_{A,B}$	$T_{A,B} \circ T_{D,E}$ is a translation ($T_{A,B} \circ T_{D,E} = T_{D,F}$ where $F = T_{A,B}(E)$.)	$T_{A,B} \circ R_{D,b}$ is a:	$T_{A,B} \circ Z_M$ is a:	$T_{A,B} \circ G_{D,E}$ is a:
$R_{C,a}$	$R_{C,a} \circ T_{D,E}$ is a:	$R_{C,a} \circ R_{D,b}$ is a: translation if $b = -a$ rotation if $b \neq -a$	$R_{C,a} \circ Z_M$ is a: reflection if M passes through C glide reflection if M does not pass through C	$R_{C,a} \circ G_{D,E}$ is a:
Z_L	$Z_L \circ T_{D,E}$ is a:	$Z_L \circ R_{D,b}$ is a: reflection if L passes through D glide Reflection if L does not pass through D	$Z_L \circ Z_M$ is a: translation if L and M are parallel rotation if L and M intersect	$Z_L \circ G_{D,E}$ is a:
$G_{A,B}$	$G_{A,B} \circ T_{D,E}$ is a:	$G_{A,B} \circ R_{D,b}$ is a:	$G_{A,B} \circ Z_M$ is a:	$G_{A,B} \circ G_{D,E}$ is a: translation if $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$ are parallel rotation if $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$ intersect in a single point