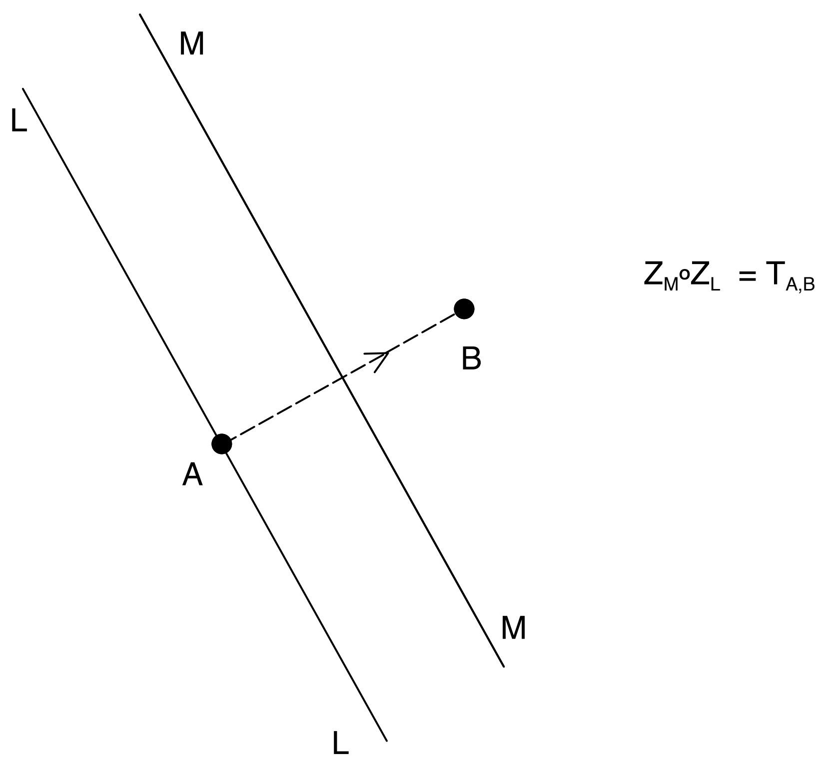
Lesson 10: The Decomposition of Rigid Motions into Reflections

The Activities and Homework Problems in Lesson 9 show how reflections can be composed to produce translations, rotations and glide reflections. We now summarize these facts in three observations.

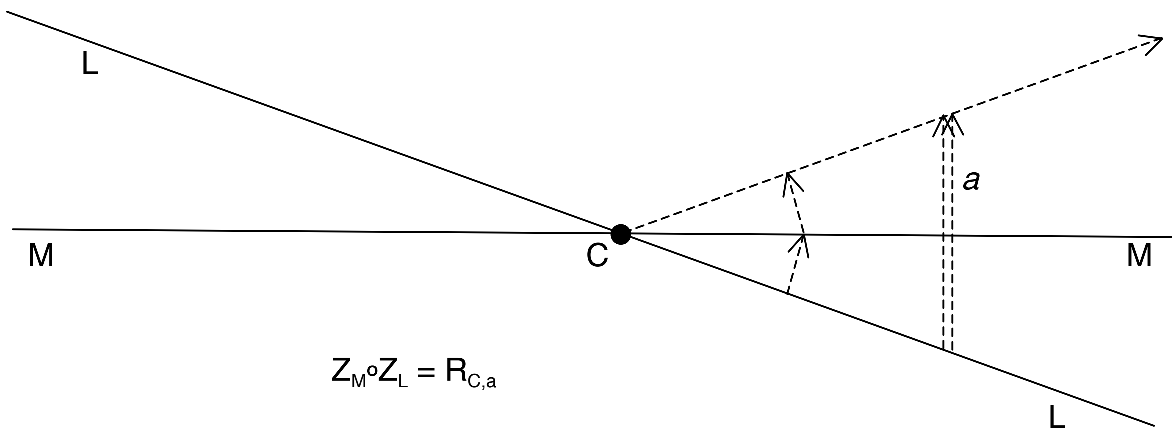
Observation 1. If L and M are parallel lines in a plane ∏, then the composition $Z_M \circ Z_L$ is a translation $T_{A,B}$ with the following properties.

- The ray $\overrightarrow{AB}$ is perpendicular to the lines L and M and points in the direction from L to M.
- The distance AB is twice the (perpendicular) distance between the lines L and M.



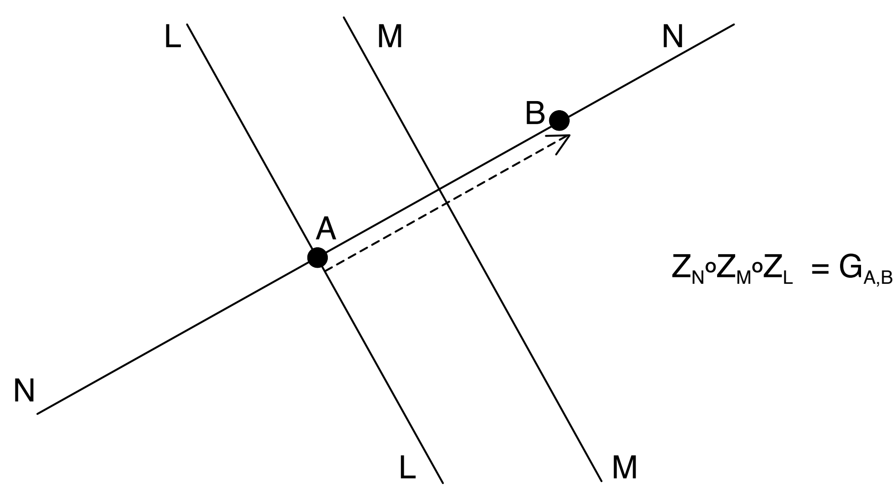
Observation 2. If L and M are lines in a plane ∏ that intersect at a point C, then the composition $Z_M \circ Z_L$ is a rotation $R_{C,a}$ with the following properties.

- The oriented angle with oriented measure *a* has vertex C.
- The oriented measure *a* of this oriented angle has absolute value $|a|$ equal to twice the measure of the smaller of the two angles made by the lines L and M.
- The orientation of this oriented angle points from line L to line M determined at the smaller of the two angles made by the lines L and M.



Observation 3. If L, M and N are lines in a plane ∏ such that L and M are both perpendicular to N, then the composition $Z_N \circ Z_M \circ Z_L$ is a glide reflection $G_{A,B}$ with the following properties.

- The points A and B lie on the line N so that the ray $\overrightarrow{AB}$ points in the direction from L to M.
- The distance AB is twice the (perpendicular) distance between the lines L and M.



We can view these three observations from a reversed perspective. The reversed perspective will reveal that every translation, rotation and glide reflections can be *decomposed* into two or three reflections.

Observation 1 Reversed. Every translation $T_{A,B}$ of a plane $\prod$ can be expressed as the composition of two reflections Z_L and Z_M ($T_{A,B} = Z_M \circ Z_L$) where

- the lines L and M are perpendicular to the ray $\overrightarrow{AB}$,
- the ray $\overrightarrow{AB}$ points in the direction from L to M, and
- the (perpendicular) distance between L and M is half the distance AB between the points A and B.

Observation 2 Reversed. Every rotation $R_{C,a}$ of a plane $\prod$ can be expressed as the composition of two reflections Z_L and Z_M ($R_{C,a} = Z_M \circ Z_L$) where

- the lines L and M both pass through the point C,
- the measure of the smaller angle between L and M is one-half the absolute value $|a|$ of the oriented measure a of the oriented angle,
- the orientation of this oriented angle points from L to M through the smaller of the two angles made by the lines L and M.

Observation 3 Reversed. Every glide reflection $G_{A,B}$ of a plane $\prod$ can be expressed as the composition of three reflections Z_L, Z_M and Z_N ($G_{A,B} = Z_N \circ Z_M \circ Z_L$) where

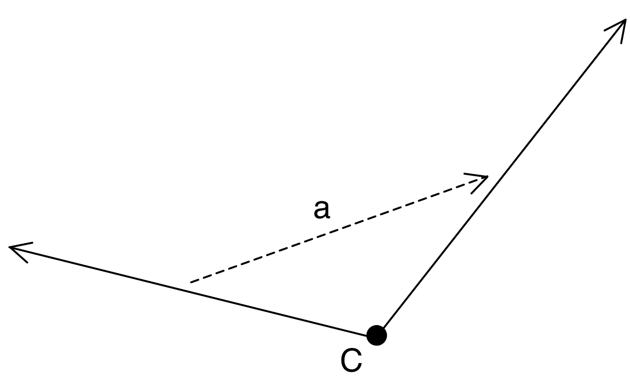
- the line N passes through the points A and B,
- the lines L and M are perpendicular to the ray $\overrightarrow{AB}$,
- the ray $\overrightarrow{AB}$ points in the direction from L to M, and
- the (perpendicular) distance between L and M is half the distance AB between the points A and B.

In the following activity, you are asked to apply these three observations.

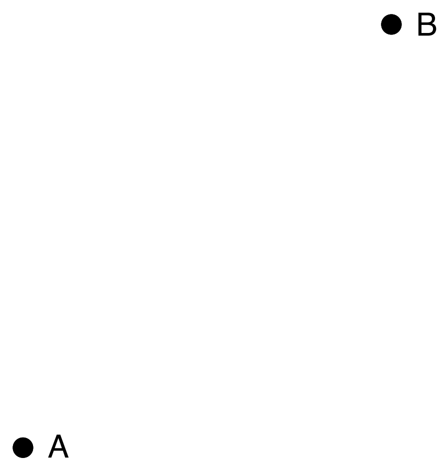
Activity 1. Groups should carry out the following three activities and report their results to the class.

a) Points A and B are already drawn in the following figure. Draw two lines L and M in this figure so that $T_{A,B} = Z_M \circ Z_L$.

b) The oriented angle with vertex C and oriented measure *a* is already drawn in the following figure. Draw two lines L and M in this figure so that $R_{C,a} = Z_M \circ Z_L$.



c) Points A and B are already drawn in the following figure. Draw three lines L, M and N in this figure so that $G_{A,B} = Z_N \circ Z_M \circ Z_L$.

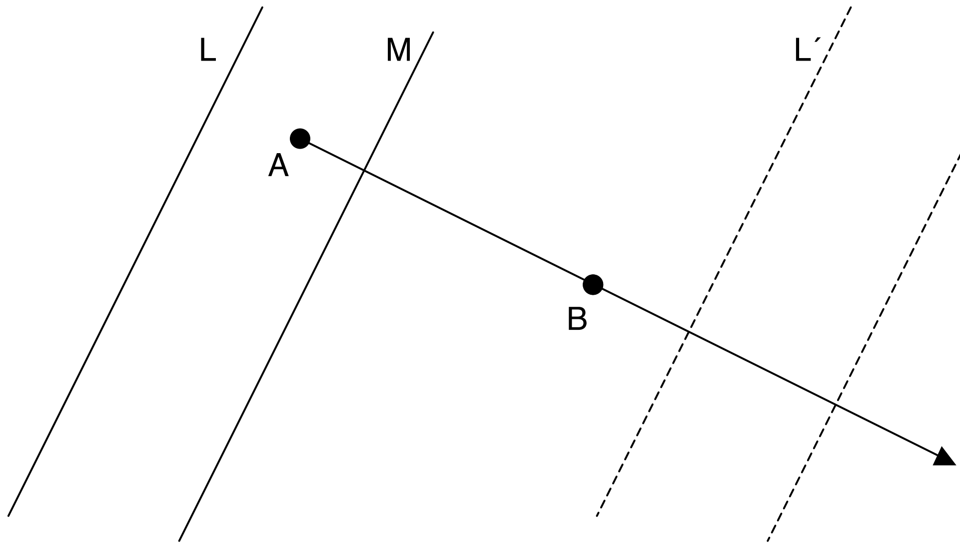


We note that Observation 1 and Observation 1 Reversed can be combined to give us further information. Suppose L, M, L´ and M´ are four parallel lines in a plane Π such that the (perpendicular) distance from L to M equals the (perpendicular) distance from L´ to M´ and any ray that is perpendicular to these lines and points from L to M also points from L´ to M´. Let $\overrightarrow{AB}$ be any ray that is perpendicular to the lines L and M, that points from L to M, such that AB equals twice the (perpendicular) distance from L to M. Then it is also the case that $\overrightarrow{AB}$ is a ray that is perpendicular to the lines L´ and M´, that points from L´ to M´, such that AB equals twice the (perpendicular) distance from L´ to M´. Therefore, $Z_M \circ Z_L = T_{A,B}$ and $Z_{M'} \circ Z_{L'} = T_{A,B}$. Hence, $Z_M \circ Z_L = Z_{M'} \circ Z_{L'}$.

These remarks prove the following statement.

The Sliding Trick. Suppose that L and M are two parallel lines in a plane Π. Let L´ and M´ be two lines in Π that are parallel to L and M such that the (perpendicular) distance from L to M equals the (perpendicular) distance from L´ to M´ and any ray that is perpendicular to these lines and points from L to M also points from L´ to M´. (Think of the lines L´ and M´ as having been obtained by *sliding* the lines L and M.) Then

$$Z_M \circ Z_L = Z_{M'} \circ Z_{L'}$$

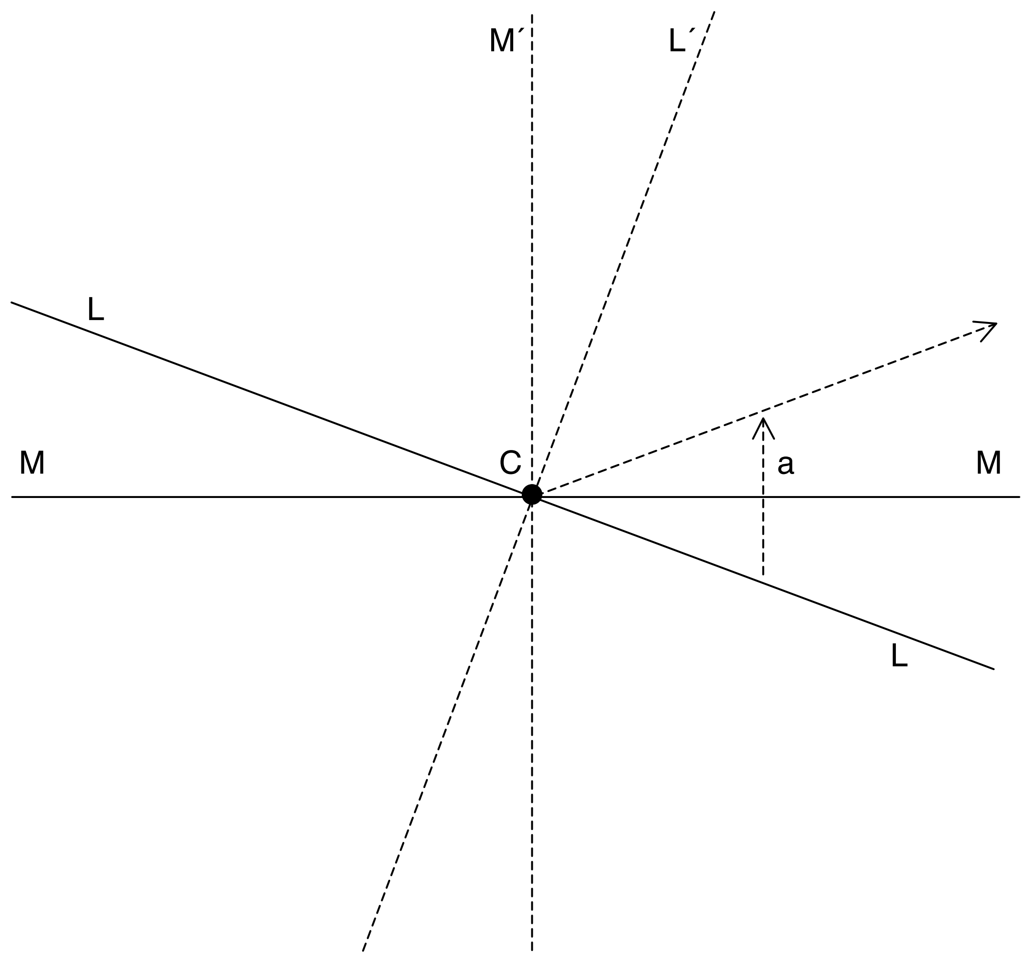


In the same way that Observation 1 and Observation 1 Reversed lead to the Sliding Trick, we can combine Observation 2 and Observation 2 Reversed gains further information. Suppose L, M, L´ and M´ are four lines in a plane ∏ that all pass through the same point such that the smaller angle from L to M has the same orientation and oriented angle measure as the smaller angle between L´ and M´. Let C be the point where lines L and M intersect and let *a* equal twice the oriented angle measure of the smaller oriented angle from L to M. Then C is also the point where lines L´ and M´ intersect and *a* equals twice the oriented angle measure of the smaller oriented angle from L´ to M´. Therefore, $Z_M \circ Z_L = R_{C,a}$ and $Z_{M'} \circ Z_{L'} = R_{C,a}$. Hence, $Z_M \circ Z_L = Z_{M'} \circ Z_{L'}$.

These remarks prove the following statement.

The Twisting Trick. Suppose that L and M are two in a plane ∏ that intersect at a point C. Let L´ and M´ be two lines in ∏ that also pass through the point C such that the smaller angle from L to M has the same orientation and oriented angle measure as the smaller angle from L´ to M´. (Think of the lines L´ and M´ as having been obtained by *twisting* the lines L and M.) Then

$$Z_M \circ Z_L = Z_{M'} \circ Z_{L'}$$



The Sliding Trick and the Twisting Trick tell us when we can express the composition of two reflections as the composition of two other reflections. The equations provided by the Sliding and Twisting Tricks can be *algebraically* transformed into equations that tell us that the composition of three reflections is equal to a single reflection. We describe this algebraic transformation in the *Algebraic Lemma* stated below. The algebraic fact needed to achieve this transformation is that reflections are their own inverses. In other words, if L is a line in a plane ∏, then $Z_L^{-1} = Z_L$. This implies that if L is a line in a plane ∏, then $Z_L \circ Z_L = id_\Pi$.

Algebraic Lemma. If J, K, L and M are lines in a plane ∏ and if $Z_J \circ Z_K = Z_L \circ Z_M$, then $Z_L \circ Z_J \circ Z_K = Z_M$ and $Z_J \circ Z_K \circ Z_M = Z_L$.

Proof. We are given the equation $Z_J \circ Z_K = Z_L \circ Z_M$. If we compose both sides of this equation on the left with Z_L, we get $Z_L \circ (Z_J \circ Z_K) = Z_L \circ (Z_L \circ Z_M)$. Regrouping the expression on the right side of this equation, we get:

$$Z_L \circ Z_J \circ Z_K = (Z_L \circ Z_L) \circ Z_M = id_\Pi \circ Z_M = Z_M.$$

On the other hand, if we compose both sides of the equation $Z_J \circ Z_K = Z_L \circ Z_M$ on the rightwith Z_M, we get $(Z_J \circ Z_K) \circ Z_M = (Z_L \circ Z_M) \circ Z_M$. Regrouping the expression on the right side of this equation, we get:

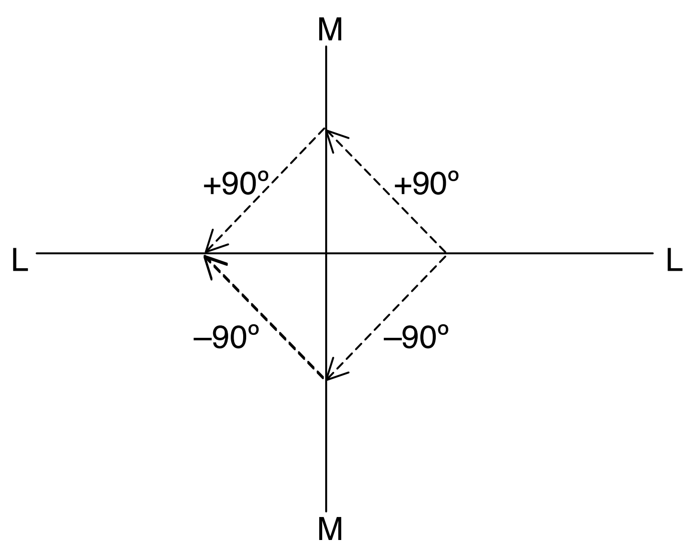
$$Z_J \circ Z_K \circ Z_M = Z_L \circ (Z_M \circ Z_M) = Z_L \circ id_\Pi = Z_L.$$

This completes the proof of the Algebraic Lemma.

We extract one additional piece of information from Observation 2 and Observation 2 Reversed. Suppose L and M are perpendicular lines that intersect at the point C. Then both the counterclockwise and clockwise oriented angles **from L to M** have the same unoriented angle measure: 90°. Thus, the counterclockwise oriented angle **from L to M** has oriented angle measure +90°, while the clockwise oriented angle **from L to M** has oriented angle measure –90°. The first of these two statements implies that $Z_M \circ Z_L = R_{C,180}$, while the second statement implies $Z_M \circ Z_L = R_{C,-180}$. There is no contradiction in these two statements because $R_{C,180} = R_{C,-180}$; in other words, a counterclockwise rotation around the point C through 180° is the same as a clockwise rotation about C through 180°. Thus, $Z_M \circ Z_L = R_{C,180} = R_{C,-180}$. We can apply the same analysis to the two oriented angles **from M to L**. Both these angles have the same unoriented angle measure (90°); one of these angles has oriented angle measure +90° while the other has oriented angle measure –90°. Thus, $Z_L \circ Z_M = R_{C,180} = R_{C,-180}$. We conclude that $Z_M \circ Z_L = Z_L \circ Z_M$. In the language of algebra: Z_L and Z_M *commute.*

These remarks prove the following statement.

The Commuting Trick. If L and M are two *perpendicular* lines in a plane Π, then $Z_M \circ Z_L = Z_L \circ Z_M$.



Activity 2. Each group should use the Sliding and Twisting Tricks to solve the following problems and report its results to the class.

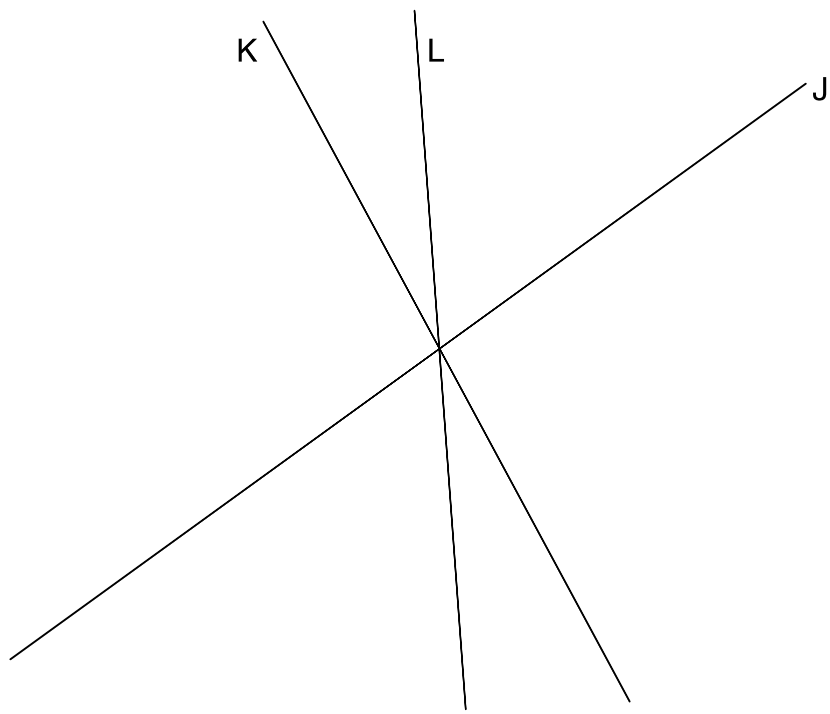
a) In the figure below, the lines J, K and L are drawn.

i) Draw a line M such that $Z_K \circ Z_J = Z_M \circ Z_L$.

ii) Draw a line N such that $Z_K \circ Z_J = Z_L \circ Z_N$.

iii) Complete the following two equations by filling in the blanks with the name of the appropriate reflection:

$Z_K \circ Z_J \circ Z_L =$ _______ $Z_L \circ Z_K \circ Z_J =$ _______



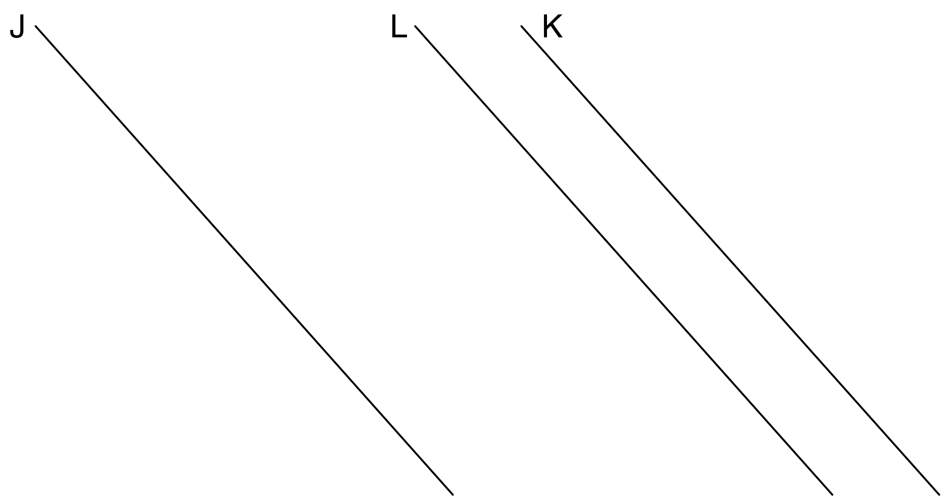
b) In the figure below, the lines J, K and L are drawn.

i) Draw a line M such that $Z_K \circ Z_J = Z_M \circ Z_L$.

ii) Draw a line N such that $Z_K \circ Z_J = Z_L \circ Z_N$.

iii) Complete the following two equations by filling in the blanks with the name of the appropriate reflection:

$$Z_K \circ Z_J \circ Z_L = \_\_\_\_\_\_ \qquad Z_L \circ Z_K \circ Z_J = \_\_\_\_\_\_$$



We will end the exposition in this lesson with a discussion of the Classification Theorem of Rigid Motions of a Plane. We recall the statement of this theorem.

Classification Theorem for Rigid Motions of a Plane. Every rigid motion of a plane is either a translation, a rotation, a reflection or a glide reflection.

The proof of this theorem is usually done in two steps:

Step 1. Prove that every rigid motion of a plane can be expressed as the composition of three or fewer reflections.

Step 2. Prove that every rigid motion which is the composition of three or fewer reflections must be either a translation, a rotation, a reflection or a glide reflection.

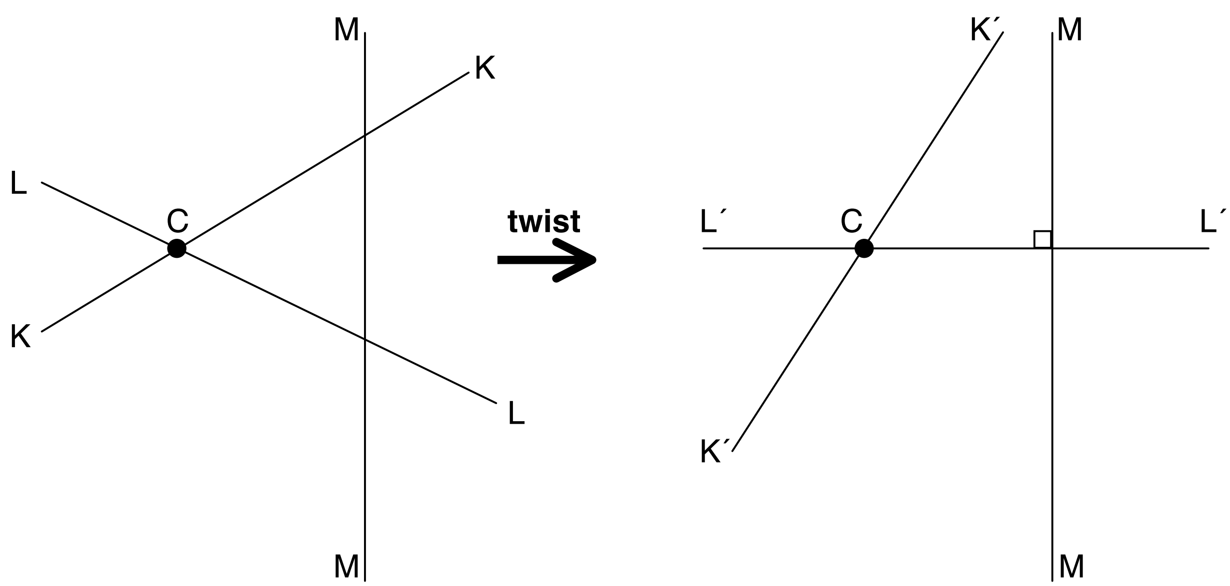
We will not discuss Step 1 of the proof in this course.

We have already made some observations that make progress on Step 2. For example, it is obvious that the composition of one reflection is simply a reflection.

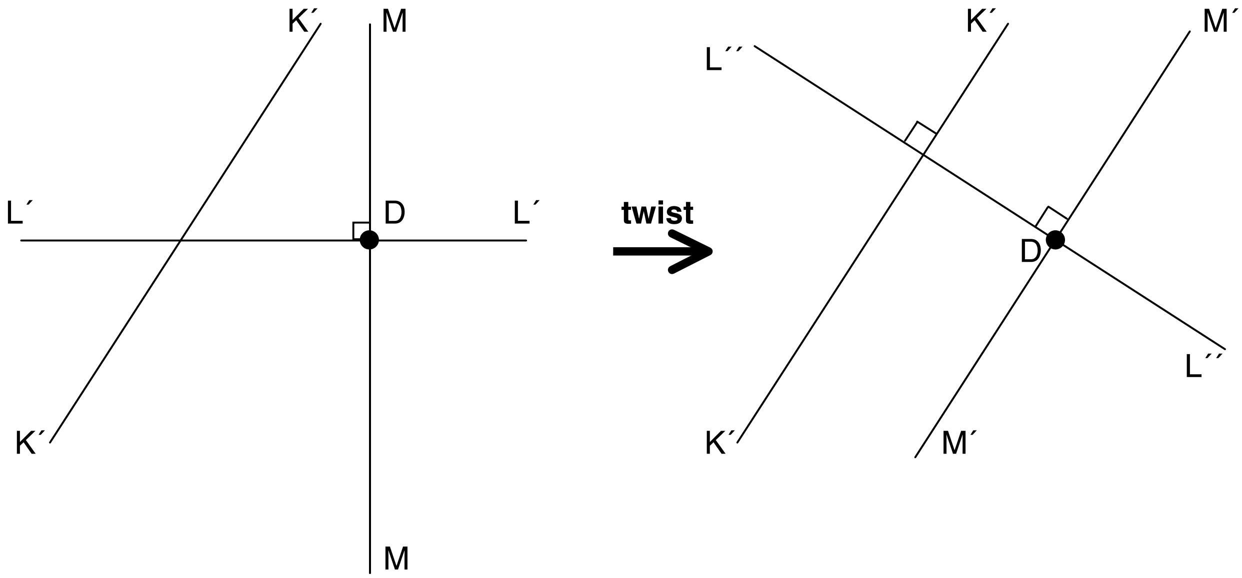
Furthermore, we know that the composition of two reflections is either a translation or a rotation: the composition of two reflections in *parallel* lines is a translation, while the composition of two reflections in *intersecting* lines is a rotation. In Activity 2, we saw that if three lines are *concurrent* (i.e., all three lines pass through a common point), then the composition of the corresponding reflections is a reflection. We also saw that if three lines are *parallel*, then the composition of the corresponding reflections is also a reflection. So to complete Step 2 of the proof of the Classification Theorem of Rigid Motions of a Plane, it remains to consider the composition of three reflections in lines that are neither concurrent nor parallel.

We will now complete Step 2 of the proof of the Classification Theorem of Rigid Motions of a Plane by show that if K, L and M are three lines that are neither concurrent nor parallel, then the composition $Z_M \circ Z_L \circ Z_K$ is a glide reflection.

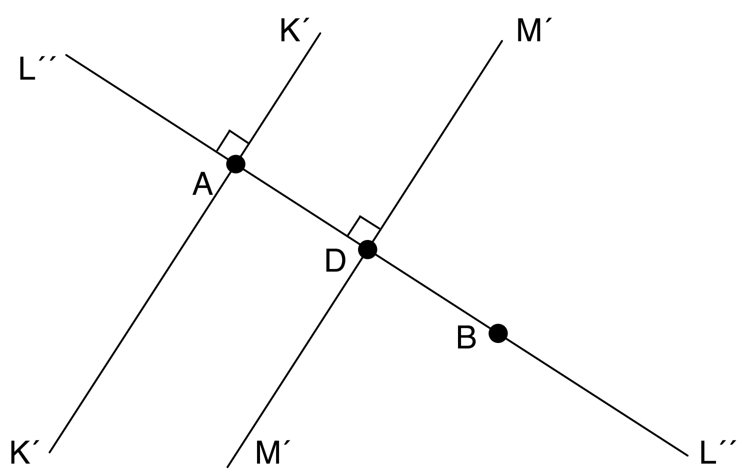
Let K, L and M be three lines that are neither parallel nor concurrent. Since the three lines are not all parallel, then at least two of them must intersect. Let us assume that lines K and L intersect. (If lines K and L were parallel but lines L and M intersect, then we could present a proof that is similar to the one we are about to describe.) Let us call the point where the lines K and L intersect "C". Use the Twisting Trick to twist lines K and L around C to lines K´ and L´ so that L´ is perpendicular to M. Then the Twisting Trick implies $Z_L \circ Z_K = Z_{L'} \circ Z_{K'}$. Hence, by substitution, $Z_M \circ Z_L \circ Z_K = Z_M \circ Z_{L'} \circ Z_{K'}$.



Let us call the point where L´ and M intersect (perpendicularly) "D". Use the Twisting Trick a second time to twist lines L´ and M around D to lines L´´ and M´ so that L´´ is perpendicular to K´. Then the Twisting Trick implies $Z_M \circ Z_{L'} = Z_{M'} \circ Z_{L''}$. Hence, by substitution, $Z_M \circ Z_{L'} \circ Z_{K'} = Z_{M'} \circ Z_{L''} \circ Z_{K'}$. Therefore, $Z_M \circ Z_L \circ Z_K = Z_{M'} \circ Z_{L''} \circ Z_{K'}$.

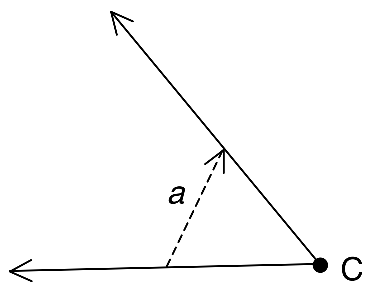


Since L'' and M' are perpenndicular, then the Commuting Trick implies $Z_{M'} \circ Z_{L''} = Z_{L''} \circ Z_{M'}$. Hence, by substitution, $Z_{M'} \circ Z_{L''} \circ Z_{K'} = Z_{L''} \circ Z_{M'} \circ Z_{K'}$. Therefore, $Z_M \circ Z_L \circ Z_K = Z_{L''} \circ Z_{M'} \circ Z_{K'}$. Since K' and M' are both perpendicular to L'', then K' and M' are parallel. Therefore, $Z_{M'} \circ Z_{K'}$ is a translation. We can name this translation $T_{A,B}$ if we let "A" denote the point where lines K' and L'' intersect, and we let "B" denote the point on the line L'' with the properties that the ray $\overrightarrow{AB}$ points in the direction from K' to M' and the distance AB is twice the (perpendicular) distance between lines K' and M'. (Hence, D is the midpoint of the line segment $\overline{AB}$.) Now $Z_{M'} \circ Z_{K'} = T_{A,B}$. Also, the lines L'' and $\overleftrightarrow{AB}$ coincide. Therefore, $Z_{L''} = Z_{\overleftrightarrow{AB}}$. Hence, by substitution, $Z_{L''} \circ Z_{M'} \circ Z_{K'} = Z_{\overleftrightarrow{AB}} \circ T_{A,B}$. Therefore, $Z_M \circ Z_L \circ Z_K = Z_{\overleftrightarrow{AB}} \circ T_{A,B}$. Now observe that the composition $Z_{\overleftrightarrow{AB}} \circ T_{A,B}$ is the glide reflection $G_{A,B}$: $Z_{\overleftrightarrow{AB}} \circ T_{A,B} = G_{A,B}$. Therefore, $Z_M \circ Z_L \circ Z_K = G_{A,B}$. This finishes



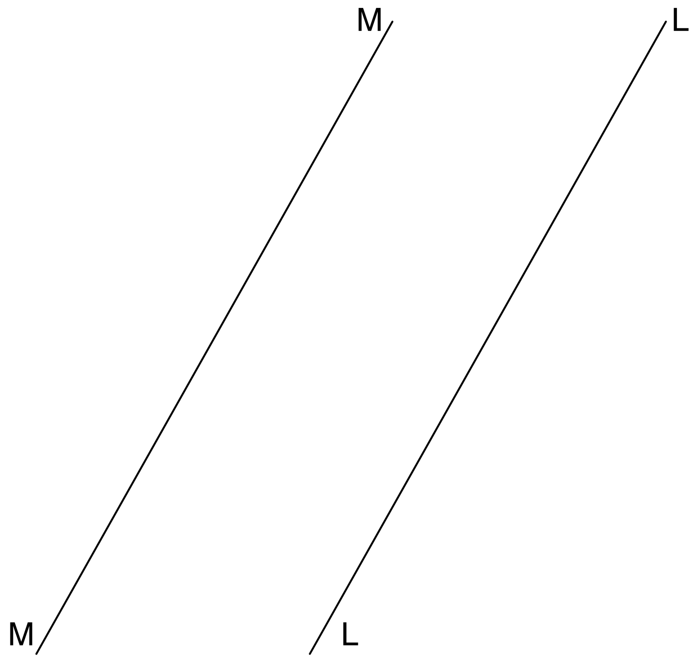
the proof that if K, L and M are three lines that are neither concurrent nor parallel, then the composition $Z_M \circ Z_L \circ Z_K$ is a glide reflection. Step 2 of the proof of the Classification Theorem for Rigid Motions of a Plane is now complete.

Homework Problem 1. The point C and the oriented angle with oriented angle measure *a* are drawn below. Draw and label lines L and M so that $R_{C,a} = Z_M \circ Z_L$.

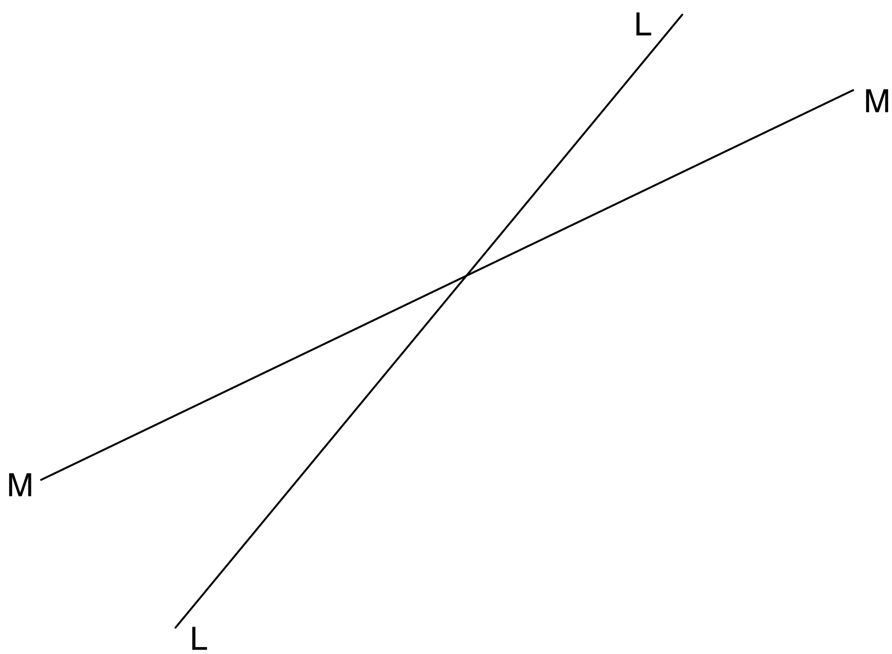


Homework Problem 2. The points A and B are drawn below. Draw and label lines L and M so that $T_{A,B} = Z_M \circ Z_L$.

Homework Problem 3. Lines L and M are drawn below. Draw and label points A and B so that $Z_M \circ Z_L = T_{A,B}$.



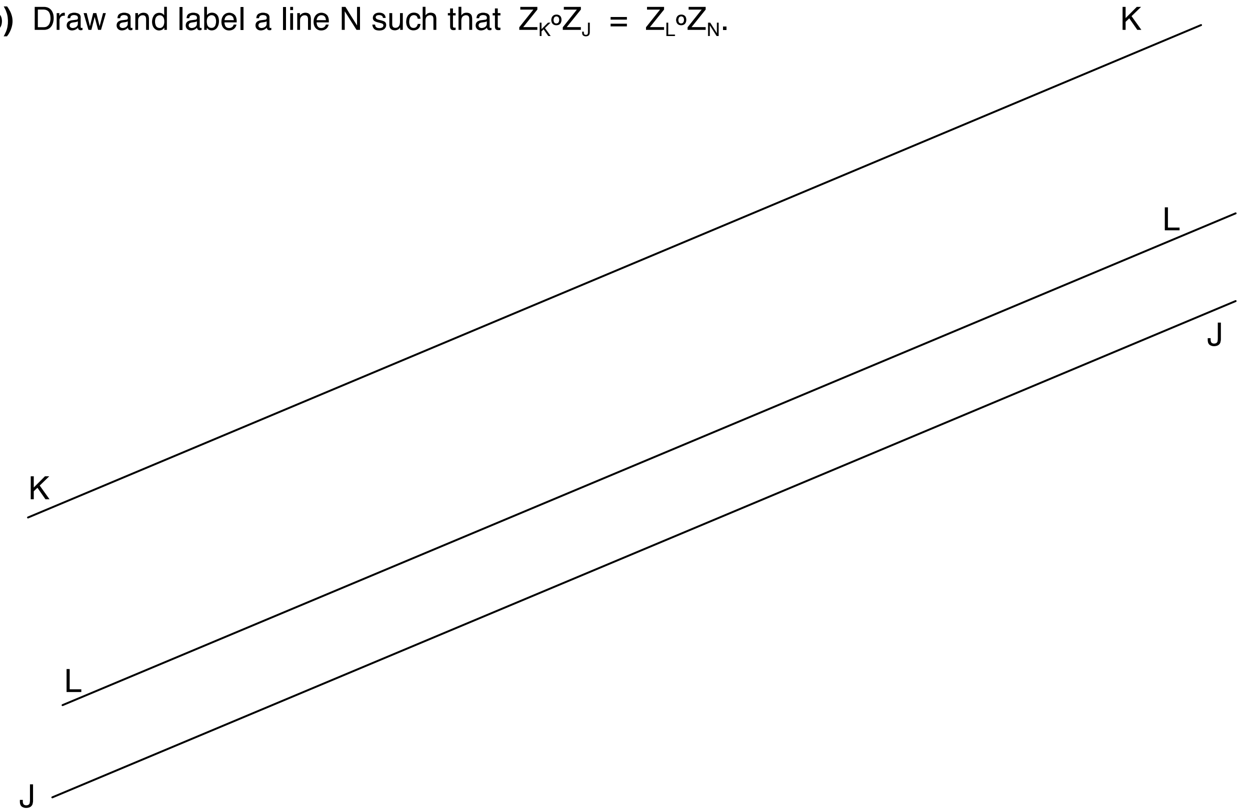
Homework Problem 4. Lines L and M are drawn below. Draw and label the point C and the oriented angle of oriented angle measure *a* so that $Z_M \circ Z_L = R_{C,a}$.



Homework Problem 5. Parallel lines J, K and L are drawn below.

a) Draw and label a line M such that $Z_K \circ Z_J = Z_M \circ Z_L$.

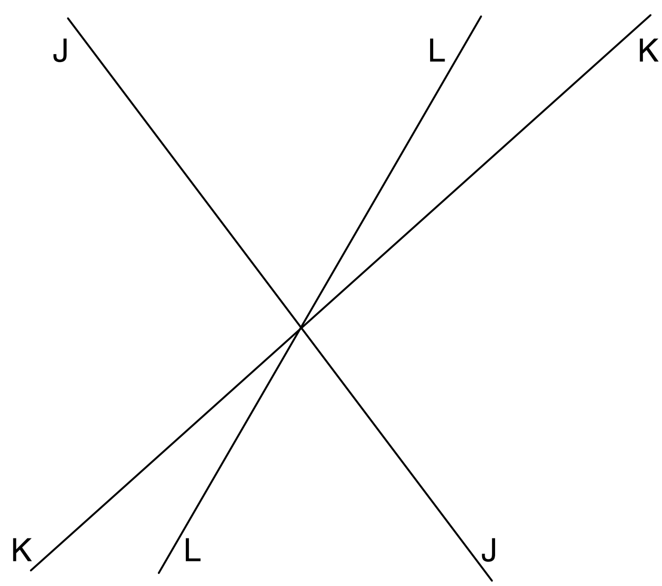
b) Draw and label a line N such that $Z_K \circ Z_J = Z_L \circ Z_N$.



Homework Problem 6. Concurrent lines J, K and L (all passing through the same point) are drawn below.

a) Draw and label a line M such that $Z_K \circ Z_J = Z_M \circ Z_L$.

b) Draw and label a line N such that $Z_K \circ Z_J = Z_L \circ Z_N$.

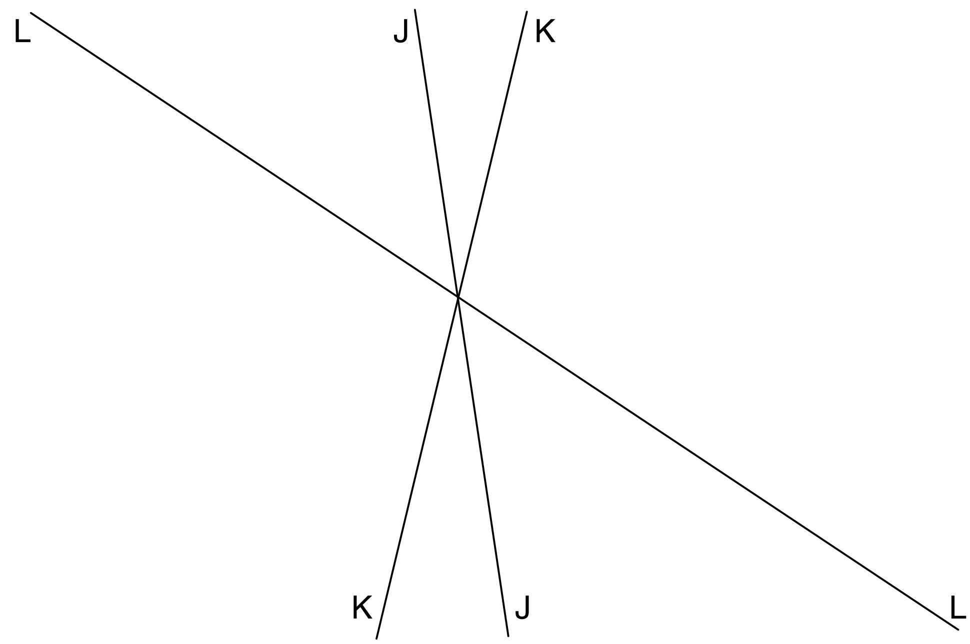


Homework Problem 7. Concurrent lines J, K and L (all passing through the same point) are drawn below.

a) Draw and label a line M such that $Z_L \circ Z_K \circ Z_J = Z_M$.

b) Draw and label a line N such that $Z_L \circ Z_J \circ Z_K = Z_N$.

c) Draw and label a line S such that $Z_J \circ Z_L \circ Z_K = Z_S$.



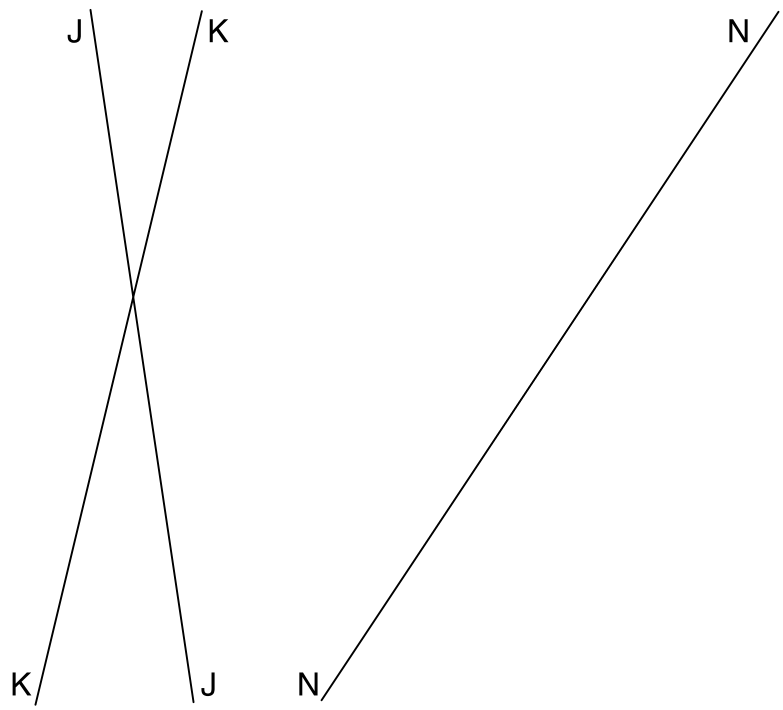
Homework Problem 8. Parallel lines J, K and L are drawn below.

a) Draw and label a line M such that $Z_L \circ Z_K \circ Z_J = Z_M$.

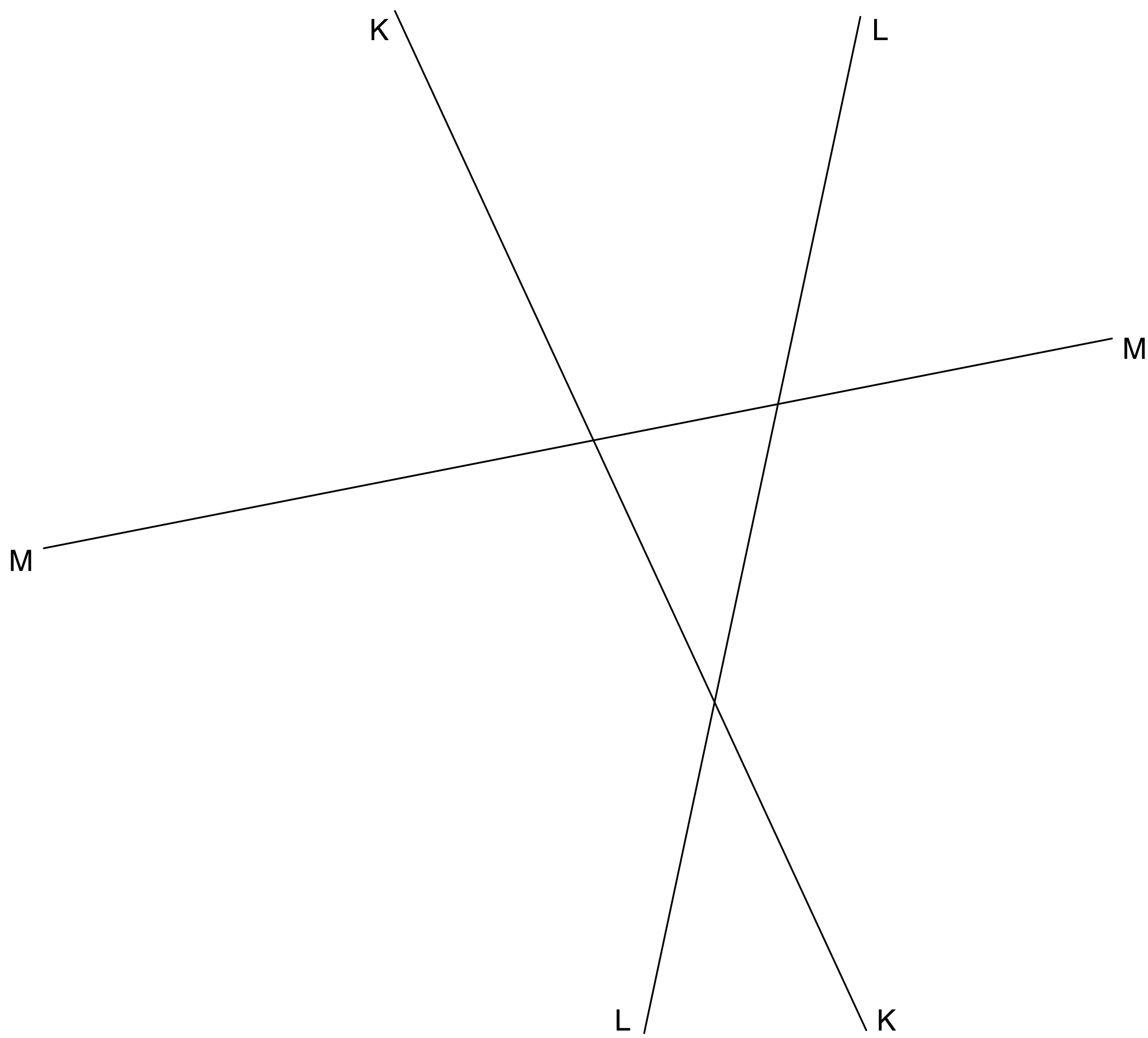
b) Draw and label a line N such that $Z_L \circ Z_J \circ Z_K = Z_N$.

c) Draw and label a line S such that $Z_J \circ Z_L \circ Z_K = Z_S$.

Homework Problem 9. Intersecting lines J and K and a line N are drawn below.
a) Draw and label lines L and M so that L is perpendicular to N and $Z_K \circ Z_J = Z_M \circ Z_L$.
b) Prove $Z_K \circ Z_J \circ Z_L = Z_M$.
c) Prove $Z_M \circ Z_K \circ Z_J = Z_L$.



Homework Problem 10. In the text above, a procedure is described which proves that if K, L and M are three lines in a plane ∏ that are neither parallel nor concurrent, then the composition $Z_M \circ Z_L \circ Z_K$ is a glide reflection. Three lines K, L and M that are neither parallel nor concurrent are drawn below. Imitate the procedure described above in the drawing below to locate and label two points A and B such that $Z_M \circ Z_L \circ Z_K = G_{A,B}$. Draw and label all lines and points created during the procedure as carefully and accurately as possible.



Homework Problem 11. Consider the following statement. "If L and M are two lines in a plane ∏ that do not coincide and are not perpendicular, then $Z_M \circ Z_L \neq Z_L \circ Z_M$." Is this statement true? Justify your answer.